Math 426
Homework 2
Due 26 September 2022

1. Let (X, τ) be a topological space and $A \subset X$ a subset. Recall that the *subspace topology* τ_A is defined as follows: $U \in \tau_A$ if there exists some $V \in \tau$ such that $U = V \cap A$.

(a) Prove that τ_A defines a topology.

(b) Prove that $\tau_A \subset \tau$ if and only if A is open in X.

(c) Prove that $C \subset A$ is closed in the subspace topology on A (i.e., is the complement $A \setminus U$ for some $U \in \tau_A$) if and only if there exists some closed subset K of X such that $C = K \cap A$.

2. Suppose $\{X_j\}_{j \in J}$ and $\{Y_j\}_{j \in J}$ are two families of topological spaces and that $\{f_j : X_j \to Y_j\}_{j \in J}$ is a set of continuous functions between them. Prove there exists a unique function

$$f : \prod_{j \in J} X_j \to \prod_{j \in J} Y_j$$

with the property that $\mathrm{proj}_j f(x) = f_j(\mathrm{proj}_j x)$ for all $x \in \prod_{j \in J} X_j$, and prove that this function f is continuous. (Hint: it may be helpful to use the universal property of the product, Remark 1.56).

3. Let $\{X_i\}_{i \in I}$ be a family of spaces and let $\mathrm{proj}_i : \prod_{i \in I} X_i \to X_i$ denote the projection maps.

(a) Prove that for any family of subsets $\{A_i \subset X_i\}$,

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i}$$

(b) Give an example to show that the analogous statement is false when the closure is replaced by the interior.

(c) Suppose that the spaces X_i are all Hausdorff. Prove that $\prod_{i \in I} X_i$ is Hausdorff.