1. Let $X$ be a first-countable topological space. Prove that a subset $A$ of $X$ is closed if it is sequentially closed. Specifically, suppose $x \in \cl{A}$. Produce a sequence of elements $a_n \in A$ such that $(a_n) \to x$.

Choose a countable local base $\{U_n\}$ for $x$ and define $W_n = U_1 \cap U_2 \cap \cdots \cap U_n$. Each $W_n$ is an open neighbourhood of $x$ and therefore $W_n \cap A \neq \emptyset$. For each $n$, choose $a_n \in W_n \cap A$. This gives a sequence in $A$ with the following property: if $V \ni x$ is an open neighbourhood, then $V \supseteq U_k$ for some $k$ and therefore $V \supseteq W_{k+d}$ for all $d \geq 0$. Therefore $V \ni a_{k+d}$ for all $d \geq 0$. Consequently, $(a_n) \to x$ as required. □

2. A number of ‘universal properties’ are asserted in the class notes but not proved. The proofs of these all have a similar flavour, and it is best if you do them yourself. This exercise makes you work through one such proof in detail, and use the property to show the space enjoying the property is determined by it up to homeomorphism. Needless to say, you cannot assert the universal property of the quotient in proving these results—although for other homework exercises, doing so is fine and even encouraged.

Suppose $X$ is a topological space and $\sim$ is an equivalence relation on $X$. The definition of the topological space $X/\sim$ and the quotient function $q: X \to X/\sim$ was given in the notes.

(a) Suppose that $Y$ is some topological space, $f: X \to Y$ is a continuous function and suppose that $f(x) = f(x')$ whenever $x \sim x'$. Prove that there is a continuous function $\bar{f}: X/\sim \to Y$ such that $f = \bar{f} \circ q$ and $\bar{f}$ is uniquely determined by this equation. This is the universal property of the quotient.

(b) Now suppose that $p: X \to Z$ is some other continuous function satisfying $p(x) = p(x')$ whenever $x \sim x'$ and having the universal property of the quotient: specifically whenever $f: X \to Y$ is a continuous function such that $f(x) = f(x')$ for all $x \sim x'$, then there is a unique continuous function $g: Z \to Y$ such that $f = g \circ p$. Prove that there exists a unique continuous function $\phi: X/\sim \to Z$ such that $\phi \circ q = p$.

(c) Prove that $\phi$ is a homeomorphism.

(a) Note that $q: X \to X/\sim$ is surjective by construction. We may define a function $\tilde{f}: X/\sim \to X$ as follows: given any $q(x) \in X/\sim$, we can find $x \in q(x)$; set $\tilde{f}(q(x)) = f(x)$. The hypothesis on $f$
says \( f(x) = f(x') \) whenever \( x \sim x' \) ensures that \( \tilde{f}(q(x)) \) does not depend on the specific \( x \in q(x) \) that was chosen.

For any \( x \in X \), we have \( \tilde{f}(q(x)) = f(x) \), which is to say \( \tilde{f} \circ q = f \).

To show \( \tilde{f} \) is continuous, consider an arbitrary open subset \( U \subset Y \). Then \( \tilde{f}^{-1}(U) \) is a subset of \( X/\sim \). We know that \( q^{-1}(\tilde{f}^{-1}(U)) = f^{-1}(U) \), which is open. The definition of the co-induced topology on \( X/\sim \) tells us that \( \tilde{f}^{-1}(U) \) is open in \( X/\sim \). Since \( U \) was arbitrary, this shows \( \tilde{f} \) is continuous.

Now let us consider the question of uniqueness. Suppose \( g : X/\sim \to Y \) is a function for which \( g \circ q = f \). Then for every \( q(x) \in X/\sim \), we must have \( g(q(x)) = f(x) = \tilde{f}(q(x)) \). Hence \( g = \tilde{f} \).

(b) In particular, \( p : X \to Z \) is a continuous function that has the property that \( p(x) = p(x') \) whenever \( x \sim x' \). Therefore the universal property in the last part applies and we deduce that there exists a continuous function \( \phi : X/\sim \to Z \) that is uniquely determined by the property \( \phi \circ q = p \).

(c) Similarly to the previous part, there exists a continuous function \( \psi : Z \to X/\sim \) that is uniquely determined by the property \( \psi \circ p = q \).

Now consider the continuous function \( \psi \circ \phi : X/\sim \to X/\sim \). This function makes the diagram below commute

\[
\begin{array}{ccc}
  X & \xrightarrow{\phi} & Z \\
  \downarrow{p} & & \downarrow{\psi} \\
  X/\sim & \xrightarrow{\psi \circ \phi} & X/\sim
\end{array}
\]

We can remove certain objects from this, leaving a diagram that is still commutative:

\[
\begin{array}{ccc}
  X & \xrightarrow{p} & X/\sim \\
  \downarrow{\phi} & & \downarrow{\psi \circ \phi} \\
  X/\sim & \xrightarrow{\psi \circ \phi} & X/\sim
\end{array}
\]

This shows that \( \psi \circ \phi : X/\sim \to X/\sim \) is a continuous function of the kind that is asserted to exist by the universal property of the quotient \( X/\sim \) applied to the continuous function \( p : X \to X/\sim \). By the uniqueness part of the universal property, \( \psi \circ \phi \) is uniquely determined by the fact that it makes (1) commute.

The identity function \( \text{id} : X/\sim \to X/\sim \) also makes this diagram commute:

\[
\begin{array}{ccc}
  X & \xrightarrow{p} & X/\sim \\
  \downarrow{\text{id}} & & \downarrow{\text{id}} \\
  X/\sim & \xrightarrow{\text{id}} & X/\sim
\end{array}
\]

We conclude that \( \text{id} = \psi \circ \phi \).

A completely symmetric argument shows that \( \phi \circ \psi \) is the identity on \( Z \) as well. Therefore both \( \phi \) and \( \psi \) are homeomorphisms.
3. If $X$ is a Hausdorff topological space, we say that $X$ is perfectly normal if, for every closed set $A$, there exists a continuous function $f_A : X \rightarrow [0, 1]$ such that $f_A^{-1}(0) = A$. Prove that metric spaces are perfectly normal. Hint: it may help to define a function $X \rightarrow [0, \infty)$ by

$$
\text{dist}_A(x) = \inf\{d(x, a) \mid a \in A\}
$$

and prove that $\text{dist}_A(x)$ is continuous.

Given a set $A$ in a metric space $X$, we define $\text{dist}_A : X \rightarrow \mathbb{R}$ by the formula $\text{dist}_A(x) = \inf\{d(x, a) \mid a \in A\}$. We prove that $\text{dist}_A$ is continuous by an $\varepsilon$-$\delta$ argument.

Choose $\varepsilon > 0$. Given $x, y \in X$ such that $d(x, y) < \varepsilon$, we calculate

$$
\text{dist}_A(y) = \inf\{d(y, a) \mid a \in A\} \leq \inf\{d(y, x) + d(x, a) \mid a \in A\} = d(x, y) + \text{dist}_A(x) < \text{dist}_A(x) + \varepsilon,
$$

from this and a symmetric argument, we see that $|\text{dist}_A(x) - \text{dist}_A(y)| < \varepsilon$.

This establishes continuity of $\text{dist}_A$.

Suppose $\text{dist}_A(x) = 0$. Then every open ball $B(x, \varepsilon)$ must meet $A$, so that $x \in \bar{A}$. If $A$ is closed, we deduce that $\text{dist}_A(x) = 0$ implies $x \in A$.

There is the minor inconvenience that $\text{dist}_A$ may be unbounded, while $f_A$ is supposed to be bounded. We may define

$$
f_A(x) = \frac{\text{dist}_A(x)}{1 + \text{dist}_A(x)}
$$

to produce an $f_A$ meeting the conditions of the question.

This shows that metric spaces are perfectly normal. $\square$