Math 426
Homework 7
Due 7 November 2022

1. For a topological space X, let $\pi_0(X)$ denote the set of path components of X. Given a map $f : X \to Y$ of spaces, we may attempt to define a function $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ by $\pi_0(f)(P_x) = P_{f(x)}$.

(a) Prove that $\pi_0(f)$ is indeed well defined. It is customary to write f_* instead of $\pi_0(f)$.

(b) Suppose $f \simeq g : X \to Y$ are homotopic maps. Prove that $f_* = g_*$.

(c) Deduce that π_0 actually yields a functor $\pi_0 : \mathbf{H} \to \mathbf{Set}$. You do not need to give details here. Just list the further properties that π_0 must satisfy in order to be a functor.

(d) Suppose $f : X \to Y$ is a homotopy equivalence. Use the theory above to show that $f_* : \pi_0(X) \to \pi_0(Y)$ is a bijection.

2. Let S denote a compact version of the topologist's-sine-curve:

$$S = \{(0, y) \in \mathbf{R}^2 \mid -1 \leq y \leq 1\} \cup \{(x, y) \in \mathbf{R}^2 \mid 0 < x \leq 1,\ y = \sin x^{-1}\}.$$

You may assume without proof that S has two path components, as indicated in lecture. Let $L = \{(0, y) \in \mathbf{R}^2 \mid -1 \leq y \leq 1\}$ denote the closed-interval part of S. This is one of the path components. I remark in passing that L is contractible and is a closed subspace of S.

(a) Produce, with proof, a homeomorphism $S/L \to [0,1]$.

(b) Prove that S is not homotopy equivalent to S/L.

3. In each of the following cases, use deformation retracts to show that the two spaces indicated are homotopy equivalent to one another. Pictures may be sufficient proof, provided they describe a correct argument and I can understand them.

(a) $S^2 \vee S^1$ and $S^2 \cup L$ where L is a line segment joining the north and south poles of S^2.

(b) $\mathbf{R}^3 \setminus \{(x, y, z) \mid x^2 + y^2 = 1,\ z = 0\}$ and $S^2 \vee S^1$.