Math 426
Homework 8
Due 21 November 2022

1. This question takes place in some unspecified category. Prove that if the morphisms $g \circ f$ and $h \circ g$ in $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ are isomorphisms, then so are f, g, h.

Deduce that if $s : M \to N$, and $t : N \to M$ are two morphisms such that $s \circ t$ and $t \circ s$ are isomorphisms, then s and t are isomorphisms.

2. Recall that if $H : X \times I \to X$ is a homotopy, then the notation H_t denotes the map $H_t(x) = H(x, t)$. If X is a space and Y_1, Y_2 are two subspaces, then an *ambient isotopy* of X taking Y_1 to Y_2 is a homotopy $H : X \times I \to X$ with the properties:

- $H(x, 0) = x$ for all $x \in X$;
- For all $t \in [0, 1]$, the map $H_t : X \to X$ is a homeomorphism;
- $H_1|_{Y_1} : Y_1 \to X$ has image exactly equal to Y_2 (i.e., $H_1|_{Y_1}$ is a homeomorphism from Y_1 to Y_2).

Let X be a space. Prove that ambient isotopy defines an equivalence relation on subspaces of X.

3. For a topological space X and a natural number n, the n-fold *symmetric product*, $\mathrm{Sym}^n X$, is the quotient of X^n by the permutation action of the symmetric group S_n. Specifically

$$(x_1, \ldots, x_n) \sim (x_{\pi(1)}, \ldots, x_{\pi(n)})$$

if $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a bijection. You do not need to prove this is an equivalence relation.

As in all homework questions, you may assume the results of previous homework questions. This problem requires the results of Homework 6, q3, and Homework 5, q3. In particular, you may assume that the spaces $\mathbf{C}\mathrm{P}^n$ are Hausdorff compactifications of $\mathbf{C}^n$, and that every point in $\mathbf{C}\mathrm{P}^n$ has a neighbourhood that is homeomorphic to $\mathbf{C}^n$. This implies that $\mathbf{C}\mathrm{P}^n$ is locally compact.

Write the point $(a_0, a_1, \ldots, a_{n-1})$ of $\mathbf{C}^{n+1} \setminus \{\mathbf{0}\}$ as $a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + a_n z^n$ where z is a formal variable. You may assume that the polynomial-multiplication map $m : (\mathbf{C}^2 \setminus \{0\})^n \to \mathbf{C}^{n+1} \setminus \{0\}$ sending an n-tuple of linear terms $((b_1 + c_1 z), \ldots, (b_n + c_n z))$ to the product $\prod_{i=1}^n (b_i + c_i z)$ is continuous.

(a) Produce, with proof, a continuous surjection $(\mathbf{C}\mathrm{P}^1)^n \to \mathbf{C}\mathrm{P}^n$. You may assume Theorem 9.5 from the notes.

(b) Produce, with proof, a homeomorphism $\mathrm{Sym}^n \mathbf{C}\mathrm{P}^1 \to \mathbf{C}\mathrm{P}^n$. What does the inverse of this homeomorphism say about the roots of polynomials?