I. Recall that a Hausdorff space $X$ is perfectly normal if, for every closed set $A$, there exists a continuous function $f_A : X \to [0,1]$ such that $f_A^{-1}(0) = A$. In such a space, for any two disjoint closed sets $A, B$, there exists a continuous function

$$f : X \to [0,1], \quad f = \frac{f_A}{f_A + f_B}$$

for which $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

Suppose $X$ is perfectly normal, that $C$ is a closed subset of $X$ and $g : C \to [-1,1]$ is a continuous function.

(a) By considering $g^{-1}([-1,-1/3])$ and $g^{-1}([1/3,1])$, construct a continuous function $h_1 : X \to [-1/3,1/3]$ such that $|h_1(c) - g(c)| \leq 2/3$ for all $c \in C$.

(b) Produce a sequence $(h_n)$ of continuous functions $h_n : X \to [-1 + (2/3)^n, 1 - (2/3)^n]$ such that

i. $|h_n(c) - g(c)| \leq (2/3)^n$ for all $c \in C$.

ii. $|h_n(x) - h_{n-1}(x)| \leq (1/3)(2/3)^{n-1}$ for all $x \in X$.

(a) The sets $A = g^{-1}([-1,-1/3])$ and $B = g^{-1}([1/3,1])$ are closed in $X$ and disjoint, and so, since $X$ is perfectly normal, we can produce a continuous $h_1 : X \to [-1/3,1/3]$ that takes the value $-1/3$ on $A$, the value $1/3$ on $B$ and values in-between elsewhere. We can estimate $|h_1(c) - g(c)|$ by dividing into three distinct cases. If $c \in A$, then $h_1(c) = -1/3$ and $g(c) \in [-1,-1/3]$, so $|h_1(c) - g(c)| \leq 2/3$. If $c \in B$, then $h_1(c) = 1/3$ and $g(c) \in [1/3,1]$, so $|h_1(c) - g(c)| \leq 2/3$. Finally if $c \in C - (A \cup B)$ then $g(c) \in (-1/3,1/3)$ and $h_1(c) \in (-1/3,1/3)$ so $|h_1(c) - g(c)| \leq 2/3$, as required.

(b) We proceed by induction. Set $h_0 = 0$. The previous part of the question established the case of $n = 1$. Let us suppose $h_{n-1}$ has been constructed meeting the conditions for $n-1$. Then let us consider $g_n = g - h_{n-1}$. The conditions on $h_{n-1}$ ensure that $g_n : C \to [-2(3)^{n-1},2(3)^{n-1}]$. By the same argument used in the previous part, we can construct a continuous function $f_n : X \to [-2(3)^{n-1}(1/3),(2(3)^{n-1}(1/3)]$ such that $|f_n(c) - g_n(c)| \leq (2/3)^n$ for all $c \in C$. Now set

$$h_n = h_{n-1} + f_n.$$

Finally can carry out some estimates:

- First,

$$|h_n(x)| \leq |h_{n-1}(x)| + |f_n(x)| \leq (1 - (2/3)^{n-1}) + (2/3)^{n-1}(1/3) = 1 - (2/3)^n$$

This implies that $h_n : X \to [-1 + (2/3)^n,1 - (2/3)^n]$, as required.
• Second, if $c \in C$, then
  $$|h_n(c) - g(c)| = |h_{n-1}(c) + f_n(c) - g_n(c) - h_{n-1}(c)| = |f_n(c) - g_n(c)| \leq (2/3)^n$$
as required.

• Third, $|h_n(x) - h_{n-1}(x)| = |f_n(x)| \leq (1/3)(2/3)^{n-1}$, as required.

\[ \square \]

2. You may use the results of the previous problem in answering this one.

(a) Suppose $(f_n : X \to \mathbb{R})$ is a sequence of functions, where $X$ is a topological space. We say that $(f_n)$ converges to a function $f : X \to \mathbb{R}$ uniformly if, for all $x \in X$ and all $\epsilon > 0$, there exists some $N_\epsilon \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for all $n > N_\epsilon$. Suppose you are given a sequence of continuous functions $f_n : X \to \mathbb{R}$ converging uniformly to $f : X \to \mathbb{R}$. Prove that $f$ is continuous.

(b) Suppose $X$ is a perfectly normal topological space and $C \subset X$ is a closed subset. Suppose $g : C \to [-1, 1]$ is a continuous function. Construct a continuous function $h_\infty : X \to [-1, 1]$ such that $h_\infty(c) = g(c)$ for all $c \in C$.

(a) Consider an open set $U \subset \mathbb{R}$. We show that $f^{-1}(U)$ is open in $X$. To do this, suppose $x \in f^{-1}(U)$. We produce an open set $V \ni x$ such that $f(V) \subset U$.

Choose $c > 0$ sufficiently small so that $(f(x) - 3c, f(x) + 3c) \subset U$. Choose some $n$ sufficiently large that $|f(y) - f_n(y)| < \epsilon$ for all $y$. Let $V$ be the open set $f^{-1}_n((f(x) - \epsilon, f(x) + \epsilon))$. We show $f(V) \subset U$. For any $y \in V$, we have

$$|f(y) - f(x)| = |f(y) - f_n(y) + f_n(y) - f_n(x) + f_n(x) - f(x)| \leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)| < \epsilon + \epsilon + \epsilon = 3\epsilon$$

so that $f(y) \in (f(x) - 3\epsilon, f(x) + 3\epsilon) \subset U$, as required. This proves that $f$ is continuous.

(b) Using the results of question 1, we know that there exists a sequence of continuous functions $h_n : X \to [-1, 1]$ satisfying certain conditions defined there. Define a function $h_\infty : X \to [-1, 1]$ by $h_\infty(x) = \lim_{n \to \infty} h_n(x)$. This exists because property (ii) of Question 1 ensures $(h_n(x))$ is a Cauchy sequence:

$$n^{+d-1} \frac{1}{3^n} \leq \sum_{i=n}^{n+d-1} \frac{2^i}{3^n} = \frac{2^{n-1} d - 1}{3^n} \leq \frac{2^n}{3^n}$$

and the upper bound can be made arbitrarily small.

Observe that $h_\infty(c) = g(c)$ if $c \in C$, since $|h_n(c) - g(c)| \to 0$ as $n \to \infty$.

We claim that $h_n \to h$ uniformly. Using the result of the previous part, this will prove that $h$ is continuous. This is similar to the proof of the Cauchy property. For all $x \in X$, for all $n \in \mathbb{N}$ and all $d \in \mathbb{N}$:

$$|h_\infty(x) - h_n(x)| \leq |h_\infty(x) - h_{n+d}(x)| + |h_n(x) - h_{n+d}(x)| < |h_\infty(x) - h_{n+d}(x)| + \frac{2^n}{3^n}$$
using an estimate we established earlier.

Taking the limit as \( d \to \infty \) gives us:

\[
|h_\infty(x) - h_n(x)| \leq \frac{2n}{3^n}.
\]

The bound is independent of \( x \) and tends to 0 as \( n \to \infty \), so we see that convergence of \( h_n \to h_\infty \) is indeed uniform.

The result proved in this question is called the “Tietze Extension Theorem”. In fact, one does not need the space \( X \) to be perfectly normal, merely normal. If the space is assumed to be normal, one may use a result called Urysohn’s lemma to produce the functions required. Since metric spaces are perfectly normal, however, assuming perfect normality is good enough for almost all purposes.

To see that the closure hypothesis is necessary, consider the function \( g : (-\infty,0) \cup (0,\infty) \to [-1,1] \) given by \( g(x) = -1 \) if \( x < 0 \) and \( g(x) = 1 \) if \( x > 0 \). This function is continuous, but it is not possible to define \( h(0) \) in such a way as to make \( h : \mathbb{R} \to [-1,1] \) continuous, since \( \lim_{x \to 0^-} h(x) = -1 \) and \( \lim_{x \to 0^+} h(x) = 1 \).

\[\Box\]

3. Give \( \mathbb{Q} \) the usual topology. Let \( C \subset \mathbb{Q} \) be a compact subset. Prove that \( C \) is nowhere dense in \( \mathbb{Q} \). It may help to consider \( C \) as a subset of \( \mathbb{R} \). You may assume that \( \mathbb{Q} \) is dense in \( \mathbb{R} \).

Since \( \mathbb{Q} \) is Hausdorff and \( C \) is compact, the subset \( C \) is closed in \( \mathbb{Q} \). In order to show it is nowhere dense, therefore, it suffices to show that \( C \) has empty interior. That is, we must show that there is no open set contained in \( C \), and since \( \mathbb{Q} \) is metric, it is sufficient to show that there is no open ball in \( C \). Open balls in \( \mathbb{Q} \) are sets of the form \((a,b) \cap \mathbb{Q}\) where \( a < b \).

Suppose for the sake of contradiction that \((a,b) \cap \mathbb{Q} \subset C\). We include \( \mathbb{Q} \) in \( \mathbb{R} \), with the usual topology. Since \( C \) is compact and \( \mathbb{R} \) is Hausdorff, \( C \) must be closed as a subset of \( \mathbb{R} \). Therefore

\[
(a,b) \cap \mathbb{Q} \subset C
\]

the closure being taken in \( \mathbb{R} \). We refer to an analysis course to assure us that \((a,b) \cap \mathbb{Q} = [a,b]\), so we arrive at \([a,b] \subset C \subset \mathbb{Q}\). Between \( a \) and \( b \), however, there is an irrational number. Therefore \([a,b] \not\subset \mathbb{Q}\). This contradiction shows that \( C \) has empty interior, as required. \[\Box\]