1. This question takes place in some unspecified category. Prove that if the morphisms $g \circ f$ and $h \circ g$ in $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ are isomorphisms, then so are $f, g, h$.

Deduce that if $s : M \to N$, and $t : N \to M$ are two morphisms such that $s \circ t$ and $t \circ s$ are isomorphisms, then $s$ and $t$ are isomorphisms.

First, a lemma.

**Lemma 0.1.** Suppose $\alpha : X \to Y$ is a morphism in a category such that $\alpha$ has a left inverse $\beta : Y \to X$ for which $\beta \circ \alpha = \text{id}_X$ and a right inverse $\gamma : Y \to X$ for which $\alpha \circ \gamma = \text{id}_Y$, then $\beta = \gamma$ and $\alpha$ is an isomorphism.

*Proof.*

$$\beta = \beta \circ (\alpha \circ \gamma) = (\beta \circ \alpha) \circ \gamma = \gamma$$

which establishes the first claim. Then $\beta \circ \alpha = \text{id}_X$ and $\alpha \circ \beta = \text{id}_Y$, establishing the second. □

Now we prove $g$ is an isomorphism. It has a left-inverse: $(h \circ g)^{-1} \circ h \circ g = \text{id}_X$ and a right-inverse: $g \circ f \circ (g \circ f)^{-1} = \text{id}_Y$. The lemma proves that $g$ is an isomorphism.

Next we show that $f$ is an inverse for the isomorphism $j := (g \circ f)^{-1} \circ g$. it is immediate that $f$ is a right inverse for $j$. Since $j$ is an isomorphism, it has a left inverse $j^{-1}$ as well, and the lemma tells us that $f = j^{-1}$, so that $f$ is an isomorphism.

Similarly, $h$ is a left inverse for the isomorphism $k := g \circ (h \circ g)^{-1}$, and therefore $h = k^{-1}$ by the lemma.

Finally, if we apply the preceding results in the case where $f = h = s$ and $g = t$, we see that if $s \circ t$ and $t \circ s$ are isomorphisms, then $s$ and $t$ are isomorphisms. □

2. Recall that if $H : X \times I \to X$ is a homotopy, then the notation $H_t$ denotes the map $H_t(x) = H(x, t)$. If $X$ is a space and $Y_1$, $Y_2$ are two subspaces, then an *ambient isotopy* of $X$ taking $Y_1$ to $Y_2$ is a homotopy $H : X \times I \to X$ with the properties:

- $H(x, 0) = x$ for all $x \in X$;
- For all $t \in [0, 1]$, the map $H_t : X \to X$ is a homeomorphism;
- $H_t|Y_1 : Y_1 \to X$ has image exactly equal to $Y_2$ (i.e., $H_t|Y_1$ is a homeomorphism from $Y_1$ to $Y_2$).

Let $X$ be a space. Prove that ambient isotopy defines an equivalence relation on subspaces of $X$. 


Reflexivity is immediate: the homotopy \( H(x, t) = x \) is an ambient isotopy taking \( Y_1 \) to \( Y_1 \).

Symmetry: suppose \( H \) is an ambient isotopy taking \( Y_1 \) to \( Y_2 \). Consider \( H' : X \times I \to X \) defined by \( H'(x, t) = H(H_1^{-1}(x), 1 - t) \). We verify that this is an ambient isotopy taking \( Y_2 \) to \( Y_1 \).

- \( H'(x, 0) = H(H_1^{-1}(x), 1) = H_1 \circ H_1^{-1}(x) = x \) for all \( x \in X \).
- \( H'(x, t) = H(H_1^{-1}(x), 1 - t) \) for all \( x \), so that \( H'_1 : X \to X \) is the composite of two homeomorphisms and is therefore itself a homeomorphism.
- Note that \( H'_1 = H_0 \circ H_1^{-1} = H_1^{-1} \), so that \( H'_1 \) restricted to \( Y_2 \) is a homeomorphism of \( Y_2 \) to \( Y_1 \), being the inverse of \( H_1 \).

Transitivity: suppose \( H \) is an ambient isotopy taking \( Y_1 \) to \( Y_2 \) and \( H' \) is an ambient isotopy taking \( Y_2 \) to \( Y_3 \). Define \( H'' : X \times I \to X \) by

\[
H''(x, t) = H'(H(x, t), t).
\]

We calculate \( H''(x, 0) = H'(H(x, 0), 0) = H'(x, 0) = x \) for all \( x \). Second, \( H''_1 = H_1' \circ H_1 \) is a composite of homeomorphisms, and therefore is a homeomorphism. Third, \( H'_1|_{Y_1} : Y_1 \to Y_2 \) is surjective, as is \( H''_1|_{Y_2} : Y_2 \to Y_3 \). The composite of two surjections is a surjection, so that \( H''_1|_{Y_1} : Y_1 \to Y_3 \) is surjective, as desired.

\[\square\]

3. For a topological space \( X \) and a natural number \( n \), the \( n \)-fold symmetric product, \( \text{Sym}^n X \), is the quotient of \( X^n \) by the permutation action of the symmetric group \( S_n \). Specifically

\[(x_1, \ldots, x_n) \sim (x_{\pi(1)}, \ldots, x_{\pi(n)})\]

if \( \pi : \{1, \ldots, n\} \to \{1, \ldots, n\} \) is a bijection. You do not need to prove this is an equivalence relation.

As in all homework questions, you may assume the results of previous homework questions. This problem requires the results of Homework 6, q3, and Homework 5, q3. In particular, you may assume that the spaces \( \mathbb{C}P^n \) are Hausdorff compactifications of \( \mathbb{C}^n \), and that every point in \( \mathbb{C}P^n \) has a neighbourhood that is homeomorphic to \( \mathbb{C}^n \). This implies that \( \mathbb{C}P^n \) is locally compact.

Write the point \((a_0, a_1, \ldots, a_{n-1})\) of \( \mathbb{C}^{n+1} \setminus \{0\} \) as \( a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + a_n z^n \) where \( z \) is a formal variable. You may assume that the polynomial-multiplication map \( m : (\mathbb{C}^2 \setminus \{0\})^n \to \mathbb{C}^{n+1} \setminus \{0\} \) sending an \( n \)-tuple of linear terms \((b_1 + c_1 z), \ldots, (b_n + c_n z))\) to the product \( \prod_{i=1}^{n} (b_i + c_i z) \) is continuous.

(a) Produce, with proof, a continuous surjection \((\mathbb{C}P^1)^n \to \mathbb{C}P^n\). You may assume Theorem 9.5 from the notes.

(b) Produce, with proof, a homeomorphism \( \text{Sym}^n \mathbb{C}P^1 \to \mathbb{C}P^n \). What does the inverse of this homeomorphism say about the roots of polynomials?

(a) In a previous homework, we showed that there is a quotient map \( q_n : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}^{n-1} \). Note that both source and target of this map are locally compact (since they are both manifolds, for instance).
We claim that the product of \( n \) copies of \( q_2 : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1 \) is a quotient map \( q_2^n : (\mathbb{C}^2 \setminus \{0\})^n \to (\mathbb{C}P^1)^n \). By use of Theorem 9.5, any map of the form

\[
\text{id}_{(\mathbb{C}P^1)^r} \times q_2 \times \text{id}_{(\mathbb{C}^2 \setminus \{0\})^{n-r-1}} : \mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1 \times (\mathbb{C}^2 \setminus \{0\}) \times \cdots \times (\mathbb{C}^2 \setminus \{0\}) \to \mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1 \times (\mathbb{C}^2 \setminus \{0\}) \times \cdots \times (\mathbb{C}^2 \setminus \{0\})
\]

is a quotient map, and it is elementary to prove that the composite of quotient maps is a quotient map, from which we see that indeed \( q_2^n \) is a quotient map, which is what we claimed.

Consider now the composite \( q_n \circ m : (\mathbb{C}^2 \setminus \{0\})^n \to \mathbb{C}P^{n-1} \). Write \( \tilde{f} \) for \( q_n \circ m \). This map sends an \( n \)-tuple of nonzero linear factors \((l_1, l_2, \ldots, l_n)\) to the equivalence class of the coefficients of the product \( l_1 l_2 \cdots l_n \) up to scalar multiple. Observe that if we replace each \( l_i \) by a nonzero scalar multiple \( \lambda_1 l_i \), then the effect is to replace the product \( p = l_1 l_2 \cdots l_m \) by \( \lambda_1 \lambda_2 \cdots \lambda_n p \), and so \( \tilde{f}(l_1, l_2, \ldots, l_n) = \tilde{f}(\lambda_1 l_1, \lambda_2 l_2, \ldots, \lambda_n l_n) \). The universal property of quotient spaces implies that there exists a unique map \( f' \) making the following diagram commute:

\[
\begin{array}{ccc}
(\mathbb{C}^2 \setminus \{0\})^n & \xrightarrow{m} & \mathbb{C}^{n+1} \setminus \{0\} \\
q_2^n \downarrow & & \downarrow q_{n+1} \\
(\mathbb{C}P^1)^n & \xrightarrow{f'} & \mathbb{C}^n
\end{array}
\]

In order to show that \( f' \) is surjective, it is sufficient to show that \( \tilde{f} \) is surjective. To see that \( \tilde{f} \) is surjective, it is sufficient to prove that \( m \) is surjective, since \( q_{n+1} \) is surjective. The assertion that \( m \) is surjective is the assertion that every complex polynomial of degree \( n \) or less can be factored completely into linear factors. This is the fundamental theorem of algebra, and does not have to be proved here.

(b) We note that \( f' : (\mathbb{C}P^1)^n \to \mathbb{C}^n \) defined above does not depend on the order of the factors. This feature is inherited from the polynomial multiplication map \( m : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}^{n+1} \setminus \{0\} \), which is commutative. The details do not need to be spelled out. Therefore, the map \( f' \) descends to a map \( f : \text{Sym}^n \mathbb{C}P^1 \to \mathbb{C}^n \).

This map \( f \) is, in fact, bijective. We show this below.

Suppose \([p] \in \mathbb{C}P^n\). There is a unique monic polynomial of degree \( d \leq n \) that represents \([p]\). Write this as

\[ p(z) = a_0 + a_1 z + \cdots + a_{d-1} z^{d-1} + z^d. \]

This polynomial factors into linear terms

\[ p(z) = (z - r_1)(z - r_2) \cdots (z - r_d)(1)(1) \cdots (1) \]

where we add in \( n - d \) copies of a trivial factor 1. The factors in this presentation are unique up to reordering, by reference to field theory or basic algebra. The factorization of \( p(z) \) yields a unique element of \( \text{Sym}^n \mathbb{C}P^1 \) mapping to \([p]\) under \( f \), namely the class of

\([1 : -r_1], [1 : -r_2], \ldots, [1 : -r_d], [0 : 1], \ldots, [0 : 1]\).
Therefore $f$ is bijective.

Since the source of $f$ is the quotient of the compact space $(\mathbb{C}P^1)^n$, it is compact. The target of $f$ is $\mathbb{C}P^n$, which was shown to be Hausdorff in a previous homework assignment. Therefore $f$ is a continuous bijection from a compact space to a Hausdorff space and is therefore a homeomorphism. The inverse of $f$ provides a homeomorphism from $\mathbb{C}P^n$, which is the space parametrizing all nonzero polynomials of degree $n$ or less (up to nonzero scalar multiples), to the space $\text{Sym}^n \mathbb{C}P^1$, which parametrizes their factorizations, up to order and again up to nonzero scalar multiples. The fact that this map is continuous implies that the factorization of a nonzero complex polynomial, and therefore the roots of a complex polynomial, depend continuously on the coefficients of the polynomial.