Math 426
Homework 1
Due 19 September 2022

1. Let $X = \{g, s\}$, and endow X with the following topology: The subsets $\{\emptyset, X, \{g\}\}$ are open. Give $[0, 1]$ the usual metric topology.

(a) Suppose $f : X \to [0, 1]$ is a continuous function such that $f(s) = 0$. Show that $f(g) = 0$.

(b) Produce, with proof, a nonconstant continuous function $f : [0, 1] \to X$.

2. Let (X, d) be a metric space. Recall that a sequence (x_n) in X is said to be a *Cauchy sequence* if, for all $\epsilon > 0$, there exists some $N_\epsilon \in \mathbf{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m > N_\epsilon$. The space X is said to be *complete* if every Cauchy sequence converges in X. Given an example, with proof, of a homeomorphism $f : X \to Y$ of metric spaces where X is complete and Y is not complete.

3. Let X, Y be topological spaces and $f : X \to Y$ a function between them. As usual in this course, when a topology on a subset is not otherwise specified, the subspace topology is assumed.

(a) Suppose A, B are closed subsets of X such that $X = A \cup B$, and suppose that $f|_A : A \to Y$ and $f|_B : B \to Y$ are continuous. Prove that f is continuous.

(b) Suppose that for all $x \in X$, there exists an open set $U \ni x$ such that $f|_U$ is continuous. Prove f is continuous.

(c) Give an example, with proof, of sets X and Y and a discontinuous function $f : X \to Y$ such that for all $x \in X$, there exists a closed set $A \ni x$ such that $f|_A$ is continuous.