Math 426
Homework 6
Due 31 October 2022

1. There is a forgetful functor V : **Top** $\to$ **Set** that takes a space (X,τ) to its underlying set and a map $f : X \to Y$ to the function f, viewed now as a function between sets.

Determine with proof all functors Φ : **Set** $\to$ **Top** having the property that $V \circ \Phi$ is the identity functor, i.e., the functor sending a set X to the set X and the function $f : X \to Y$ to the function $f : X \to Y$. Possible hint: consider $\Phi(\{0,1\})$, and for each function $f : \{0,1\} \to \{0,1\}$, consider $\Phi(f)$.

2. By a *discrete* set in $\mathbf{R}^n$, we mean a subset $D \subset \mathbf{R}^n$ such that the subspace topology on D is discrete. Throughout this question n is an integer greater than 1.

(a) If D is discrete in $\mathbf{R}^n$, prove D is countable.

(b) Let F be a countable set of points in $\mathbf{R}^n$ with the usual topology. Prove that $\mathbf{R}^n \setminus F$ is path connected.

3. Recall the definition of $\mathbf{k}\mathrm{P}^n$ from the previous homework. Recall also the definitions of open sets $V_i \subset \mathbf{k}^{n+1} \setminus \{\mathbf{0}\}$ and $U_i \subset \mathbf{k}\mathrm{P}^n$, and the homeomorphisms $f_i : U_i \to \mathbf{k}^n$.

(a) By considering the compact subset $S \subset \mathbf{k}^{n+1}$ given by $|z_0|^2 + \cdots + |z_n|^2 = 1$, prove that $\mathbf{k}\mathrm{P}^n$ is compact.

(b) Prove that $\mathbf{k}\mathrm{P}^n$ is Hausdorff.

> Note: this question can be difficult. There are several strategies that I know. One is to show that $\mathbf{k}\mathrm{P}^n$ is highly symmetric, so that any two points in $\mathbf{k}\mathrm{P}^n$ must both lie in a subspace homeomorphic to U_0. Another strategy is to prove that there is a quotient map $p : S \to \mathbf{k}\mathrm{P}^n$, so that for any two points $x, y \in \mathbf{k}\mathrm{P}^n$, the inverse images $p^{-1}(x)$ and $p^{-1}(y)$ are disjoint closed subsets of S, which are therefore at a positive distance apart, and can be included in saturated disjoint open neighbourhoods.

(c) Prove that the embedding $\mathbf{k}^n \approx U_0 \subset \mathbf{k}\mathrm{P}^n$ is a compactification of $\mathbf{k}^n$.