Math 426
Homework 4
Due 11 October 2022

1. Recall that a Hausdorff space X is *perfectly normal* if, for every closed set A, there exists a continuous function $f_A : X \to [0,1]$ such that $f_A^{-1}(0) = A$. In such a space, for any two disjoint closed sets A, B, there exists a continuous function

$$f : X \to [0,1], \qquad f = \frac{f_A}{f_A + f_B}$$

for which $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

Suppose X is perfectly normal, that C is a closed subset of X and $g : C \to [-1,1]$ is a continuous function.

(a) By considering $g^{-1}([-1,-1/3])$ and $g^{-1}([1/3,1])$, construct a continuous function $h_1 : X \to [-1/3,1/3]$ such that $|h_1(c) - g(c)| \le 2/3$ for all $c \in C$.

(b) Produce a sequence (h_n) of continuous functions $h_n : X \to [-1+(2/3)^n, 1-(2/3)^n]$ such that

i. $|h_n(c) - g(c)| \le (2/3)^n$ for all $c \in C$.
ii. $|h_n(x) - h_{n-1}(x)| \le (1/3)(2/3)^{n-1}$ for all $x \in X$.

2. You may use the results of the previous problem in answering this one.

(a) Suppose $(f_n : X \to \mathbf{R})$ is a sequence of functions, where X is a topological space. We say that (f_n) converges to a function $f : X \to \mathbf{R}$ *uniformly* if, for all $x \in X$ and all $\epsilon > 0$, there exists some $N_\epsilon \in \mathbf{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for all $n > N_\epsilon$. Suppose you are given a sequence of continuous functions $f_n : X \to \mathbf{R}$ converging uniformly to $f : X \to \mathbf{R}$. Prove that f is continuous.

(b) Suppose X is a perfectly normal topological space and $C \subset X$ is a closed subset. Suppose $g : C \to [-1,1]$ is a continuous function. Construct a continuous function $h_\infty : X \to [-1,1]$ such that $h_\infty(c) = g(c)$ for all $c \in C$.

3. Give $\mathbf{Q}$ the usual topology. Let $C \subset \mathbf{Q}$ be a compact subset. Prove that C is nowhere dense in $\mathbf{Q}$. It may help to consider C as a subset of $\mathbf{R}$. You may assume that $\mathbf{Q}$ is dense in $\mathbf{R}$.