Math 426
Homework 10
Due 7 December 2022

1. Suppose $f : S^1 \to S^1$ is a continuous function. Define the *degree* of f as follows: Choose a basepoint $p \in S^1$, and let ι denote a generator of the infinite cyclic group $\pi_1(S^1, p)$. There is an induced homomorphism $f_* : \pi_1(S^1, p) \to \pi_1(S^1, f(p))$. Choose a path γ in S^1 from $f(p)$ to p. Then conjugation by the class of γ in $\Pi(S^1)$ gives us an isomorphism

$$\phi_\gamma : \pi_1(S^1, f(p)) \to \pi_1(S^1, p), \quad \phi_\gamma(\delta) = \gamma^{-1} \cdot \delta \cdot \gamma.$$

The composite $\gamma^{-1} \cdot f_*(\iota) \cdot \gamma = \iota^d$ for some integer d, since ι generates the infinite cyclic group $\pi_1(S^1, p)$. Define the *degree of f* to be this integer d.

(a) Fix a point p. Prove that the degree of f does not depend on the choice of the path γ.

(b) Prove that the degree of f does not depend on the choice of p either.

(c) Prove that two maps $f, f' : S^1 \to S^1$ are homotopic (without reference to basepoints) if and only if they have the same degree.

(d) Prove that the degree is multiplicative, in that $\deg(f \circ g) = \deg(f)\deg(g)$.

2. In this question, it is useful to know that S^1 is a topological abelian group. Most naturally, you can embed $S^1 \subset \mathbf{C}^\times$, endowing S^1 with a group operation given by complex multiplication. Alternatively, you can view each point of S^1 as corresponding to an angle $[0, 2\pi)$, and the group operation as given by addition of angles modulo 2π. You may assume this group operation, and inversion in the group, are continuous functions. The identity element of the group structure on S^1 will be denoted 1, the usual notation for the complex number $1 + 0i$.

(a) Suppose that $h : S^1 \to \mathbf{R}$ is a continuous function such that $h(-x) = -h(x)$ for all $x \in S^1$. Prove that $h(x) = 0$ for some $x \in S^1$.

(b) Suppose $f : S^1 \to S^1$ is a map of even degree (see the previous question for the definition of degree) and such that $f(1) = 1$. Prove that f has a 'square root', in that there exists a continous map $g : S^1 \to S^1$, also satisfying $g(1) = 1$, and such that $g(x)^2 = f(x)$ for all $x \in S^1$.

(c) Suppose that $f : S^1 \to S^1$ is an *odd function* in that $f(-x) = -f(x)$ for all $x \in S^1$. Prove that the degree of f is an odd integer. Hint: reduce to the case where $f(1) = 1$. Suppose for the sake of contradiction that f has even degree. Consider the continuous function $x \mapsto g(x)/g(-x)$, where g is as in the previous part.

(d) Suppose that $h : S^2 \to \mathbf{R}^2$ is a function such that $h(-x) = -h(x)$ for all $x \in S^2$. Prove that $h(x) = (0, 0)$ for some $x \in S^2$.

3. Fix a natural number $n \geq 2$. Let D^2 denote the unit disk in $\mathbf{R}^2$, and S^1 the boundary circle. Consider the quotient $q : D^2 \to X$ given by the relation

$$(\cos\theta, \sin\theta) \sim (\cos(\theta + 2r\pi/n), \sin(\theta + 2r\pi/n)) \qquad \text{for all } r \in \mathbf{Z}.$$

The map q is a homeomorphism onto its image when restricted to the interior of D^2, and on $S^1 = \partial D^2$ is an n-fold covering map $q|_{S^1} : S^1 \to Y$, where $Y \approx S^1$. You may assume all this without proof.

(a) Prove that the inclusion $i : Y \hookrightarrow X \setminus \{(0,0)\}$ is a deformation retract.

(b) Choose a basepoint x_0 in X. Calculate $\pi_1(X, x_0)$.