Linear and quadratic approximation

November 11, 2013

Definition: Suppose f is a function that is differentiable on an interval I containing the point a. The *linear approximation* to f at a is the linear function

$$L(x) = f(a) + f'(a)(x - a), \quad \text{for } x \text{ in } I.$$

Now consider the graph of the function and pick a point P not he graph and look at the tangent line at that point. As you zoom in on the tangent line, notice that in a small neighbourhood of the point, the graph is more and more like the tangent line. In other words, the values of the function f in a neighbourhood of the point are being *approximated* by the tangent line.

Suppose f is differentiable on an interval I containing the point a. The change in the value of f between two points a and $a+\Delta x$ is approximately $\Delta y = f'(a).\Delta x$, where $a+\Delta x$ is in I.

Uses of Linear approximation

- To approximate f near $x = a$, use

  $$f(x) \approx L(x) = f(a) + f'(a)(x - a).$$

  Alternately,

  $$f(x) - f(a) \approx f'(a)(x - a).$$

- To approximate the change Δy in the dependent variable given a change Δx in the independent variable, use

  $$\Delta y \approx f'(a)\Delta x.$$

Differentials

Let f be differentiable on an interval containing x. A small change in x is denoted by the differential dx. The corresponding change in $y = f(x)$ is approximate by the differential $dy = f'(x)dx$; that is

$$\Delta y = f(x + dx) - f(x) \approx dy = f'(x)dx.$$

Quadratic approximation

Recall that if a function f is differentiable at a point a, then it can be approximated near a by its tangent line. We say that the tangent line provides a *linear approximation* to f at the point a. Recall also that the tangent line at the point $(a, f(a))$ is given by

$$y - f(a) = f'(a)(x - a) \text{ a } \textit{or } y = f(a) + f'(a)(x - a).$$

The linear approximation function $p_1(x)$ is the polynomial

$$p_1(x) = f(a) + f'(a)(x - a)$$

of degree one; this is just the equation to the tangent line at that point. This polynomial has the property that it matches f in value and slope at the point a. In other words,

$$p_1(a) = f(a) \text{ and } p_1'(a) = f'(a).$$

We would like to do better, namely we would also like to get a better approximation to the function in that we match concavity of f as well at this point. Recall that the concavity information is coded in the second derivative. We thus create a quadratic approximating polynomial $p_2(x)$ of degree two defined by:

$$p_2(x) = f(a) + f'(a)(x - a) + c_2(x - a)^2; \;\; p_2(x) = p_1(x) + c_2(x - a)^2,$$

where $c_2 = 1/2f''(a)$. So we have

$$p_2(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2 = p_1(x) + \frac{1}{2}f''(a)(x - a)^2.$$

Note that

$$p_2(a) = f(a), \; p_2'(a) = f'(a), \; p_2''(a) = f''(a),$$

where we assume that f and its first and second derivatives exist at a. The polynomial $p_2(x)$ is the quadratic approximating polynomial for f at the point a. The quadratic approximation gives a better approximation to the function near a than the linear approximation.

In solving linear approximation problems, you should first look for the function $f(x)$ as well as the point a, so that you can approximate f at a point close to a. The advantage of linear approximation is the following; the function f that one is considering might be very complicated, but *near* a point a in its domain, we are able to estimate the value of the function using a polynomial of degree one (i.e. a linear function) which is far simpler than the original function. For example, let us consider the following

PROBLEM: Approximate the number $\sqrt[4]{1.1}$.

We use the fact that a curve lies very close to its tangent (linear approximation) near the point of tangency. So let $f(x) = \sqrt[4]{x}$. First we look for a point a close to 1.1, such that $f(a)$ is easy to compute, but $f(x)$ is difficult to compute for x close to a. So we use the tangent line at $(a, f(a))$ and use it as an approximation to the curve $y = f(x)$.

Clearly, 1 is close to 1.1 and $\sqrt[4]{1}$ is easy to compute! So we take $a = 1$. Then the tangent line is

$$y = f(a) + f'(a)(x - a).$$

As $f(a) = 1$ and $f'(a) = \frac{1}{4a^{3/4}}$. The equation to the tangent line is

$$f(a) + f'(a)(x - a),$$

and

$$f(x) \approx f(a) + f'(a)(x - a).$$

Putting $a = 1$ and $x = 1.1$, we have

$$\sqrt[4]{1.1} \approx \sqrt[4]{1} + \frac{1}{4.1^{3/4}}(1.1 - 1) = 1 + \frac{1}{4}(0.1) = 1.025.$$

Underestimates and overestimates

If $f(x)$ is concave up in a neighbourhood of $(a, f(a))$, then the tangent line lies below the graph of f, and the approximated value is an *underestimate*. In this case, the value obtained by linear approximation is less than the actual value of $f(x)$ in a neighbourhood of a. If the tangent line lies above the graph of f, then the approximated value is an *overestimate*. We remake that linear approximation gives good estimates when x is close to a but the accuracy of the approximation gets worse when the points are farther away from 1. Also, a calculator would give an approximation for $\sqrt[4]{1.1}$, but linear approximation gives an approximation over a small interval around 1.1.

Percentage Error

Suppose you have used linear or quadratic approximation centred around a to approximate $f(c)$, for a point c close to a.The *percentage error* is calculated using the formula

$$\frac{(True\ value) - (approximated\ value)}{(True\ Value)} x100.$$

Thus if you have used linear approximation using the linear polynomial, the approximated value at a point c close to a is given by $p_1(c)$ and the percentage error is given by

$$\frac{f(c) - p_1(c)}{f(c)} \times 100.$$

If you have used quadratic approximation using the formula $p_2(x)$, we would use $p_2(c)$ instead of p_1 in the above formula.

Taylor series

Recall that

$$p_1(x) = f(a) + f'(a)(x-a)$$

is the linear approximating polynomial for $f(x)$ centred around a and

$$p_2(x) = f(a) + f'(a)(x-a) + f''(a)/2(x-a)^2$$

is the quadratic polynomial. The polynomial $p_1(x)$ is the equation of the tangent line at the centre point $(a, f(a)$, and these are used to estimate values of $f(x)$ for points x that are close to a. For example, you should check that for $f(x) = \operatorname{Sin} x$, with centre $a = 0$, we have $p_1(x) = x$. The quadratic approximation gives a better estimate for $f(x)$ for x near a when compared with the linear polynomial $f_1(x)$.

Suppose a function f has derivatives $f^{(k)}(a)$ of all orders at the point a. The *Taylor polynomial* of order (or degree) n is defined by

$$p_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$$

where $c_k = \frac{f^{(k)}(a)}{k!}$.

Of course, $p_1(x)$ and $p_2(x)$ are as above. If we let $n \to \infty$, we obtain a power series, called the *Taylor series* for f centred at a. The special case of a Taylor series centred at 0 is called a *Maclaurin series*. The higher order Taylor polynomials give better and better approximations for $f(x)$ in a neighbourhood of the centre a.

Errors

Definition: The error in the linear approximation of a function $f(x)$ is M where $|R(x)| = |f(x) - p_1(x)| \leq M$.

To find M, we should find an upper bound for the difference between the actual value and the approximated value $p_1(x)$. Notice that the difference between p_1 and f increases as we mover farther away from the centre a. Notice also that the difference increases if the function bends away from the tangent. In other words, the larger the absolute value of the second derivative at a, $\mid f''(a) \mid$, the greater the deviation of $f(x)$ from the tangent line. The linear approximation at a is more accurate for f, when the rate of change of f', which is nothing else but f'' is smaller.

Remainder

In general, write $f(x) = p_n(x) + R_n(x)$, where $n = 1$ or 2 so that $p_1(x)$ is the linear approximating polynomial and $p_2(x)$ is the quadratic approximation polynomial. To estimate the remainder term $R_n(x)$, we find a number M such that $\mid f^{(n+1)}(c) \mid \leq M$, for all c between a and x inclusive, Then the remainder satisfies

$$\mid R_n(x) \mid = \mid f(x) - p_n(x) \mid \leq M \frac{\mid x-a \mid^{n+1}}{(n+1)!},$$

where $p_1(x)$ (respectively $p_2(x)$) is the linear (respectively quadratic) approximating polynomial centred around a. A similar formula holds for the remainder term for the n-th order Taylor polynomial.

Derivatives of inverse trigonometric functions

$$
\begin{aligned}
\frac{d}{dx}(sin^{-1}x) &= \frac{1}{\sqrt{1-x^2}} \text{ for } -1 < x < 1. \\
\frac{d}{dx}(cos^{-1}x) &= \frac{-1}{\sqrt{1-x^2}} \text{ for } -1 < x < 1. \\
\frac{d}{dx}(tan^{-1}x) &= \frac{1}{\sqrt{1+x^2}} \text{ for } -\infty < x < \infty. \\
\frac{d}{dx}(cot^{-1}x) &= \frac{1}{\sqrt{1+x^2}} \text{ for } -\infty < x < \infty. \\
\frac{d}{dx}(sec^{-1}x) &= \frac{1}{|x|\sqrt{x^2+1}} \text{ for } |x| > 1. \\
\frac{d}{dx}(cosec^{-1}x) &= \frac{1}{|x|\sqrt{x^2-1}} \text{ for } |x| > 1.
\end{aligned}
$$