1. Let $X = \{g, s\}$, and endow $X$ with the following topology: The subsets $\emptyset, X, \{g\}$ are open. Give $[0,1]$ the usual metric topology.

(a) Suppose $f : X \to [0,1]$ is a continuous function such that $f(s) = 0$. Show that $f(g) = 0$.

(b) Produce, with proof, a nonconstant continuous function $f : [0,1] \to X$.

(a) The metric topology on $[0,1]$ is Hausdorff, so that $\{0\}$ is closed. The function $f$ is continuous, so that $f^{-1}(\emptyset)$ must be a closed subset of $X$, and $g \in f^{-1}(\emptyset)$. Based on the description of the topology, any closed set of $X$ containing $g$ must be all of $X$. It follows that $f^{-1}(\emptyset) = X$, i.e., $f(x) = 0$ for all $x \in X$.

(b) Let us set $f(0) = s$ and $f(x) = g$ for all $x > 0$. There are three open sets in $X$: $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}([0,1]) = X$ are trivially satisfied. The only nontrivial condition that must be satisfied is that $f^{-1}(g)$ should be open, and $f^{-1}(g) = (0,1]$ is indeed open in $[0,1]$.

\[ \square \]

**Note 0.1.** Here is a sketch of an alternative solution.

Take $[0,1]$ and consider the subspace $(0,1]$. Form the quotient $q : [0,1] \to [0,1]/(0,1]$. Observe that $[0,1]/(0,1]$ consists of two points: the point corresponding to 0 and the point corresponding to $(0,1]$. Let us name these “s” and “g” respectively (the letters stand for “special” and “generic”). The quotient topology on $(g, s)$ is such that $\{g\}$ is open because $q^{-1}(g) = (0,1]$ which is open in $[0,1]$. There are no other nontrivial open sets.

In other words, we may identify the space $X$ in the question with the quotient $[0,1]/(0,1]$. By construction there exists a continuous surjective $q : [0,1] \to X$ which is certainly not constant. This settles part (b).

For part (a): a continuous function $\tilde{f} : X \to [0,1]$ is equivalent (universal property of the quotient) to a function $f : [0,1] \to [0,1]$ such that $f|_{(0,1]}$ is constant. Since $(0,1]$ is dense in $[0,1]$, if $Y$ is a Hausdorff space, then any continuous function $[0,1] \to Y$ that agrees with a constant function on $(0,1]$ must actually agree with that constant function on all of $[0,1]$, i.e., it must be constant. That implies that, for any Hausdorff space $Y$, every continuous function $X \to Y$ is actually constant. This is more than enough to settle part (a).
2. Let \((X, d)\) be a metric space. Recall that a sequence \((x_n)\) in \(X\) is said to be a \textit{Cauchy sequence} if, for all \(\epsilon > 0\), there exists some \(N_\epsilon \in \mathbb{N}\) such that \(d(x_n, x_m) < \epsilon\) for all \(n, m > N_\epsilon\). The space \(X\) is said to be \textit{complete} if every Cauchy sequence converges in \(X\). Given an example, with proof, of a homeomorphism \(f : X \to Y\) of metric spaces where \(X\) is complete and \(Y\) is not complete.

There are many examples. Consider \(f : \mathbb{R} \to (-\pi/2, \pi/2)\) given by \(f(x) = \arctan(x)\). This is differentiable, therefore continuous, and has a differentiable inverse \(f^{-1}(x) = \tan x\). On the other hand, a sequence in \((-\pi/2, \pi/2)\) converging to \(\pi/2\) is Cauchy, but has no limit in \((-\pi/2, \pi/2)\), so the bounded interval is not complete, whereas \(\mathbb{R}\) is well known to be complete. \(\square\)

3. Let \(X, Y\) be topological spaces and \(f : X \to Y\) a function between them. As usual in this course, when a topology on a subset is not otherwise specified, the subspace topology is assumed.

(a) Suppose \(A, B\) are closed subsets of \(X\) such that \(X = A \cup B\), and suppose that \(f|_A : A \to Y\) and \(f|_B : B \to Y\) are continuous. Prove that \(f\) is continuous.

(b) Suppose that for all \(x \in X\), there exists an open set \(U \ni x\) such that \(f|_U\) is continuous. Prove \(f\) is continuous.

(c) Give an example, with proof, of sets \(X\) and \(Y\) and a discontinuous function \(f : X \to Y\) such that for all \(x \in X\), there exists a closed set \(A \ni x\) such that \(f|_A\) is continuous.

(a) First a small lemma.

\textbf{Lemma 0.2.} If \(A \subset X\) is a closed subset and \(K \subset A\) is closed in the subspace topology on \(A\), then \(K\) is a closed subset of \(X\).

\textit{Proof.} Since \(K \subset A\) is closed, there exists an open set \(W \subset X\) such that \(W \cap A = A \setminus K\). But now \((X \setminus W) \cap A = A \setminus (W \cap A) = K\) is the intersection of two closed subsets of \(X\) and is therefore closed. \(\square\)

For the main part of the answer, we proceed as follows. It is sufficient to prove that \(f^{-1}(K)\) is closed in \(X\) for all closed \(K \subset Y\). Consider \(f|_A^{-1}(K)\). This is a closed set in \(A\), since \(f|_A\) is continuous. Similarly, \(f|_B^{-1}(K)\) is closed in \(B\). Since \(A\) and \(B\) are themselves closed in \(X\), the lemma says that \(f|_A^{-1}(K)\) and \(f|_B^{-1}(K)\) are closed subsets of \(X\). But

\[ C = f|_A^{-1}(K) \cup f|_B^{-1}(K) \]

consists exactly of those elements \(x\) that are in \(A \cup B = X\) such that \(f(x) \in K\). That is, \(C = f^{-1}(K)\), and since this is closed, we are done.

(b) Let \(V\) be an open subset of \(Y\). We will show that for all \(x \in f^{-1}(V)\), there exists an open set \(W_x \ni x\) and \(W_x \subset f^{-1}(V)\). This suffices by Proposition 1.9.
Suppose \( x \in f^{-1}(V) \). There exists some open \( U \ni x \) such that \( f|_U : U \to Y \) is continuous. Therefore \( f|_U^{-1}(V) \) is an open subset of \( U \). We may write \( f|_U^{-1}(V) = V' \cap U \) where \( V' \subset X \) is open. The intersection of two open sets is open, so that \( f|_U^{-1}(V) \) is open in \( X \). Defining \( W_x = f|_U^{-1}(V) \), we see that \( W_x \) has the required property.

(c) Consider any discontinuous function \( f : \mathbb{R} \to \mathbb{R} \). For instance, one may take \( f(x) = 0 \) if \( x < 0 \) and \( f(x) = 1 \) if \( x \geq 1 \). The singleton sets \( \{x\} \subset \mathbb{R} \) are closed, and every function whose domain is a singleton is trivially continuous (the only topology on the singleton is the discrete topology). Therefore, for each \( x \), the set \( \{x\} \) plays the part of \( A \).