Math 426
Homework 5
Due 17 October 2022

1. If $\mathbf{x} \in \mathbf{R}^{n+1}$, then the notation $\|\mathbf{x}\|_2$ means the usual euclidean norm of $\mathbf{x}$, i.e., $\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^{n+1} x_i^2}$.

The *n-dimensional sphere* is the set

$$S^n = \{\mathbf{x} \in \mathbf{R}^{n+1} \mid \|\mathbf{x}\|_2 = 1\},$$

given the subspace topology.

You may assume that there is a continuous function $f : \mathbf{R}^n \to S^n$ given by

$$f(\mathbf{x}) = f(x_1, \dots, x_n) = \left(\frac{\|\mathbf{x}\|_2^2 - 1}{\|\mathbf{x}\|_2^2 + 1}, \frac{2x_1}{\|\mathbf{x}\|_2^2 + 1}, \frac{2x_2}{\|\mathbf{x}\|_2^2 + 1}, \dots, \frac{2x_n}{\|\mathbf{x}\|_2^2 + 1} \right).$$

It is helpful to know that the image of f consists of $S^n \setminus \{(1, 0, 0, \dots, 0)\}$.

Prove this is a one-point compactification of $\mathbf{R}^n$. Hint: first construct an inverse map $g : f(\mathbf{R}^n) \to \mathbf{R}^n$. You can rely on standard facts about continuous functions and should not have to devote effort to proving g is continuous.

2. Suppose $q : X \to Y$ is a surjective function and X is a topological space. The *quotient topology* on Y has the following definition: U is open in Y if and only if $q^{-1}(U)$ is open in X. If Y has the quotient topology, then q may be called a *quotient map*.

A subspace $A \subset X$ is *saturated with respect to* q if $q^{-1}(q(A)) = A$. Suppose A is an open saturated subspace of X. Prove that $q(A)$ is an open subspace in Y and $q|_A : A \to q(A)$ is a quotient map.

3. Throughout, the notation $\mathbf{k}$ denotes either $\mathbf{R}$ or $\mathbf{C}$. Here $\mathbf{C}$ is homeomorphic to $\mathbf{R}^2$. You may assume the field operations of addition, multiplication and inversion in $\mathbf{k}$ or $\mathbf{k} \setminus \{0\}$ are continuous.

Fix an integer $n \geq 1$. Define a space, the *projective space of dimension n over* $\mathbf{k}$, denoted $\mathbf{k}\mathrm{P}^n$, as the quotient space of $\mathbf{k}^{n+1} \setminus \{\mathbf{0}\}$ by the equivalence relation

$$(z_0, z_1, \dots, z_n) \sim (\lambda z_0, \lambda z_1, \dots, \lambda z_n),$$

for all $\lambda \in \mathbf{k} \setminus \{0\}$. Write q for the quotient map

$$q : \mathbf{k}^{n+1} \setminus \{\mathbf{0}\} \to \mathbf{k}\mathrm{P}^n.$$

Write the equivalence class of $(z_0, z_1, \dots, z_n)$ in $\mathbf{k}\mathrm{P}^n$ as $[z_0 : z_1 : \cdots : z_n] = q(z_0, \dots, z_n)$.

For each $i \in \{0, \dots, n\}$, write V_i for the open subset of $\mathbf{k}^{n+1}$ where $z_i \neq 0$. Write U_i for $q(V_i) \subset \mathbf{k}\mathrm{P}^n$.

(a) Show that the sets V_i are saturated.

(b) Consider the function $f_i : U_i \to \mathbf{k}^n$ given by $f_i([z_0 : z_1 : \cdots : z_n]) = (z_i^{-1} z_0, z_i^{-1} z_1, \dots, \widehat{z_i^{-1} z_i}, \dots, z_i^{-1} z_n)$, where $\hat{}$ indicates omission. Prove that f_i is continuous.

(c) Construct a continuous inverse for f_i, showing that it is a homeomorphism.