Math 426 Notes

Ben Williams

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CHAPTER 1

Topological spaces

1. Basic definitions

DEFINITION 1.1. Let X be a set and $\tau \subset \mathscr{P}X$ a subset of the power set of X, i.e., τ is a set of subsets of X. We say (X,τ) is a *topological space* and τ is a *topology* on X if the following axioms are satisfied:

(1) $\emptyset \in \tau$ and $X \in \tau$.
(2) If $\{U_i\}_{i\in I} \subset \tau$, then $\bigcup_{i\in I} U_i \in \tau$. That is, τ is closed under taking arbitrary unions.
(3) If $\{U_i\}_{i=1}^n \subset \tau$ is a finite subset, then $\bigcap_{i=1}^n U_i \in \tau$. That is, τ is closed under taking finite intersections.

NOTATION 1.2. The sets $U_i \in \tau$ are called *open sets* for the topology τ. A set $C = X \setminus U$ where U is open is called *closed*. The closed sets satisfy a dual set of axioms:

(1) $\emptyset$ and X are both closed.
(2) An arbitrary intersection of closed sets is closed.
(3) A finite union of closed sets is closed.

Here are two propositions with useful facts about inverse images and images. The proofs are routine.

PROPOSITION 1.3. *Let X and Y be two sets and $f : X \to Y$ a function between them. Suppose $\{B_i\}_{i\in I}$ is a set of subsets of Y. Then*

(1) $f^{-1}(\emptyset) = \emptyset$ *and* $f^{-1}(Y) = X$*;*
(2) $f^{-1}\left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I} f^{-1}(B_i)$*;*
(3) $f^{-1}\left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in I} f^{-1}(B_i)$*.*

PROPOSITION 1.4. *Let X and Y be two sets and $f : X \to Y$ a function between them. Suppose $\{A_i\}_{i\in I}$ is a set of subsets of X. Then*

(1) $f(\emptyset) = \emptyset$*.*
(2) $f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i)$*;*
(3) $f\left(\bigcap_{i\in I} A_i\right) \subset \bigcap_{i\in I} f(A_i)$*.*

Observe that $f(X) = Y$ if and only if f is surjective. If f is injective, then the inclusion in Proposition 1.4 (3) is actually an equality, but in general it may be a strict inclusion. For instance, if x_0, x_1 are two distinct elements of X for which $f(x_0) = f(x_1) = y$, then $f(\{x_0\}) \cap f(\{x_1\}) = \{y\}$, whereas $f(\{x_0\} \cap \{x_1\}) = f(\emptyset) = \emptyset$.

DEFINITION 1.5. Let (X,d) be a metric space. That is $d : X \times X \to [0,\infty)$ is a metric satisfying the usual metric axioms:

(1) $d(x,y) = d(y,x)$
(2) $d(x,z) \le d(x,y) + d(y,z)$
(3) $d(x,y) = 0$ if and only if $x = y$

Given a point $x \in X$ and $r > 0$, we define the *open ball around x of radius r*, denoted $B(x,r)$ to be $\{y \in X \mid d(x,y) < r\}$.

DEFINITION 1.6. If (X,d) is a metric space, we can define an associated *metric topology* τ_d on X as follows: A set U is open in the metric topology if: for any point $u \in U$, there exists some $\epsilon > 0$, possibly depending on u, such that the ball $B(u;\epsilon) \subset U$.

It is an easy exercise to prove that this really is a topology as defined above.

PROPOSITION 1.7. *Let $B(x,r)$ be an open ball in a metric space. Then $B(x,r)$ is an open set.*

PROOF. Let $y \in B(x,r)$ be a point. We know that $d(y,x) < r$. Write $d(y,x) = r - \epsilon$ for some ϵ. We claim that $B(y,\epsilon) \subset B(x,r)$. Suppose $z \in B(y,\epsilon)$, then

$$d(x,z) \le d(x,y) + d(y,z) < r - \epsilon + \epsilon = r$$

so that $z \in B(x,r)$. □

2. Separation axioms

NOTATION 1.8. If (X,τ) is a topological space and $x \in X$ is a point, then a subset $V \subset X$ containing x is called a *neighbourhood* of x if there is some open set $U \ni x$ such that $U \subset V$. The term *open neighbourhood* of x means an open set containing x.

DEFINITION 1.9. A topological space (X,τ) is T_1 if all singleton subsets $\{x\} \subset X$ are closed.

Here is an argument we use often.

PROPOSITION 1.10. *A subset V in a topological space (X,τ) is open if and only if V is a neighbourhood of x (i.e., V it contains an open set having x as an element) for all $x \in V$.*

PROOF. In one direction, if V is open then, then it is trivially a neighbourhood of each of its points.

In the other direction, for each $x \in V$ we can find some open set U_x such that $x \in U_x \subset V$. Then we can write

$$V = \bigcup_{x \in V} U_x,$$

which is open. □

DEFINITION 1.11. A topological space is T_2 or *Hausdorff* if, for any $x \neq y \in X$, there exist open sets $U \ni x$ and $V \ni y$ such that $U \cap V = \emptyset$.

REMARK 1.12. This is clearly a stronger condition than the T_1 property. That is, points in Hausdorff spaces are closed.

PROPOSITION 1.13. *Every metric topology is Hausdorff.*

PROOF. Let $x \neq y$ be two points in a metric space. We can write $d(x, y) = 2\epsilon$ where $\epsilon > 0$. Then $B(x,\epsilon) \cap B(y,\epsilon) = \emptyset$, by using the triangle inequality. □

EXAMPLE 1.14. Let X be set. Give X the *indiscrete topology* where the open sets consist only of $\emptyset$ and X. If X contains at least 2 elements, then this topology is not T_1 and therefore it is not Hausdorff and therefore it is not metric.

EXAMPLE 1.15. Let X be a set and define the *cofinite topology* on X as follows: $U \subset X$ is open if $X \setminus U$ is finite or if $U = \emptyset$. This always yields a topology. If $x \neq y$ are two elements in X, then $X \setminus \{y\}$ is an open set containing x but not containing y, so the topology is T_1.

If X is infinite, and if $U \ni x$ and $V \ni Y$ are open neighbourhoods, then $(X \setminus U) \cup (X \setminus V)$ is a finite set. In particular, it is not all of X. Any z in the complement lies in both U and V, and so $U \cap V \neq \emptyset$. Therefore, this topology is not Hausdorff (and in particular, is not metric).

EXAMPLE 1.16. A useful, if uncomplicated, topology is the *discrete topology*. Here every set is open, and consequently every set is closed. It is a metric topology, being induced by the discrete metric $d(x, y) = 1$ if $x \neq y$ for instance.

DEFINITION 1.17. A topological space X is *regular* if for all point $p \in X$ and all closed sets $C \subset X \setminus \{p\}$, there exists disjoint open sets $U \ni p$ and $V \supseteq C$ such that $U \cap V = \emptyset$.

REMARK 1.18. There exist Hausdorff spaces that are not regular, and there exist regular spaces that are not Hausdorff (e.g., the indiscrete topology). A space that is both Hausdorff (T_2) and regular is called a *T_3-space*.

DEFINITION 1.19. A topological space X is *normal* if, for all disjoint closed subsets C_1, C_2, there exists open subsets $U_1 \supset C_1$ and $U_2 \supset C_2$ such that $U_1 \cap U_2 = \emptyset$.

REMARK 1.20. A Hausdorff normal space is a *T_4-space*. Clearly, a T_4-space is a T_3-space.

EXAMPLE 1.21. It is not easy to dream up a space that is T_3 but not T_4, but they exist. For instance, if we take $\mathbf{R}$, but give it the right-half-open interval topology from Example 1.48, and then form $\mathbf{R} \times \mathbf{R}$, the resulting space, the "half-open square topology" is T_3 but not T_4. I don't know a way to prove this that doesn't involve the Baire category theorem.

3. Continuous functions

The purpose of topological spaces is to define continuity. First we recall the metric-space definition of continuity.

DEFINITION 1.22. Let (X_1, d_1) and (X_2, d_2) be two metric spaces, and let $f : X_1 \to X_2$ be a function. We say that f is *continuous* if, for all $x \in X_1$ and all $\epsilon > 0$, there exists some $\delta > 0$ such that

$$d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon.$$

And now, the topological definition of continuity.

DEFINITION 1.23. Let (X_1, τ_1) and (X_2, τ_2) be topological spaces and let $f : X_1 \to X_2$ be a function. We say f is *continuous* if, for all open sets $U \subset X_2$, the preimage set $f^{-1}(U) = \{x \in X_1 \mid f(x) \in U\}$ is open in X_1.

PROPOSITION 1.24. *Let (X_1, d_1) and (X_2, d_2) be metric spaces and $f : X_1 \to X_2$ be a function. Then f is continuous in the metric-space sense if and only if it is continuous in the topological sense.*

PROOF. Suppose f is metric-continuous. Let $U \subset X_2$ be an open set. If U is empty, then there is nothing to check. Suppose $x \in f^{-1}(U)$, so that $f(x) \in U$. Since U is open, there is some $\epsilon > 0$ such that $B(f(x), \epsilon) \subset U$. Then choose an associated δ, so that $d(x, y) < \delta$ implies $d(f(x), f(y)) < \epsilon$. This last condition is equivalent to saying that $f(y) \in B(f(x), \epsilon)$, so that $y \in f^{-1}(U)$. We have shown that $B(x, \delta) \subset f^{-1}(U)$. Since $f^{-1}(U)$ contains an open ball around each of its points, it is an open set.

Conversely, suppose f is topologically continuous. Let $x \in X_1$ and $\epsilon > 0$. The set $B(f(x), \epsilon)$ is open in X_2, and therefore $f^{-1}(B(f(x), \epsilon))$ must be open in X. That implies that there exists some δ such that $B(x, \delta) \subset f^{-1}(B(f(x), \epsilon))$, which is a restatement of the ϵ-δ-continuity condition. □

REMARK 1.25. We will assume therefore that functions from calculus classes etc. that you might reasonably have proved to be continuous are continuous.

PROPOSITION 1.26. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions between topological spaces. Then $g \circ f$ is continuous.*

PROOF. Once you observe that $(g \circ f)^{-1}(U) = g^{-1}(f^{-1}(U))$, this is immediate. □

PROPOSITION 1.27. *Let $f : X \to Y$ be a function between topological spaces. Then f is continuous if and only if f^{-1} takes closed sets to closed sets.*

PROOF. Once you observe that $f^{-1}(Y \setminus Z) = X \setminus f^{-1}(Z)$, this is immediate. □

PROPOSITION 1.28. *Let X be a topological space with the discrete topology, and Y a topological space. Then every function $f : X \to Y$ is continuous. Conversely, let Z be a topological space with the indiscrete topology. Then every function $g : Y \to Z$ is continuous.*

DEFINITION 1.29. A function $f : X \to Y$ is *open* (resp. closed) if $f(U)$ is open (resp. closed) whenever U is open (resp. closed) in X.

Both open and closed conditions on functions do arise in topology, but neither is comparable in importance to continuity.

DEFINITION 1.30. A continuous function $f : X \to Y$ with a continuous inverse $f^{-1} : Y \to X$ is called a *homeomorphism*. This is the topological version of an isomorphism.

The proof of the following statement is elementary.

LEMMA 1.31. *If $f : X \to Y$ is a continuous bijective function between topological spaces, then the following are equivalent:*

(1) f is a homeomorphism;

(2) f is open;
(3) f is closed.

EXAMPLE 1.32. Let X be a set with at least two elements. Let (X,i) denote the space X with the indiscrete topology and (X,d) the space with the discrete topology. Then $\mathrm{id}:(X,d)\to(X,i)$ is continuous but not open or closed, whereas $\mathrm{id}:(X,i)\to(X,d)$ is open and closed, but not continuous.

The first of these two maps is an example of a continuous bijective function that is not a homeomorphism.

EXAMPLE 1.33. Let $S^1=\{(x,y)\in\mathbf{R}^2 \mid x^2+y^2=1\}$, given the metric topology for the usual metric on subsets of the plane.

The function $f:[0,2\pi)\to S^1$ given by $f(\theta)=(\cos\theta,\sin\theta)$ is a continuous and bijective function since both $\cos\theta$ and $\sin\theta$ are differentiable, but it is not a homeomorphism. For example, the set $[0,\pi)$ is an open subset of the domain, but $f([0,\pi))$ is not open in S^1.

4. Generating topologies

DEFINITION 1.34. Let (X,τ) be a topological space and let $x\in X$ be a point. We say that a collection $\mathscr{B}_x$ of open neighbourhoods is a *local base* for τ at x or a *system of open neighbourhoods of x* if, for all open $U\ni x$, there exists at least one $B\in\mathscr{B}_x$ such that $x\in B\subset U$.

EXAMPLE 1.35. The most common example of this is the system of balls in metric spaces. Let $x\in X$ be a point in a metric space and let $(a_n)_n\to 0$ be a sequence converging to 0, for instance $(1,1/2,1/3,\dots)$. Then the family of balls $\{B(x,a_i)\}_{i=1}^{\infty}$ is a local base for the topology on X at x.

Note that this family is *countable*, that is, it can be indexed by the natural numbers.

PROPOSITION 1.36. *Suppose X is a topological space equipped with local bases $\mathscr{B}_x$ at each point x. Suppose Y is a topological space and $f:X\to Y$ is a function. Suppose that for all $x\in X$ and all $U\in\mathscr{B}_x$, the set $f(U)$ is a neighbourhood of $f(x)$. Then the function f is open.*

PROOF. Suppose V is an open set of X. We wish to show $f(V)$ is open. Suppose $y\in f(V)$, then there exists some $x\in V$ such that $f(x)=y$. We can find an open neighbourhood $U\ni x$ such that $U\subset V$ and then $f(U)\subset f(V)$, but $f(U)$ contains a neighbourhood of $f(x)=y$ by hypothesis. Therefore $f(V)$ contains a neighbourhood of the arbitrarily-chosen point y, so that $f(V)$ is open by Proposition 1.10. □

DEFINITION 1.37. Let X be a topological space. If each point $x\in X$ admits a countable local base, then we say X is *first countable*.

Every metric space is first countable.

DEFINITION 1.38. Let (X,τ) be a topological space and let $\mathscr{B}$ be a set of open subsets of X. If $\mathscr{B}$ has the property that it contains a local base for τ at every point, then $\mathscr{B}$ is a *base* (or basis) for the topology τ.

EXAMPLE 1.39. The prototypical example of a base is the collection of all balls of all radii in a metric topology.

DEFINITION 1.40. Let X be a topological space. If X has a countable base, then X is *second countable*.

EXAMPLE 1.41. The spaces $\mathbf{R}^n$ with the usual topology are second countable. This is perhaps surprising, but one can take as a countable base the set of all balls $B(q;1/n)$ where $n \in \mathbf{N}$ and where q has only rational-number coefficients.

Note that a base for a topology is not a wholly arbitrary set of open subsets. For instance, if $U, V \in \mathscr{B}$ are sets in a base, and if $x \in U \cap V$, then there exists some $W_x \in \mathscr{B}$ such that $x \in W_x \subset U \cap V$. Using Proposition 1.10, we deduce that $U \cap V$ can be written as a union of sets from $\mathscr{B}$. By an induction argument, every finite intersection of sets from $\mathscr{B}$ can be expressed as a union of sets from $\mathscr{B}$.

DEFINITION 1.42. Let $\mathscr{S}$ be a set of subsets of a set X. Let $\mathscr{B}$ denote the set of all finite intersections of sets in $\mathscr{S}$. We say $\mathscr{S}$ is a *subbase* for a topology τ if $\mathscr{S}$ consists of open sets of τ and if $\mathscr{B}$ is a base for this topology. That is, for any open set $U \in \tau$ and any $u \in U$, there exists a finite intersection $x \in S_1 \cap \cdots \cap S_n \subset U$.

Whereas the condition of being a base of a topology places a condition on a set, any collection of subsets can form a subbase.

CONSTRUCTION 1.43. Let X be a set and let $\mathscr{S}$ be a set of subsets of X. Let $\mathscr{B}$ denote the set of finite intersections of sets in $\mathscr{S}$ and let τ denote the set of unions of all sets in $\mathscr{B}$. Then τ is a topology, and $\mathscr{S}$ is a subbase of τ.

The proof that τ actually is a topology is left as an exercise.

NOTATION 1.44. We say that τ is the topology *generated* by $\mathscr{S}$.

REMARK 1.45. In this definition, we use the convention that the intersection of an empty set of subsets of X is X itself. In order to avoid using this convention, some might prefer to impose a condition on a subbase $\mathscr{S}$ that the union of all sets in the subbase is X itself.

Subbases can be used to detect continuity of functions:

PROPOSITION 1.46. *Let $f : X \to Y$ be a function between topological spaces. Let $\mathscr{S}$ generate the topology on Y, and suppose that for all $U \in \mathscr{S}$ that $f^{-1}(U)$ is open in X. Then f is continuous.*

PROOF. Use $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$ and $f^{-1}(\bigcup U_i) = \bigcup f^{-1}(U_i)$. □

In contrast, checking on a subbase is not enough to prove openness of a function. One needs to check on a base, and Proposition 1.36 is one way of articulating this.

EXAMPLE 1.47. Give $\mathbf{R}$ its usual topology, and let $A = \{0,1\}$ with the indiscrete topology. Define a function $f : \mathbf{R} \to A$ by declaring $f(x) = 0$ if $x \in \mathbf{Z}$ and $f(x) = 1$ otherwise. Let $\mathscr{S}$ be the subbase consisting of the intervals $\{(x,\infty) \mid x \in \mathbf{R}\}$ and $\{(-\infty, x) \mid x \in \mathbf{R}\}$. Note that $f((x,\infty) = A$, which is open, and similarly $f((\infty, x)) = A$. Therefore the function f has the property that it takes open sets from the subbase to open sets. On the other hand $f((2,3)) = \{1\}$, which is not open, so the function is not open.

An easy corollary of Proposition 1.36 is that we can verify openness of a function by checking on a base, however.

EXAMPLE 1.48. We can describe a new topology on $\mathbf{R}$ using a subbase. The *right half-open interval topology* is defined as the topology generated by all intervals $[a,b) \subset \mathbf{R}$. The set of all such intervals actually forms a base, not just a subbase, for this topology.

Since $\bigcup_{n=1}^{\infty}[a+1/n,b) = (a,b)$, the right half-open interval topology is a refinement of the usual topology on $\mathbf{R}$. It is therefore Hausdorff, but it turns out not to be metric.

5. Induced topologies

Subspace topologies.

DEFINITION 1.49. Suppose (X,τ) is a topological space and that $A \subset X$ is a subset. The *subspace topology* on A is the topology τ_A on A determined as follows: A set $U \subset A$ is open (resp. closed) in A if there exists an open (resp. closed) subset $V \subset X$ such that $V \cap A = U$.

The proof that these open sets really do define a topology is left as an exercise.

EXAMPLE 1.50. If X is a topological space and A is a subset, then sets that are open in the subspace topology on A need not be open when considered as subsets of X. A similar warning applies to closed subsets.

For instance, if $X = \mathbf{R}$ with the usual topology and $A = [0,1)$, then $(-1/2,1/2) \cap A = [0,1/2)$, so that $[0,1/2)$ is open in the subspace topology on A. It is clearly not open in $\mathbf{R}$. Similarly, $[1/2,3/2] \cap A = [1/2,1)$ is closed in the subspace topology on A, but is obviously not closed in $\mathbf{R}$.

REMARK 1.51. The subspace topology has an important property. Suppose X is a topological space and A a subset of X, given the subspace topology. Suppose $f : Y \to X$ is a function between topological spaces such that $\mathrm{im}(f) \subset A$. Then the induced function $f : Y \to A$ is continuous if and only if $f : Y \to X$ is continuous.

DEFINITION 1.52. An *embedding* $i : X \to Y$ is a continuous function such that i induces a homeomorphism $X \to i(X)$. An embedding may be an *open embedding* if it has an open image, a *closed embedding* if it has closed image. It is an (unassigned) exercise to show that an open (resp. closed) embedding is an open (resp. closed) function.

Some embeddings are neither open nor closed. For instance, the inclusion of a subspace is an embedding, and need not be open or closed.

NOTATION 1.53. Given two topologies τ_1, τ_2 on the same set X, we say τ_1 is *finer* than τ_2, and τ_2 is *coarser* than τ_1, if $\tau_1 \supseteq \tau_2$. Equivalently, the identity map $\mathrm{id} : (X,\tau_1) \to (X,\tau_2)$ is continuous.

CONSTRUCTION 1.54. The subspace topology is a special case of a more general construction, that of the induced topology. Here is a general pattern: suppose X is a set, and that $\{Y_i,\tau_i\}_{i\in I}$ is a family of topological spaces, and suppose $\{f_i : X \to Y_i\}$ is a family of functions. Then the topology on X *induced* by the f_i is the coarsest topology, i.e., fewest open sets, such that the f_i are all continuous.

The subspace topology on $A \subset X$ is the topology induced by the inclusion map $i : A \to X$.

Product Topologies.

CONSTRUCTION 1.55. A special case of the induced topology is the *product topology*. Suppose $\{(X_i, \tau_i)\}$ is a family of topological spaces. Write $Y = \prod_{i\in I} X_i$ for the cartesian product of the X_i as a set, and write $\mathrm{proj}_i : Y \to X_i$ for the projection. That is, an element $y \in Y$ is uniquely determined by the values $\mathrm{proj}_i(y) \in X_i$. The *product topology* on Y is the topology induced by the maps proj_i.

Less formally, the product topology on $Y = \prod_{i\in I} X_i$ is the topology generated by the sets $\mathrm{proj}_i^{-1}(U)$ as i ranges over I and U ranges over the open sets of X_i. In fact, we can restrict attention to open sets in subbases for the X_i, if need be.

REMARK 1.56. Suppose $(X_1, d_1), \dots, (X_n, d_n)$ is a finite set of metric spaces. We can put product metrics on $Y = \prod_{i=1}^n X_i$, as well as a product topology. The product metrics may be defined in many ways, but for definiteness, we take

$$d((y_1, \dots, y_n), (x_1, \dots, x_n)) = \sum_{i=1}^n d_i(x_i, y_i).$$

It is an exercise to show that the product metric induces the product topology.

REMARK 1.57. Given a family of topological spaces $\{(X_i, \tau_i)\}_{i\in I}$, the product set $Y = \prod_{i\in I} X_i$ equipped with the product topology is called the *product space*. It is an example of a universal construction.

Let Z be a topological space. A continuous function $f : Z \to Y$ gives rise, by composition with proj_i, to a family of continuous functions $f_i : Z \to Y \to X_i$. The functions $\{f_i\}$ determine the map f uniquely. Moreover, given any family of continuous functions $\{g_i : Z \to X_i\}$, there is a unique continuous function $g : Z \to Y$ such that $g_i = \mathrm{proj}_i \circ g$ for all i.

EXAMPLE 1.58. It is important to understand what product topologies look like. Let us start with the common and relatively easy case of two topological spaces: X and Y. The product topology on $X \times Y$ is such that the projection maps $\mathrm{proj}_1 : X \times Y \to X$ and $\mathrm{proj}_2 : X \times Y \to Y$ are both continuous. In fact, it is the coarsest topology with the property that these maps are continuous.

If $U \subset X$ is open (resp. closed) then $\mathrm{proj}_1^{-1}(U) \subset X \times Y$ is open (resp. closed). A less formal way of writing $\mathrm{proj}_1^{-1}(U)$ is $U \times Y$. Similarly if $V \subset Y$ is open, then $X \times V$ is open in $X \times Y$. Putting the two ideas together: $U \times V = U \times Y \cap X \times V$ is an open set in $X \times Y$. In fact, open sets of this form make up a base for the product topology. They do not, however, comprise all the open sets in the topology, in general, since $(U_1 \times V_1) \cup (U_2 \times V_2)$ is not generally of the form $U_3 \times V_3$.

EXAMPLE 1.59. The real fun with product topologies comes when there is an infinite product of topological spaces. Suppose $X_1, X_2, \dots$ are countably infinitely many topological spaces (for instance). Write $X = X_1 \times X_2 \times \dots$. Suppose we have a family $U_i \subset X_i$ of open sets in X_i. Then $\mathrm{proj}_i^{-1}(U_i) = X_1 \times \cdots \times X_{i-1} \times U_i \times X_{i+1} \times \dots$. These sets are open in the product topology, as is any finite intersection of them, but there is no reason to expect $U_1 \times U_2 \times U_3 \times \dots$ to be open unless almost all the U_i equal their respective X_i.

You can get a topology on $\prod_{i\in I} X_i$ by declaring a basis of sets $\prod_{i\in I} U_i$ where $U_i \subset X_i$ is open for each i. This topology is called the *box* topology, and it is finer than the product topology.

PROPOSITION 1.60. *Let $\{X_i\}_{i\in I}$ be a family of topological spaces. The projection functions* $\mathrm{proj}_i : \prod_{i\in I} X_i \to X_i$ *are open.*

PROOF. Using Proposition 1.36, it suffices to show that $\mathrm{proj}_i(U)$ is open as the sets U range over a base for $\prod_{i\in I} X_i$. One base is given by sets of the form $U = \prod_{i\in I} U_i$ where $U_i \subset X_i$ is open and all but finitely many U_i are equal to X_i. If U is not empty, then $\mathrm{proj}_i(U) = U_i$, which is open. If U is empty, then $\mathrm{proj}_i(U) = \emptyset$. □

6. Coinduced topologies

CONSTRUCTION 1.61. Suppose X is a topological space and $f : X \to Y$ is a function. The *coinduced* topology on Y is the finest topology (most open sets) on Y such that f is continuous.

This is especially useful when f is a surjective function, in which case it is called the *quotient topology*.

The generalization to a family of maps $f_i : X_i \to Y$ is not difficult.

One common case of this is when $Y = \coprod_{i\in I} X_i$, the disjoint union of the spaces X_i. Then the topology on Y is such that $U \cap Y$ is open if and only if $U \cap X_i$ is open in X_i for all i.

EXAMPLE 1.62. One simple, but technically coinduced, topology is as follows. Suppose (X,τ) and (Y,σ) are topological spaces. There are two maps $X \to X \cup Y$ and $Y \to X \cup Y$. The coinduced topology on $X \cup Y$ is the finest topology making both these maps continuous.

As a special case, if X and Y are disjoint, then the coinduced topology has as its open sets all sets $U \cup V$ where $U \subset X$ and $V \subset Y$ are each open.

DEFINITION 1.63. As a special case of the above, if X is a topological space, let X_+ denote the disjoint union of X and a point $+$, given the coinduced topology. Observe that a point can have only one topology on it. The open sets of X_+ are the sets U_+ and U when $U \subset X$ is open.

EXAMPLE 1.64. Suppose (X,τ) is a topological space and $A \subset X$ is a nonempty subspace. We define the *quotient space* X/A given by collapsing A to a point.

The idea is to collapse all of A to a single point. As a set, X/A is a disjoint union $(X \setminus A) \cup \{*\}$. If A is not empty, then we define a surjective function $q : X \to X/A$ by $q(a) = *$ if $a \in A$ and $q(x) = x$ otherwise. Then give X/A the quotient topology for the map $q : X \to X/A$.

Tracing through the definitions: X/A consists of $X \setminus A$ and one more point, $*$ which represents "all of A". What sets are open in X/A? Exactly the sets U for which $q^{-1}(U)$ is open in X. There are two possibilities for U: either $* \in U$ or $* \notin U$. If $* \in U$, then $q^{-1}(U)$ is an open subset of X containing A. If $* \notin U$, then $q^{-1}(U)$ is an open subset of X that is disjoint from A.

The space X/A has a universal property. First of all, it is more than just a space: it is equipped with a point we think of as special "$*$". We need a definition. A *pointed topological space* is a pair (Y, y_0) consisting of a topological space Y and a point $y_0 \in Y$, sometimes called the *basepoint*. If (Y, y_0) and (Z, z_0) are pointed spaces, then a function $f : Y \to Z$ such that $f(y_0) = z_0$) is said to be *based*.

Having made this definition, we can say that $(X/A, *)$ is a pointed space that satisfies all the following:

(1) There is a continuous function $q : X \to X/A$ such that $A \subset q^{-1}(*)$.

(2) If (Y, y_0) is any pointed space such that there exists a continuous function $f : X \to Y$ such that $A \subset f^{-1}(y_0)$, then there is a unique based continuous function $f' : X/A \to Y$ satisfying $f = f' \circ q$.

Heuristically, this universal property tells us that if you want to take $A \subset X$ and collapse it to a single point, then X/A is the minimally-destructive way of doing this.

REMARK 1.65. We use this universal property to inform us what $X/\varnothing$ should be, thus defining X/A no matter what A is. The pair $(X/\varnothing, *)$ should be a pointed space with the following properties:

(1) There is a continuous function $q : X \to X/\varnothing$ such that $\varnothing \subset q^{-1}(*)$—the second condition is vacuous.
(2) If (Y, y_0) is any pointed space such that there exists a continuous function $f : X \to Y$ such that $\varnothing \subset f^{-1}(y_0)$—another vacuous condition—, then there is a unique based continuous function $f' : X/A \to Y$ satisfying $f = f' \circ q$.

Think this through to discover that $(X/\varnothing, *)$ is a based space equipped with a continuous function $f : X \to X/\varnothing$ that collapses nothing, and so that the distinguished point $*$ is both open and closed. That is: $X/\varnothing$ is another construction of X_+.

EXAMPLE 1.66. As a special case of the above, observe that there is a map $f : [0,1] \to S^1$ given by $f(x) = (\cos 2\pi x, \sin 2\pi x)$. We will assume the calculus fact that f is continuous. We observe that $f(0) = f(1)$, so that there is an induced map $f' : [0,1]/\{0,1\} \to S^1$. This map is also bijective.

With some work, we can see that f' is actually a homeomorphism—this can be done in a clever way later.

EXAMPLE 1.67. Suppose X is a topological space and $\sim \subset X \times X$ is an equivalence relation on X. Then there is a surjective map of sets $q : X \to X/\sim$, where $q(x)$ is the equivalence class of x. We endow $X/\sim$ with the quotient topology, and call it the quotient space of X by $\sim$.

The quotient space here has a universal property: $q : X \to X/\sim$ is a continuous function such that $q(x) = q(x')$ whenever $x \sim x'$, and if $f : X \to Y$ is a continuous function such that $f(x) = f(x')$ whenever $x \sim x'$, then there exists a unique continuous function $f' : X/\sim \to Y$ such that $f = f' \circ q$.

EXAMPLE 1.68. The previous construction is used especially when there is a group acting on X: say $G \times X \to X$, and $x \sim y$ if there exists $g \in G$ such that $gx = y$. Then we abuse notation and write X/G for the quotient space.

Beware that if $H \subset G$ is a subgroup of a group, and if G also happens to be a topological space, then the notation G/H is ambiguous. It could either mean the result of collapsing the subspace $H \subset G$ to a single point, or it might mean the quotient space given by identifying $g \sim hg$ whenever $h \in H$. In practice, the group-action quotient is usually what is meant.

REMARK 1.69. Quotient spaces are frequently not Hausdorff, even when X is metric. In this respect, quotient constructions are unlike all the other constructions we have seen so far.

CHAPTER 2

Closure and sequence methods

1. Closure

DEFINITION 2.1. Let X be a topological space and let A be a subset of X. The *closure* of A in X, written $\overline{A}$, is the intersection of all closed sets that contain A.

PROPOSITION 2.2. *The closure operator on subsets of a topological space X has the following properties:*

(1) $\overline{A}$ is closed.
(2) $A \subset \overline{A}$, with equality if and only if A is closed.
(3) if $A \subset B$, then $\overline{A} \subset \overline{B}$
(4) $\overline{\overline{A}} = \overline{A}$.

PROOF. (1) Since the intersection of closed subsets is closed, this is immediate.
(2) This is the case because $\overline{A}$ is the intersection of closed sets, all containing A.
(3) Any closed set containing B is a closed set containing A. Therefore $\overline{A}$ is the intersection of a family of sets that contains all closed sets containing B. The result follows.
(4) This is immediate, since $\overline{A}$ is closed.

□

PROPOSITION 2.3. *With notation as before, $x \in \overline{A}$ if and only if, for every open neighbourhood $U \ni x$, the set $U \cap A \neq \emptyset$.*

PROOF. Consider the statement "every open neighbourhood U of x satisfies $U \cap A \neq \emptyset$". This is logically equivalent to "for every open set U such that $U \cap A = \emptyset$, the element x is not in U", which is equivalent to "for every closed set C such that $A \subset C$, the element x is in C" which is equivalent to "$x \in \bar{A}$". □

REMARK 2.4. Given a sequence of inclusions of sets $Z \subset Y \subset X$, where X is a topological space, it may be the case that the closure of Z in the subspace topology on Y is different from the closure of Z in the topology on X. On the other hand, if $Y \subset X$ is closed, then the two notions of closure coincide.

2. Interior and boundary

DEFINITION 2.5. Suppose $A \subset X$. Let A°, the *interior* of A, denote the union of all open $U \subset A$.

This concept is dual to that of closure. It is immediate that $A^\circ \subset A$, with equality if and only if A is open.

DEFINITION 2.6. Suppose $A \subset X$. Let ∂A, the *boundary of* A, denote $\overline{A} - A^\circ$.

PROPOSITION 2.7. *With notation as above, a point $x \in X$ lies in ∂A if and only if every neighbourhood $U \ni x$ satisfies $U \cap A \neq \emptyset$ and $U \cap (X - A) \neq \emptyset$.*

PROOF. The proof of this is an exercise. □

PROPOSITION 2.8. *Let X be a topological space and let A be a subspace. Then there is a division of X into three disjoint subsets: A°, ∂A and $(X - A)^\circ$. Moreover $\bar{A} = A^\circ \cup \partial A$.*

PROOF. The division of X into three parts is really as follows: let $x \in X$ be a point. Then exactly one of the following three cases must obtain:

(1) There is some open $U \ni x$ such that $U \subset A$. In this case, $x \in A^\circ$.
(2) There is some open $U \ni x$ such that $U \subset X - A$. In this case, $x \in (X - A)^\circ$.
(3) For every open $U \ni x$, both $U \cap A$ and $U \cap X - A$ are not empty. In this case, $x \in \partial X$.

The closure of A consists of those x for which every open neighbourhood meets A, by Proposition 2.3. Therefore $\overline{A}$ is the complement of $(X - A)^\circ$. The result follows. □

COROLLARY 2.9. $\partial A = \partial(X - A)$.

3. Density

DEFINITION 2.10. We say a subset $A \subset X$ is *dense* if $\overline{A} = X$. We say A is *sparse* or *nowehere dense* if $(\overline{A})^\circ = \emptyset$.

REMARK 2.11. The set A is dense in X if X is the only closed set containing A. The contrapositive is that the only open set $U \subset X \setminus A$ is the empty set. Therefore a set A is dense if and only if $U \cap A$ is nonempty whenever U is a nonempty open subset.

EXAMPLE 2.12. In $[0,1]$, the subset $\mathbf{Q} \cap [0,1]$ is dense, and countable.

On the other hand, define the Cantor set $C \subset [0,1]$ to consist of those numbers that can be written without any 1s in base 3. Then C is nowhere dense, and uncountable. Note that base-3 representations, like decimal representations, are not always unique. For instance

$$0.1 = 0.0222\ldots.$$

The set C is closed: it is the intersection of the closed sets:

$$C_n = \{x \in [0,1] \mid \text{Some base-3 representation of } x \text{ has } n\text{-th digit different from 1.}\}.$$

The set C is uncountable by Cantor's diagonal argument. Finally, the set C does not contain any open intervals: between any two real numbers, one can always find a third real number with a unique base-3 representation that eventually consists only of 1s.

PROPOSITION 2.13. *Let X be a topological space, let $A \subset X$ be a dense subset, and let Y be a Hausdorff topological space. Suppose $f, g : X \to Y$ are two continuous functions such that $f(a) = g(a)$ for all $a \in A$. Then $f = g$.*

In proving this, we use some lemmas that are useful in their own right.

DEFINITION 2.14. Let Y be a topological space. Write Δ_Y for the *diagonal* subset of $Y \times Y$ consisting of points (y, y).

LEMMA 2.15. *Let Y be a topological space. Then Y is Hausdorff if and only if the diagonal subset $\Delta_Y \subset Y \times Y$ consisting of points (y, y) is a closed subset.*

PROOF. Suppose Y is Hausdorff. Let (x, y) be a non-diagonal point. Then there exists open $U \ni x$ and $V \ni y$ such that $U \cap V = \emptyset$ and $U \times V$ therefore gives a subset of $Y \times Y$ containing (x, y) and disjoint from Δ_Y. It follows that $Y \times Y \setminus \Delta_Y$ is open.

Suppose Δ_Y is closed. Let (x, y) be a point not in the diagonal. Then there exists some open sets $U \ni x$ and $V \ni y$ such that $(U \times Y) \cap (Y \times V)$ does not meet the diagonal. But this implies that $U \cap V = \emptyset$, as required. □

REMARK 2.16. There is an obvious function $d : Y \to Y \times Y$ given by $d(y) = (y, y)$. Specifically, this function is given by the universal property of products applied to $\mathrm{id}_Y : Y \to Y$ twice, as in the commutative diagram below:

$$\begin{tikzcd} Y \arrow[dr, dashed, "d"] \arrow[drr, bend left, "\mathrm{id}_Y"] \arrow[ddr, bend right, "\mathrm{id}_Y"'] & & \\ & Y \times Y \arrow[r, "\mathrm{proj}_1"'] \arrow[d, "\mathrm{proj}_2"] & Y \\ & Y. & \end{tikzcd}$$

Therefore d is a continuous function. The image of d is exactly the diagonal Δ_Y. The restriction of either projection $\mathrm{proj}_i : Y \times Y \to Y$ to the subset Δ_Y gives a continuous inverse to $d : Y \to \Delta_Y$. It follows that d is an embedding. The embedding d is called the *diagonal embedding*.

Lemma 2.15 says that a topological space Y is Hausdorff if and only if the diagonal embedding is a closed embedding (as defined in Definition 1.52).

LEMMA 2.17. *Let $f, g : X \to Y$ be two continuous functions, and suppose Y is Hausdorff. The set of points $x \in X$ such that $f(x) = g(x)$ is closed in X.*

PROOF. The universal property of the product says that the function $(f, g) : X \to Y \times Y$ given by $(f, g)(x) = (f(x), g(x))$ is continuous. The inverse image $(f, g)^{-1}(\Delta_Y)$ is therefore closed in X. But $(f, g)^{-1}(\Delta_Y)$ is precisely the set of $x \in X$ such that $f(x) = g(x)$. □

Of course, you can do this much more directly if you like.

PROOF OF PROPOSITION. The set of points where $f(x) = g(x)$ is closed, and contains the dense subset A. Therefore it is all of X. □

4. Sequences

I assume you know what a sequence $(x_1, x_2, \ldots)$ in a topological space X means. We'll denote sequences by x_n, or, when it is necessary to specify the indexing variable, $(x_n)_n$.

DEFINITION 2.18. A sequence $(x_n)_n$ in X *converges* to $x \in X$ if, for all open $U \ni x$, there exists some $N \in \mathbf{N}$ such that $x_i \in U$ for all $i > N$.

In this formulation, it is clear that the notion of convergence given here is a generalization of the familiar notion from analysis.

Here is a different way of conceptualizing convergence.

NOTATION 2.19. Unless we say otherwise, the set $\mathbf{N} \cup \{\infty\}$ will be given a topology where $\{n\}$ is open for all $n \in \mathbf{N}$, and a set $U \ni \infty$ is open if and only if it contains some tail: $\{n, n+1, n+2, \ldots\}$.

PROPOSITION 2.20. *The subspace topology on* $\mathbf{N} \subset \mathbf{N} \cup \{\infty\}$ *is discrete, and* $\mathbf{N}$ *is dense in* $\mathbf{N} \cup \{\infty\}$.

REMARK 2.21. A sequence x_n is a function $x : \mathbf{N} \to X$: the notation x_n is a conventional way of writing $x(n)$. Since $\mathbf{N}$ carries the discrete topology, all such functions are continuous.

PROPOSITION 2.22. *A sequence* $x : \mathbf{N} \to X$ *extends to a continuous function* $\hat{x} : \mathbf{N} \cup \{\infty\} \to X$ *if and only if* $x_n \to \hat{x}(\infty)$.

In this formulation, the following is an immediate consequence of the fact that composites of continuous functions are continuous.

PROPOSITION 2.23. *Let* x_n *be a sequence in* X *and suppose* x_n *converges to* x. *Let* $f : X \to Y$ *be a continuous function. Then* $f(x_n)$ *converges to* $f(x)$ *in* Y.

REMARK 2.24. Similar tricks with composite functions $\mathbf{N} \cup \{\infty\} \to \mathbf{N} \cup \{\infty\}$ can be used to show that if $x_n \to x$ and if y_n is a subsequence of x_n, then $y_n \to x$. You can also do this directly. You probably get enough of this sort of thing in analysis lectures.

We know from analysis lectures that the limit of a convergent sequence is unique in a metric space. Unfortunately, this does not generalize to non-Hausdorff spaces.

EXAMPLE 2.25. Consider an infinite set X with the cofinte topology. Let x_n be a sequence in X in which $x_i \neq x_j$ for all $i \neq j$. Such a sequence exists because there exists an injective map $\mathbf{N} \to X$.

Let $y \in X$. Consider any open $U \ni y$. The set $X \setminus U$ consists of only finitely many elements, and therefore only finitely many elements of (x_n) lie outside U. In particular, some tail of the sequence (x_n) lies entirely inside U, and so $x_n \to y$. But y was arbitrary.

This shows that even in a T_1 topological space, limits of sequences may not be unique.

PROPOSITION 2.26. *Let* X *be a Hausdorff topological space and let* x_n *be a sequence in* X. *Suppose* $x_n \to y$ *and* $x_n \to z$. *Then* $y = z$.

PROOF. The two limits can be encoded as continuous functions

$$\begin{tikzcd}
 & \mathbf{N} \cup \{\infty\} \arrow[dr, "\hat{y}"] & \\
\mathbf{N} \arrow[ur] \arrow[dr] & & X \\
 & \mathbf{N} \cup \{\infty\} \arrow[ur, "\hat{z}"'] &
\end{tikzcd}$$

Here $\hat{y}(0) = y$ and $\hat{z}(0) = z$, whereas $\hat{y}|_{\mathbf{N}} = \hat{z}|_{\mathbf{N}}$. Since $\mathbf{N}$ is dense in $\mathbf{N} \cup \{\infty\}$, and X is Hausdorff, it follows that $\hat{y} = \hat{z}$. □

Of course, one can prove this in a lower-level way, by working directly with points and sets. You have surely all done this in analysis courses.

Our function-based approach to sequence convergence gives us a slick proof of the following:

PROPOSITION 2.27. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces, let X denote the product. Let (x_j) be a sequence in X. Let $y \in X$ be an element. Then $(x_j) \to y$ if and only if $\pi_i(x_j) \to \pi_i(y)$ for all i.*

That is, a sequence in a product space converges to y if and only if all the projections converge to the appropriate projections of y.

In a metric space, the closed sets admit a characterization in terms of limits of sequences. This carries over to all first-countable spaces.

DEFINITION 2.28. Let X be a topological space and A a subspace. Say that A is *sequentially closed* if it has the following property: if $(a_n)_n$ is a sequence in A that converges to $x \in X$, then $x \in A$.

PROPOSITION 2.29. *Let X be a topological space and A a closed subspace. Then A is sequentially closed.*

PROOF. Consider the commutative diagram of continuous maps

$$\begin{tikzcd} \mathbf{N} \arrow[r] \arrow[d, "a"] & \mathbf{N} \cup \{\infty\} \arrow[d, "\hat{x}"] \\ A \arrow[r] & X \end{tikzcd}$$

which implements $(a_n)_n \to x$. If A is closed in X, then $\hat{x}^{-1}(A)$ is closed in $\mathbf{N}$. Since $\hat{x}^{-1}(A) \supseteq \mathbf{N}$, it follows that $\hat{x}^{-1}(A) \supseteq \overline{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$. □

PROPOSITION 2.30. *Suppose X is a first-countable topological space and A is a sequentially closed subspace. Then A is closed.*

COROLLARY 2.31. *Let X and Y be topological spaces where X is first countable. Suppose $f : X \to Y$ is a function with the property that $x_n \to x$ in X implies $f(x_n) \to f(x)$ in Y. Then f is continuous.*

PROOF. Let $A \subset Y$ be a closed set. We will show $f^{-1}(A)$ is closed. If A is empty, there is nothing to show, so assume it is not. To show $f^{-1}(A)$ is closed, it is sufficient to show it is sequentially closed. Let x_n in $f^{-1}(A)$ be a sequence converging to $x \in X$. Then $f(x_n)$ is a sequence in A converging to $f(x)$, and since A is closed, $f(x) \in A$, which implies $x \in f^{-1}(A)$. □

5. Completions of Metric Spaces

DEFINITION 2.32. A sequence x_n in a metric space (X,d) is said to be *Cauchy* if, for all $\epsilon > 0$, there exists some $N \in \mathbf{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m > N$.

REMARK 2.33. All convergent sequences are Cauchy. The sequence $x_n = \sum_{i=1}^{n} 1/i$ is not Cauchy in $\mathbf{R}$—even though the distance between successive terms tends to 0.

DEFINITION 2.34. A metric space (X,d) is *complete* if every Cauchy sequence converges in X.

EXAMPLE 2.35. It is well known that $\mathbf{R}^n$ is complete. A product of two metric spaces is complete. A subset of a complete metric space is complete if and only if it is closed.

EXAMPLE 2.36. The metric space $\mathbf{R}$ is homeomorphic to any bounded open interval. For instance $f(x) = x/\sqrt{1+x^2}$ is a homeomorphism $f : \mathbf{R} \to (-1,1)$. This shows that completeness is not a topological, but rather a metric, property.

DEFINITION 2.37. A map of metric spaces $f : X \to Y$ is an *isometry* if $d(f(x), f(x')) = d(x, x')$ for all $x, x' \in X$. We remark that an isometry is necessarily injective.

DEFINITION 2.38. A *pseudometric space* (X,δ) is a set X equipped with a function $\delta : X \times X \to [0,\infty)$ satisfying the axioms of a metric space except that $\delta(x, y) = 0$ does not necessarily imply $x = y$.

PROPOSITION 2.39. *Let (X,d) be a metric space. Let CX denote the set of all Cauchy sequences in X, and let $(x_n), (y_n) \in CX$. Then the sequence $d(x_n, y_n)$ converges in $[0,\infty)$. If we define $\delta((x_n),(y_n)) = \lim_{n\to\infty} d(x_n, y_n)$, then δ is a pseudometric on CX.*

SKETCH OF PROOF. We claim $d(x_n, y_n)$ is a Cauchy sequence in $\mathbf{R}$. For any $\epsilon > 0$, choose N sufficiently large so that $d(x_n, x_m) < \epsilon/2$ and $d(y_n, y_m) < \epsilon/2$ for all $n, m > N$. Then $d(x_m, y_m) \le d(x_n, y_n) + \epsilon$ by the triangle inequality. By a symmetric argument $|d(x_n, y_n) - d(x_m, y_m)| < \epsilon$.

Since $\mathbf{R}$ is complete, δ is well defined. The pseudometric properties are straightforward to prove:

$$\lim_{n\to\infty} d(x_n, y_n) \le \lim_{n\to\infty} d(x_n, z_n) + \lim_{n\to\infty} d(z_n, y_n)$$

establishes the triangle inequality, for instance. □

REMARK 2.40. If (x_n) is a Cauchy sequence and $(x_{n_i})_i$ is a subsequence, then $\delta((x_n),(x_{n_i})) = 0$. This follows from the Cauchy property.

CONSTRUCTION 2.41. The relation $x \sim y$ if $\delta(x, y) = 0$ is an equivalence relation (use the triangle inequality). Write QX for the set of equivalence classes. If $\delta(x, x') = 0$ then the triangle inequality shows that $\delta(x, y) = \delta(x', y)$. Therefore δ induces a well-defined metric, also denoted δ here, on QX.

PROPOSITION 2.42. *Let (X,d) be a metric space. The metric space (QX,δ) is complete.*

SKETCH OF PROOF. Let $(x^n)_n$ denote a Cauchy sequence in QX. We can choose representatives of each (x^n) so that $d(x_i^n, x_j^n) < \max\{1/i, 1/j\}$ for all i, j—pass to a subsequence if need be.

Define a sequence by $z_n = x_n^n$. We claim that z_n a Cauchy sequence and that $x^n \to z$. First we remark that for any given n:

$$d(x_n^i, x_n^j) \le d(x_n^i, x_s^i) + d(x_s^i, x_s^j) + d(x_s^j, x_n^j) \le \delta(x^i, x^j) + 2/n$$

so that

$$d(z_n, x_j^i) \le d(x_n^n, x_n^i) + d(x_n^i, x_j^i) \le \delta(x^n, x^i) + \max\{1/i, 1/n\} + 2/n.$$

But since $(x^n)_n$ is Cauchy, it follows easily that $z = (z_n)_n$ is a Cauchy sequence. Moreover

$$\delta(z, x^i) \le \delta(x^j, x^i) + \max\{1/i, 1/j\} + 2/j$$

for all j. This shows that $x^i \to z$. □

PROPOSITION 2.43. *Define a function $\iota : X \to QX$ by $\iota(x) = (x, x, \ldots)$. Then ι is an isometry with dense image.*

PROOF. That ι is an isometry is trivial. The density of the image is proved by observing that if $(x_n) = (x_1, x_2, \ldots)$ represents an element of QX then $(\iota(x_1), \iota(x_2), \ldots)$ converges to (x_n). □

REMARK 2.44. The space QX as constructed in this proof is called the *completion* of X. We have shown that any metric space embeds isometrically as a dense subset of a complete metric space.

CHAPTER 3

Compactness

1. Elementary Theory

DEFINITION 3.1. An *open cover* $\mathscr{U}$ of a topological space X is a collection $\{U_i\}_{i\in I}$ of open sets such that $\bigcup_{i\in I} U_i = X$.

DEFINITION 3.2. A topological space X is *compact* if every open cover $\{U_i\}_{i\in I}$ contains a finite subcover $\{U_1,\dots,U_n\}$ such that $X = \bigcup_{i=1}^n U_i$.

REMARK 3.3. Sometimes the term "quasicompact" is used if X is not Hausdorff. This is the normal usage in algebraic geometry, which is unfortunate since this is also where one sees most non-Hausdorff compact spaces. In French "compact" means what "compact Hausdorff" means in English.

PROPOSITION 3.4. *Let $f : X \to Y$ be a continuous function and suppose X is compact. Then $f(X)$ is a compact subspace of Y.*

PROOF. Let $\{U_i\}_{i\in I}$ be an open cover of $f(X)$. Consider $\{f^{-1}(U_i)\}_{i\in I}$, which is an open cover of X, and therefore has a finite subcover $\{f^{-1}(U_1),\dots,f^{-1}(U_n)\}$. The set $\{U_1,\dots,U_n\}$ is the required finite subcover of $f(X)$. $\square$

PROPOSITION 3.5. *If X is a space and C_1 and C_2 are two compact subsets, then $C_1 \cup C_2$ is compact.*

PROOF. A cover of $C_1 \cup C_2$ contains a finite subset covering C_1 and a finite subset covering C_2. $\square$

PROPOSITION 3.6. *Suppose C is a closed subspace of a compact space X. Then C is compact.*

PROOF. Consider an open cover $\{U_i\}_{i\in I}$ of C. Let V_i be open in X and satisfy $U_i = C \cap V_i$. Consider $\{V_i\}_{i\in I} \cup \{X \setminus C\}$. This is an open cover of X. Any finite subcover induces a finite subcover of $\{U_i\}$. $\square$

PROPOSITION 3.7. *Suppose C is a compact subspace of a Hausdorff space X. Then C is closed in X.*

PROOF. Let $x \in X \setminus C$. For each $y \in C$ we can find disjoint open neighbourhoods $U_y \ni y$ and $V_y \ni x$. Finitely many of the U_y suffice to cover C, since it is compact, and therefore there exists an intersection of finitely many V_y that is disjoint from C. But a finite intersection of open sets is open. This proves that x has an open neighbourhood disjoint from C. $\square$

The following corollary is extremely useful.

COROLLARY 3.8. *Let $f : X \to Y$ be a continuous bijection between topological spaces where X is compact and Y is Hausdorff. Then f is a homeomorphism.*

PROOF. We prove that f is a closed map. This implies that the closed subsets of X are in bijective correspondence with the closed subsets of Y. Let $C \subset X$ be closed, then C is compact, so $f(C)$ is compact, so $f(C)$ is closed. □

PROPOSITION 3.9. *A compact Hausdorff space X is a T_4 space.*

PROOF. In a time-honoured tradition, we prove that X is T_3. The proof that it's T_4 is the same argument again.

Let p be a point and C be a closed subset disjoint from p. Since X is compact, C is compact. Since X is Hausdorff, for each $c \in C$ we can find disjoint $U_c \ni p$ and $V_c \ni c$ such that $U_c \cap V_c = \emptyset$. Finitely many V_c suffice to cover C: say $V(p) = V_{c_1} \cup \cdots \cup V_{c_n} \supseteq C$. Then $U(p) = \bigcap_{i=1}^n U_{c_i}$ is an open set disjoint from $V(p)$ and $p \in U(p)$.

If C_1 and C_2 are disjoint closed sets, for each $p \in C_1$ we can find disjoint open $U(p) \ni p$ and $V(p) \supseteq C_2$. Finitely many of the $U(p)$ suffice to cover C_1 and we use the same finite-intersection idea again to prove X is normal. □

2. The Tube Lemma

LEMMA 3.10 (Generalized Tube Lemma). *Let X and Y be topological spaces and $A \subset X$ and $B \subset Y$ be compact subsets. If N is an open subset of $X \times Y$ containing $A \times B$, then there exist open subsets $U \subset X$ and $V \subset Y$ such that $A \times B \subset U \times V \subset N$.*

PROOF. If A is empty, there is nothing to show.

Let $a \in A$. We produce a cover of $a \times B$ as follows. For each $b \in B$, we can find an open set of the form $U_b \times V_b$ such that $(a, b) \in U_b \times V_b \subset N$. By compactness, there is a finite set $\{(a, b_1), (a, b_2), \dots, (a, b_n)\}$ of such points so that the associated $U_{b_i} \times V_{b_i}$ form a cover $A \times B$. If we take $U(a) = \bigcap_{i=1}^n U_{b_i}$ and $V(a) = \bigcup_{i=1}^n V_{b_i}$, then the open set $U(a) \times V(a)$ has the following properties:

(1) It is contained in N.
(2) It contains $\{a\} \times B$.

We now repeat this procedure for all $a \in A$ to produce a cover $\{U(a) \times V(a)\}_{a \in A}$ of $A \times B$ that is contained in N. Since A is compact, we can find a finite set $\{a_1, a_2, \dots, a_r\}$ of points in A so that $\bigcup_{i=1}^r U(a_i) \supset A$. Define $U = \bigcup_{i=1}^r U(a_i)$ and $V = \bigcap_{i=1}^r V(a_i)$. Then $A \subset U$, by construction, and $B \subset V$, since $B \subset V(a)$ for all a. The set $U \times V$ is open and contains $A \times B$. It remains to verify that it is contained in N.

Suppose $(x, y) \in U \times V$. Then $x \in U(a_i)$ for at least one of the a_i chosen above, and $y \in V \subset V(a_i)$. Therefore $(x, y) \in U(a_i) \times V(a_i) \subset N$. This proves the required containment. □

COROLLARY 3.11 (The Tube Lemma). *Let X and Y be topological spaces and suppose X is compact. Let $y \in Y$. Suppose N is an open neighbourhood of $X \times \{y\}$, then there exists an open $U \ni y$ such that $X \times U \subset N$.*

This implies the following weak version of Tychanoff's theorem, Theorem 3.26.

COROLLARY 3.12. *Let X and Y be compact topological spaces. Then $X \times Y$ is compact.*

PROOF. Suppose $\mathscr{U} = \{U_i\}$ is an open cover of $X \times Y$. For each $y \in Y$, the set $X \times \{y\}$ is compact, and therefore there exists a finite subset $\mathscr{U}_y$ of $\mathscr{U}$ such that $\bigcup \mathscr{U}_y \supset X \times \{y\}$. By use of the tube lemma, with $N = \bigcup \mathscr{U}_y$, we can find an open $V_y \ni y$ such that $X \times V_y \subset \bigcup \mathscr{U}_y$.

Since Y is compact, finitely many V_y suffice to cover Y: say, $V_{y_1}, V_{y_2}, \ldots, V_{y_n}$ for some points $y_1, \ldots, y_n \in Y$. Now take the finite union of finite sets of open sets in $X \times Y$:

$$\mathscr{W} = \bigcup_{i=1}^{n} \mathscr{U}_{y_i}.$$

This is a finite subcover of $\mathscr{U}$. □

3. Compactness in Metric Spaces

The Lebesgue Covering Lemma.

DEFINITION 3.13. If X is a metric space and A is a subspace of X, then the *diameter* of A is the supremum of the set $\{d(x, y) \mid x, y \in A\}$. It is finite if and only if A is bounded.

LEMMA 3.14 (Lebesgue covering). *Let X be a compact metric space and let $\mathscr{U}$ be an open cover of X. There exists $\delta > 0$ such that whenever A is a subset of X of diameter less than δ, there exists some $U_i \in \mathscr{U}$ such that $A \subset U_i$.*

PROOF. For each $x \in X$, choose some $U_x \in \mathscr{U}$ containing x, and then choose some radius $r_x > 0$ such that $B(x, r_x) \subset U_x$.

Now consider the balls $B(x, r_x/2)$. These are open, and they form an open cover of X, so we may select a finite set of such balls that cover X, say:

$$X = B(x_1, r_{x_1}/2) \cup B(x_2, r_{x_2}/r) \cup \cdots \cup B(x_n, r_{x_n}/2).$$

Let r denote the least of the radii $r_1, r_2, \ldots, r_n$, and set $\delta = r/2$.

Let A be a nonempty set of diameter δ or less. Let $x \in A$ be a point. Then $x \in B(x_i, r_i/2)$ for some i. Suppose $y \in A$ is another point. Then $d(y, x_i) < d(y, x) + d(x, x_i) = \delta + r_i/2 \le r_i$. We have shown that $A \subset B(x_i, r_i) \subset U_{x_i}$, as required. □

The Heine–Borel Theorem. The Heine–Borel Theorem says that a subspace $C \subset \mathbf{R}^n$ is compact if and only if it is closed and bounded. This section is devoted to a generalization of this fact, Theorem 3.24.

Before we turn to the theorem itself, we note some consequences

LEMMA 3.15. *Let X be a compact topological space and $f : X \to \mathbf{R}$ be a continuous function. Then f attains a maximum value on X.*

PROOF. The image, $f(X)$, is a compact subset of $\mathbf{R}$ and is therefore closed and bounded. □

EXAMPLE 3.16. The quotient $[0,1]/\{0,1\}$ is homeomorphic to S^1. We can now prove this quickly. The interval $[0,1]$ is compact, so its surjective image $[0,1]/\{0,1\}$ is also compact. Therefore the map $f : [0,1]/\{0,1\} \to S^1$ given by $(\cos 2\pi\theta, \sin 2\pi\theta)$ is a continuous bijection with compact source and Hausdorff target.

DEFINITION 3.17. A topological space X is said to be *sequentially compact* if every sequence has a convergent subsequence.

Sequential compactness is neither stronger than nor weaker than compactness. For metric spaces, however, we will prove that the two concepts coincide.

DEFINITION 3.18. A metric space X is said to be *totally bounded* if, for all $\epsilon > 0$, one can cover X by finitely many balls $B(x_i, \epsilon)$.

REMARK 3.19. A totally bounded subspace of a metric space is necessarily bounded. Conversely, a bounded subspace of $\mathbf{R}^n$ is totally bounded. In infinite-dimensional spaces, however, bounded may not imply totally bounded. For instance, the unit ball $B(0,1)$ of ℓ_2 is not totally bounded.

PROPOSITION 3.20. *A compact metric space is totally bounded.*

PROOF. For any ϵ, the balls $B(x,\epsilon)$ form an open cover. The finite-subcover property implies total boundedness. □

PROPOSITION 3.21. *A compact metric space is complete.*

PROOF. Suppose X is a compact metric space. There is an isometric embedding $\iota : X \to QX$ where QX is complete and $\iota(X)$ is dense. Since $\iota(X)$ is compact, it is also closed. Therefore ι is a bijective isometry (a metric equivalence) and X is complete. □

PROPOSITION 3.22. *A complete, totally bounded metric space X is sequentially compact. In particular, a compact metric space is sequentially compact.*

PROOF. Let X be a totally bounded metric space. Let $(x_n^0)_n$ be a sequence. We produce a Cauchy sequence recursively. Cover X by finitely many balls of radius 1. One of these, B_1, contains a tail of the sequence. Let x^1 be the sequence that starts with x_1^0 and thereafter consists of only those terms in B_1.

Cover B_1 by finitely many balls of radius 1/2. One of these, B_2, contains a tail of the sequence x^1. Form the sequence

$$x^2 = (x_1^1, x_2^1, \text{subsequent terms in } B_2)$$

Cover B_2 by finitely many balls of radius 1/3. One of these, B_3, contains a tail of the sequence x^2. Form the sequence

$$x^3 = (x_1^2, x_2^2, x_3^2, \text{subsequent terms in } B_3)$$

Note that the sequences x^i stabilize: $x_n^i = x_n^{i+1}$ if $n \le i$. Form x^∞ as $(x_n^n)_n$. Observe that by virtue of how we constructed this, if $n < m$, then x_n^∞ and x_m^∞ lie in the same $1/n$-ball, so that x^∞ is a Cauchy subsequence of (x_n^0).

Since X is complete, this Cauchy subsequence converges and so x^0 has a convergent subsequence. □

PROPOSITION 3.23. *A sequentially compact metric space is complete and totally bounded.*

PROOF. Suppose X is sequentially compact. If (x_n) is a Cauchy sequence, then x_n converges to the limit of any convergent subsequence. Therefore X is complete.

If X is not totally bounded, we can find some $\epsilon > 0$ such that X cannot be covered by ϵ-balls. Therefore there is an infinite sequence x_n of points that are pairwise at distance at least ϵ from each other. This sequence can have no Cauchy subsequence and therefore no convergent subsequence, a contradiction. □

THEOREM 3.24. *Let (X, d) be a metric space. The following are equivalent:*

(1) X is compact,
(2) X is complete and totally bounded,
(3) X is sequentially compact.

PROOF. We have already proved that 1 implies 2 and that 2 is equivalent to 3. Let us now assume that X is sequentially compact (and therefore totally bounded). Let $\{U_i\}_{i\in I}$ be a cover. We claim that for some $n \in \mathbf{N}$, each ball $B(x, 1/n)$ is contained in some U_i.

Suppose for the sake of contradiction that this is not the case. Then let $x_n \in X$ be a sequence such that $B(x_n, 1/n)$ is not contained in any U_i. The sequence $(x_n)_n$ contains a subsequence converging, to some limit x. This x is in some U_i, and furthermore, there is some $r > 0$ such that $B(x, 2r) \subset U_i$ and such that $B(x, r)$ contains infinitely many terms of (x_n). But then for any $n > 1/r$, we have $B(x_n, 1/n) \subset U_i$, a contradiction.

Therefore the claim holds. Now, fix a radius $1/n$ such that each of the balls $B(x, 1/n)$ is contained in some U_i, depending on x. Since X is totally bounded, finitely many such balls suffice to cover X, and therefore finitely many of the U_i suffice to cover X. So X is compact. □

COROLLARY 3.25. *A subspace of $\mathbf{R}^n$ is compact if and only if it is closed and bounded.*

4. Tychanoff's Theorem

THEOREM 3.26 (Tychanoff's Theorem). *Suppose $\{X_i\}_{i\in I}$ is a family of compact topological spaces. Then the product space $\prod_{i\in I} X_i$ is compact.*

REMARK 3.27. This theorem is equivalent to the axiom of choice (in the generality in which it has been stated). The most useful case is when I is a finite set, in which case it follows from the case of the product of two compact spaces, $X \times Y$. We have already proved the result in this case in Corollary 3.12.

LEMMA 3.28. *Let X be a topological space and suppose that X is not compact. Then there exists a cover $\{U_i\}_{i\in I}$ of X that does not have a finite subcover and that is maximal with this property. I.e., for any open set $V \notin \{U_i\}_{i\in I}$, the open cover $\{U_i\}_{i\in I} \cup \{V\}$ has a finite subcover.*

PROOF. We apply Zorn's lemma. Suppose $\{\mathscr{U}_j\}_{j\in J}$ is a chain of open covers, each without a finite subcover. Then $\mathscr{V} = \bigcup_{j\in J} \mathscr{U}_j$ is an open cover of X. Suppose $\mathscr{V}$ has a finite subcover $\{U_1, U_2, \dots, U_n\}$. Then there is some $\mathscr{U}_j$ containing all these sets, a contradiction.

Since any chain of covers-without-finite-subcovers has an upper bound, there must be a maximal such chain by Zorn's lemma. □

THEOREM 3.29 (Alexander's subbase theorem). *Let X be a topological space and let $\mathscr{S}$ be a subbase for the topology on X, such that $\bigcup \mathscr{S} = X$. The space X is compact if and only if every cover $\{S_i\}_{i\in I} \subset \mathscr{S}$ has a finite subcover.*

PROOF. One direction is trivial: X is compact then *a fortiori* every subbasic cover has a finite subcover.

Suppose therefore for the sake of contradiction that every cover drawn from $\mathscr{S}$ has a finite subcover, but that X is nonetheless not compact. Let $\mathscr{U} = \{U_i\}_{i\in I}$ be a maximal open cover without a finite subcover. Consider $\mathscr{U} \cap \mathscr{S}$. This cannot form a cover of X, so there is some $x \in X$ such that x is not contained in any of the sets of $\mathscr{U} \cap \mathscr{S}$. Nonetheless, there exists some $V \in \mathscr{U}$ containing x. We can write $x \in S_1 \cap S_2 \cap \cdots \cap S_n \subset V$ where $S_i \in \mathscr{S}$ for all S. Because of the way we chose x, none of the S_i can appear in $\mathscr{U}$, so that each of the covers $\mathscr{U} \cup \{S_i\}$ strictly contains $\mathscr{U}$, and therefore each contains a finite subcover of X. These subcovers must involve S_i. There are therefore finite subcovers

$$\begin{gathered}
\{U_{1,1}, U_{1,2}, \ldots, U_{1,m_1}, S_1\} \\
\{U_{2,1}, U_{2,2}, \ldots, U_{2,m_1}, S_2\} \\
\vdots \\
\{U_{n,1}, U_{n,2}, \ldots, U_{n,m_1}, S_n\}
\end{gathered}$$

where the open sets $U_{i,j} \in \mathscr{U}$. But then $\bigcup_{i,j} U_{i,j} \cup V$ is a union of open sets in $\mathscr{U}$ and it contains every point in X, a contradiction. □

PROOF OF TYCHANOFF'S THEOREM. The product topology $X = \prod_{i\in I} X_i$ has a subbase given by all sets of the form $\pi_i^{-1}(U)$ where U is open in X_i. By virtue of Alexander's subbase theorem, it is sufficient to prove that any cover of X by sets from this subbase has a finite subcover.

Suppose $\mathscr{S}$ is an open cover of X where the open subsets are taken from the subbase above. For any coordinate i, consider the set $\mathscr{S} \cap \{\pi_i^{-1}(U) \mid U \subset X_i\}$. We claim that for at least one i, the sets U appearing here must form an open cover of X_i. Suppose not, then in each X_i, there exists some x_i which is not in any of the appropriate open $U \subset X_i$. But now consider $x \in X$ such that $\pi_i(x) = x_i$ for all i. This does not lie in any set in $\mathscr{S}$, a contradiction. Hence the claim is proved.

We may assume that there is some X_i such that the open sets $\pi_i^{-1}(U_j)$ appearing in $\mathscr{S}$ are such that the U_j cover X_i. Since X_i is compact, we can take $U_1, \ldots, U_n$ that cover X_i. Then $\pi_i^{-1}(U_1), \ldots, \pi_i^{-1}(U_n)$ cover X. □

5. Compactifications

DEFINITION 3.30. Let X be a topological space. A *compactification* of X is an embedding $i : X \to \hat{X}$ where $\hat{X}$ is a compact space and where the image of i is dense in $\hat{X}$.

REMARK 3.31. If we require both X and $\hat{X}$ to be Hausdorff, which is reasonable if we are doing geometry, and if X is already compact, then the only available compactification is X itself, up to homeomorphism. If X is not compact, however, then there may be many different compactifications.

REMARK 3.32. If $i : X \to \hat{X}$ is a compactification, then we may refer to $\hat{X}$ as "the compactification", especially when the map i is obvious. We may say the compactification has a property P, if the space $\hat{X}$ has that property. For instance, a "Hausdorff compactification" is a compactification of $i : X \to \hat{X}$ in which $\hat{X}$ is Hausdorff.

CONSTRUCTION 3.33. Let X be a topological space. Let the *one-point compactification* $X \to X \cup \{\infty\}$ be the inclusion of X into a space $X \cup \{\infty\}$ consisting of X itself and a new point "at infinity". The topology on $X \cup \{\infty\}$ is defined as follows:

- open subsets of X are open in $X \cup \{\infty\}$.
- A subset $U \ni \{\infty\}$ is open if $X \setminus (U \cap X)$ is a closed compact subset of X.

We note that this is actually a topology. Verifying this is routine.

REMARK 3.34. In spite of the name, this is not a compactification if X is already compact. In that case, $X \cup \{\infty\} = X_+$.

REMARK 3.35. If X is Hausdorff, then all compact subsets of X are closed, so the "closed" in "closed compact" is redundant. This is the most commonly-used case of the construction.

PROPOSITION 3.36. *If X is not compact, then the obvious inclusion $i : X \to X \cup \{\infty\}$ is a compactification.*

PROOF. First we observe that $i : X \to X \cup \{\infty\}$ is an embedding. It is clearly injective and easy to verify that it is continuous. In fact, it is an open map so it follows that it is an embedding.

Next we prove that $X \cup \{\infty\}$ is compact. Suppose $\{U_i\}_{i \in I}$ is a cover. Then at least one U_i contains ∞, so that $X \setminus \{U_1\}$ is compact. Therefore finitely many of the other U_i suffice to cover $X \setminus U_1$.

Finally, we verify that $X \subset X \cup \{\infty\}$ is dense. The space X is not compact itself, so that $\{\infty\}$ is not an open set. Since X is not closed in $X \cup \{\infty\}$, its closure must be $X \cup \{\infty\}$. $\square$

EXAMPLE 3.37. We have already seen a one-point compactification: $\mathbf{N} \to \mathbf{N} \cup \{\infty\}$ was used to investigate convergence of sequences .

EXAMPLE 3.38. Here is another, more troubling, example. Take $\mathbf{Q}$ with the usual topology and form the one-point compactification $\mathbf{Q} \to \mathbf{Q} \cup \{\infty\}$. The open neighbourhoods of ∞ are the complements of compact sets.

We claim that no compact set K contains the intersection of $\mathbf{Q}$ with an open interval I. If it did, we could find a sequence in $I \cap \mathbf{Q}$ converging to an irrational number, and this sequence could have no convergent subsequence in $\mathbf{Q}$.

Therefore, for any $q \in \mathbf{Q}$, every open neighbourhood $(a, b) \cap \mathbf{Q}$ meets every open neighbourhood of ∞. We deduce that $\mathbf{Q} \cup \{\infty\}$ is compact, but it is not Hausdorff.

This space has many of the desirable properties of Hausdorff spaces, however. For instance, convergent sequences have unique limits in it. Suppose K is compact Hausdorff, then the image of every continuous function $f : K \to \mathbf{Q} \cup \{\infty\}$ is closed, just as would be the case if the target were Hausdorff. This condition makes $\mathbf{Q} \cup \{\infty\}$ a *weakly Hausdorff* space.

Now we devote ourselves to finding conditions that ensure that the one-point compactification of a Hausdorff space is again Hausdorff.

DEFINITION 3.39. We will say a space X is *locally compact* if every point $x \in X$ is contained in an open set U that is itself contained in a compact set K.

PROPOSITION 3.40. *If X is a Hausdorff space then the following are equivalent:*

(1) *X is locally compact,*
(2) *every point $x \in X$ has an open neighbourhood U such that $\bar{U}$ is compact,*
(3) *every point $x \in X$ has a local base consisting of open sets U such that $\bar{U}$ is compact.*

PROOF. It is trivial that 3 implies 2 which implies 1. Therefore it suffices to show that 1 implies 3. Let $x \in X$ be a point and choose $U \subset K$ such that $x \in U$ and $U \subset K$ where U is open and K is compact. Let $\{V_i\}_{i\in I}$ be any local base at x. Then $\{U \cap V_i\}_{i\in I}$ is another local base at x and, furthermore, each of these sets is contained in K, which is closed—due to the Hausdorff property. Therefore their closures are closed and contained in K. Consequently, their closures are compact. □

PROPOSITION 3.41. *Let X be a locally compact Hausdorff space. Then the one-point compactification $X \cup \{\infty\}$ is Hausdorff.*

PROOF. Let $x \neq y$ be two points in $X \cup \{\infty\}$. These two points have disjoint open neighbourhoods in X if they both lie in X. Therefore the only case we have to check is when $y = \{\infty\}$. We can find an open $U \ni x$ such that the closure in X, denoted $\bar{U}$ is compact, and so $X \cup \{\infty\} \setminus \bar{U}$ is an open neighbourhood of $\{\infty\}$ disjoint from U. □

LEMMA 3.42. *Let $i : X \to Y$ be any Hausdorff compactification of a locally compact Hausdorff space. Then i is an open map.*

PROOF. We prove that $i(X)$ is open in Y. Let $x \in X$. There exists some open $U \ni x$ and a compact $U \subset K$ in X. We know that $i(K)$ is compact, and hence closed, in Y. Since $i(X)$ is homeomorphic to X, there exists an open $W \ni i(x)$ such that $W \cap i(X) = i(U)$. Now consider $W \setminus i(K)$. This is an open set in Y and $W \cap i(X) \subset i(K)$ implies that it is disjoint from $i(X)$. Since $i(X)$ is dense in Y, this means that it must be empty, so $W \subset i(K) \subset i(X)$. It follows $i(X)$ is open.

For any open $U \subset X$, the set $i(U) = V \cap i(X)$ for some open $V \subset Y$, since i is a homeomorphism onto its image. Since $i(X)$ is open, the map $i : X \to Y$ is open too. □

PROPOSITION 3.43. *Let X be a locally compact, Hausdorff, non-compact space and suppose $i : X \to Y$ is a compactification where Y is Hausdorff. Then the map $f : Y \to X \cup \{\infty\}$ given by*

$$f(y) = \begin{cases} y \text{ if } y \in X \\ \infty \text{ otherwise} \end{cases}$$

is continuous.

This means that $X \cup \{\infty\}$ is *final* among all Hausdorff compactifications of X.

PROOF. Let $U \subset X \cup \{\infty\}$ be an open set. We want to show that $f^{-1}(U)$ is open. There are two cases to consider.

First consider the case where $\infty \not\in U$. In this case, the argument is clear enough: $U \subset X$ is open, so $i(U) = f^{-1}(U)$ is open by Lemma 3.42.

Second, consider the case when $\infty \in U$. Then $X \cup \{\infty\} \setminus U$ is a compact subset K of X, and $i(K) = f^{-1}(X \cup \{\infty\} \setminus U$ is closed in Y. The result follows. □

DEFINITION 3.44. A commutative diagram of the form

$$\begin{tikzcd} & X \arrow[dl, "i"'] \arrow[dr, "i'"] & \\ Y \arrow[rr, "f"] & & Y' \end{tikzcd}$$

where $i : X \to Y$ and $i' : X \to Y'$ are compactifications and f is a continuous map is called a *map of compactifications of* X. A *homeomorphism of compactifications* is a map of compactifications with an inverse map, or equivalently, a map of compactifications where f is a homeomorphism.

COROLLARY 3.45. *Let X be a locally compact, Hausdorff but not compact space, and let i : $X \to Y$ be a Hausdorff compactification in which $Y \setminus i(X)$ consists of one point. Then i is homeomorphic to the one-point compactification.*

PROOF. There exists a continuous bijection $f : Y \to X \cup \{\infty\}$ in which the source is compact and the target is Hausdorff. □

EXAMPLE 3.46. Using stereographic projection, we can prove that S^n is homeomorphic to the one-point compactification of $\mathbf{R}^n$.

REMARK 3.47. A Hausdorff compactification of a locally compact Hausdorff space, $i : X \to Y$, is determined up to homeomorphism by which continuous functions $f : X \to Z$ where Z is compact Hausdorff can be extended

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Z. \\ & Y \arrow[ur, dashed] & \end{tikzcd}$$

Sometimes we can quantify over a smaller class than "all compact Hausdorff spaces": For instance, the compactification S^1 of $\mathbf{R}$ has the property that a function $f : \mathbf{R} \to [0,1]$ extends to a function $\hat{f} : S^1 \to \mathbf{R}$ if and only if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist and agree.

There is a "two-point compactification of $\mathbf{R}$", denoted $\mathbf{R} \cup \{\pm\infty\}$. This is actually homeomorphic to $[0,1]$. A function $f : \mathbf{R} \to [0,1]$ extends to $\hat{f} : \mathbf{R} \cup \{\pm\infty\} \to [0,1]$ if and only if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist, but they do not have to agree.

One might hope for a "maximal" Hausdorff compactification of a locally compact Hausdorff space X, and this does exist (assuming the axiom of choice). It is called the *Stone–Čech compactification* $iX \to \beta X$. It has the property that any continuous function $f : X \to Z$ where Z is compact Hausdorff has a unique extension to $\beta f : \beta X \to Z$.

For the rest of the course, we will largely work with locally compact Hausdorff spaces.

The one-point compactification allows us to prove the following proposition extending the characterization of locally compact Hausdorff spaces. It extends 3.40

PROPOSITION 3.48. *Let X be a Hausdorff topological space. Then X is locally compact if and only if, for all open $N \ni x$, there exists an open $U \ni x$ such that $\bar{U} \subset N$ and $\bar{U}$ is compact.*

PROOF. Suppose X is locally compact. Let $X \cup \{\infty\}$ denote the one-point compactification. This is a compact Hausdorff space, and is therefore normal. Consider the two closed sets in $X \cup \{\infty\}$:

$$(X \cup \{\infty\}) \setminus N, \quad \{x\}.$$

These are disjoint, and by normality (regularity is enough), we can find disjoint open sets U, V containing them. The set U is an open set containing ∞, so its complement is a compact subset $K \subset X$, and U was constructed to contain $X \setminus N$, so that $K \subset N$. On the other hand, V is an open set not containing ∞, and therefore $V \subset X$ and V is an open set in X. Moreover $V \subset K$, since V and U are disjoint.

But then $\bar{V}$ in X is a closed subset contained in K, which is compact, so $\bar{V}$ is compact and $\bar{V} \subset N$.

The other direction is trivial. □

PROPOSITION 3.49. *Suppose X is a locally compact Hausdorff space and $A \subset X$ is a closed subspace. Then A is also locally compact and Hausdorff.*

PROOF. The space A is certainly Hausdorff.

Suppose $a \in A$ is a point. There exists an open neighbourhood U of a in X such that the closure $\bar{U}$ in X is compact. Now take $U \cap A$. This is an open neighbourhood of a in A, and it is contained in the compact subset $\bar{U} \cap A$ of A. Since a was an arbitrary point in A, the space A is locally compact. □

PROPOSITION 3.50. *Suppose X is a locally compact Hausdorff space and $A \subset X$ is an open subspace. Then A is also locally compact and Hausdorff.*

6. Compactly generated topologies

DEFINITION 3.51. Let X be a topological space. We say a subspace $C \subset X$ is *k-closed* if $u^{-1}(C)$ is closed in K for all continuous maps $u : K \to X$ with compact Hausdorff source.

PROPOSITION 3.52. *The k-closed subsets form the closed sets of a topology on X. If the set X equipped with this topology is denoted kX, then the identity map $kX \to X$ is continuous.*

DEFINITION 3.53. We say a topology on X is *compactly generated* if every k-closed set is closed.

PROPOSITION 3.54. *If X is a topological space satisfying either of the following conditions, then X is compactly generated:*

(1) X is Hausdorff and locally compact.
(2) Every sequentially closed subset of X is closed (e.g. if X is first countable).

PROOF. (1) Suppose C is k-closed and $x \in \bar{C}$. Let K be a compact neighbourhood of x (a compact set containing an open neighbourhood). Suppose $V \ni x$ is an arbitrary open neighbourhood, then $K \cap V$ contains an open neighbourhood of x and so $K \cap$

$V \cap C \neq \emptyset$. Therefore $x \in \overline{K \cap C}$. Let $j : K \to X$ be the inclusion. Then $j^{-1}(C) = K \cap C$ is closed in K, which implies that $x \in C$.

(2) Exercise.

$\square$

CHAPTER 4

Connectedness

1. Connectedness

DEFINITION 4.1. We say a topological space X is *connected* if every function $X \to \{0,1\}$ where $\{0,1\}$ has the discrete topology is constant.

LEMMA 4.2. *Let X be a topological space. The following are equivalent:*

(1) X is connected;
(2) If $A \subset X$ is open and closed, then $A = \emptyset$ or $A = X$;
(3) Every function $X \to D$, where the target is discrete, is constant.

EXAMPLE 4.3. Let $A \subset \mathbf{R}$ be a subset of the real line. If $x < y < z$ are three points in $\mathbf{R}$ such that $x, z \in A$ but $y \notin A$, then the function $f : \mathbf{R} \setminus \{y\} \to \{0,1\}$ given by $f(t) = 0$ if $t < y$ and $f(t) = 1$ otherwise is continuous and so restricts to a continuous nonconstant function on A. So A is not connected (it is *disconnected*).

This implies that if A is a connected subset of $\mathbf{R}$, then A is an interval (including the degenerate intervals $\emptyset$ and $\{a\}$). On the other hand, if A is an interval and $f : A \to \mathbf{R}$ has the property that $f(x) = 0$ and $f(z) = 1$ (without loss of generality $x < z$) and that f is continuous, then consider the element $y = \inf\{f^{-1}(1) \cap [x,z]\}$. Since $f^{-1}(1)$ is closed, $f(y) = 1$, but since $f^{-1}(0)$ is closed, $f(y) = 0$, a contradiction.

It follows that the connected subsets of $\mathbf{R}$ are precisely the intervals.

PROPOSITION 4.4. *Let $f : X \to Y$ be a continuous surjective function, and suppose X is connected. Then Y is connected.*

PROPOSITION 4.5. *Let X be a topological space, let $\{A_i\}_{i\in I}$ be a family of connected subspaces of X such that for all $i, j \in I$, the set $A_i \cap A_j$ is nonempty and such that $\bigcup_{i\in I} A_i = X$. Then X is connected.*

COROLLARY 4.6. *Let $\{X_i\}_{i\in I}$ be a family of connected spaces. Then $X = \prod_{i\in I} X_i$ is connected.*

PROOF. If any of the sets X_i is empty, then the product is empty and there is nothing to do.

Let $f : X \to \{0,1\}$ be a continuous function that is not identically 0. First we observe that if y and y' are two points of X that differ only in one coordinate, the j-th coordinate, then $f(y) = f(y')$. This is because there is an inclusion $c_j : X_j \to \prod_{i\in I} X_i$ given by the other coordinates, and the composite $f \circ c_j$ gives a continuous function $X_j \to \{0,1\}$.

By an easy induction, if y and y'' differ in finitely many coordinates, then $f(y) = f(y'')$.

Finally, suppose $x \in X$ is such that $f(x) = 1$. Then $f^{-1}(1)$ contains a subbasic open set U around x, so that there are finitely many coordinates $i_1, \dots, i_r$ such that if x and y'' agree in these coordinates, then $y'' \in U$. In particular, $f(y'') = 1$ as well. But then for arbitrary $y \in X$, we can change finitely many coordinates to produce $y'' \in U$, so $f(y) = 1$. Therefore f is identically 1. $\square$

DEFINITION 4.7. Let X be a topological space and let $x \in X$ be point. The *connected component* C_x of x is the union of all connected subsets of X containing x.

REMARK 4.8. The connected components are connected, and if $y \in C_x$, then $C_y = C_x$.

EXAMPLE 4.9. The space $\mathbf{Q}$ shows that the connected components are not necessarily open and closed. A space in which connected components are singletons is said to be *totally disconnected.*

The space $\mathbf{Q}$ seems badly behaved from a certain point of view: one might like the connected components themselves to be both closed and open subsets of the space, but this is not the case.

REMARK 4.10. Every topological space is a union of connected components, and these components are pairwise disjoint.

DEFINITION 4.11. A space X is *locally connected* if every point $x \in X$ has a local base $\{U_i\}_{i \in I}$ such that U_i is connected.

PROPOSITION 4.12. *Let X be a locally connected space, and let C_x be the connected component of $x \in X$. Then C_x is open and closed.*

PROOF. It suffices to prove C_x is open, since then the complement is the union of the other components of X, which is also open.

The component C_x contains open neighbourhoods around each point, and is therefore open. $\square$

2. Path-connectedness

Let I denote $[0,1]$ with the usual topology throughout.

DEFINITION 4.13. A topological space X is *path-connected* if for every two points $x, y \in X$, there is a continuous function $\gamma : I \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

If the functions γ may always be chosen to be injective, then X is *arc-connected.*

PROPOSITION 4.14. *Let $f : X \to Y$ be a continuous surjective function, and suppose X is path-connected. Then Y is path-connected.*

PROPOSITION 4.15. *Let X be a topological space, let $\{A_i\}_{i \in I}$ be a family of path-connected subspaces of X such that for all $i, j \in I$, the set $A_i \cap A_j$ is nonempty and such that $\bigcup_{i \in I} A_i = X$. Then X is path-connected.*

COROLLARY 4.16. *If X is path connected, then it is connected.*

COROLLARY 4.17. *Let $\{X_i\}_{i \in I}$ be a family of path-connected spaces. Then $X = \prod_{i \in I} X_i$ is path-connected.*

DEFINITION 4.18. Let X be a topological space and let $x \in X$ be point. The *path component* P_x of x is the union of all path-connected subsets of X containing x.

REMARK 4.19. The path components are path-connected, and if $y \in P_x$, then $P_y = P_x$.

REMARK 4.20. Every topological space is a union of connected components, and these components are pairwise disjoint. Note also that for all x, the path component P_x is contained in the connected component C_x.

DEFINITION 4.21. A space X is *locally path-connected* if every point $x \in X$ has a local base $\{U_i\}_{i\in I}$ such that U_i is path-connected.

PROPOSITION 4.22. *If X is locally path connected and $U \subset X$ is an open subset, then U is locally path connected.*

We include this result mostly to highlight the fact that it does not necessarily apply to closed subsets.

PROPOSITION 4.23. *Let X be a locally path-connected space, and let $x \in X$ be a point. Then $P_x = C_x$.*

PROOF. If X is locally path-connected, then it is locally connected. Therefore Proposition 4.12 says that C_x is both open and closed in X. An argument entirely analogous to Proposition 4.12 says that, since X is locally path-connected, P_x is also open and closed in X.

In particular, P_x is both an open and a closed subset of the connected subspace C_x. Since P_x is not empty, it must be the case that $P_x = C_x$. □

COROLLARY 4.24. *If X is locally path-connected and connected, then X is path-connected.*

EXAMPLE 4.25. The topologist's sine curve S is a well-known metric space that is connected but not path connected. This set is defined as

$$S = \{(0, y) \mid -1 \le y \le 1\} \cup \{(x, \sin\frac{1}{x}) \mid x > 0\}.$$

First we show that this is connected. The subset $C = \{(x, \sin\frac{1}{x}) \mid x > 0\}$ is homeomorphic to $(0,\infty)$ and is therefore connected. Let $(0, t)$ be a point in $L = \{(0, y) \mid -1 \le y \le 1\}$, and let $f : S \to \{0,1\}$ be a continuous function taking (without loss of generality) the value 0 on C. Then there is a sequence in C converging to $(0, t)$, so that $f((0, t)) = 0$ by continuity. The function f is therefore constant. This shows S is connected.

Now we show that S is not path connected. Suppose $f : I \to S$ is a path from a point in L to $p = (1/\pi, 0)$. Consider the set $f^{-1}(L)$, which is closed in L, and therefore has a maximal element, $t_0 < 1$. Restricting, we have a continuous function $f : [t_0, 1] \to S$ with the property that $f(t_0) = (0, y_{t_0})$ and $(x_t, y_t) := f(t) \in C$ for $t > t_0$. Since f is continuous, $\lim_{t\to t_0+} x_t = 0$, and by the intermediate value theorem we can find sequences of values of t

$$(t_n^+) \quad \text{such that } x_{t_n^+} = 1/(2n\pi)$$

and

$$(t_n^-) \quad \text{such that } x_{t_n^+} = 3/(2n\pi)$$

each converging to t_0. But then $f(t_n^+) = (t_n^+, 1)$ while $f(t_n^-) = (t_n^-, -1)$, so that the y coordinate of $f(t_0)$ must be both $\lim_{n\to\infty} +1$ and $\lim_{n\to\infty} -1$.

CHAPTER 5

Homotopy

1. Basic Definitions

Homotopies. In this section, "map" will denote a continuous function and I will denote $[0,1]$.

DEFINITION 5.1. Let X and Y be topological spaces and $A \subset X$ be a subset. Suppose $f_0, f_1 : X \to Y$ are two maps such that $f_0|_A = f_1|_A$. Then a *homotopy relative to* A from f_0 to f_1 (or between f_0 and f_1) is a map $H : X \times I \to Y$ such that

$$\begin{aligned} H(x,0) &= f_0(x) \quad \forall x \in X \\ H(x,1) &= f_1(x) \quad \forall x \in X \\ H(a,t) &= f_0(a) = f_1(a) \quad \forall a \in A, \forall t \in I \end{aligned}$$

NOTATION 5.2. A homotopy relative to $\emptyset$ is called a *homotopy*.

NOTATION 5.3. If a homotopy relative to A exists from f to g, we say that f and g are *homotopic relative to A*. In the case where $A = \emptyset$, we say f and g are *homotopic*.

NOTATION 5.4. We write $f \simeq_A g$ to indicate that f is homotopic to g relative to A. This implies, among other things, that $f|_A = g|_A$. We may write $f \simeq g$ if A is understood or if A is empty.

PROPOSITION 5.5. *Let $A \subset X$ and Y be topological spaces, and consider maps $f : X \to Y$. The relation $\simeq_A$ is an equivalence relation.*

PROOF. Reflexivity: $H(x,t) = f(x)$ gives a homotopy from f to f.

Symmetry: if H gives a homotopy one way, then $H'(x,t) := H(x, 1-t)$ gives a homotopy the other way.

Transitivity: suppose H_0 is a homotopy (rel. A) from f_0 to f_1 and H_1 is a homotopy (rel. A) from f_1 to f_2. Then define

$$H(x,t) = \begin{cases} H_0(x,2t) & t \le 1/2 \\ H_1(x,2t-1) & t > 1/2 \end{cases} .$$

This gives a homotopy from f_0 to f_2 (rel. A). □

PROPOSITION 5.6. *Suppose X, Y and Z are three spaces, and $A \subset X$ and $B \subset Y$ are subspaces. Suppose $f_0, f_1 : X \to Y$ are maps such that $f_0 \simeq_A f_1$ and that $f_0(A) \subset B$. Suppose also that $g_0, g_1 : Y \to Z$ are maps such that $g_0 \simeq_B g_1$. Then $g_1 \circ f_1 \simeq_A g_0 \circ f_0$.*

PROOF. There exist homotopies H' from f_0 to f_1 and H'' from g_0 to g_1. Now consider

$$H : X \times I \to Z$$

defined by

$$H(x,t) = H''(H'(x,t),t).$$

This is the required homotopy. □

Homotopy Equivalence and the Homotopy Category. Recall that two spaces X and Y are homeomorphic if there are maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. Homotopy equivalence of spaces is the relation you get if you weaken the "=" signs to "≃".

DEFINITION 5.7. A map $f : X \to Y$ of topological spaces is a *homotopy equivalence* if there exists a map $g : Y \to X$ such that:

$$g \circ f \simeq \mathrm{id}_X, \qquad f \circ g \simeq \mathrm{id}_Y.$$

Note that a homeomorphism is a homotopy equivalence. We now show that the homotopy equivalences are exactly the isomorphisms in a certain category.

DEFINITION 5.8. The *homotopy category* **H** is a category defined as follows. The objects of **H** are the topological spaces. The morphisms in **H** from X to Y are the homotopy classes (rel. $\varnothing$) of maps $X \to Y$. We know from Proposition 5.5 that these equivalence classes are defined. We know from Proposition 5.6 that composition of equivalence classes is well defined. It is immediate from the construction that the class of $\mathrm{id}_X : X \to X$ serves as an identity morphism in this category.

NOTATION 5.9. In this course, we will write $[X,Y]$ for the set of morphisms $X \to Y$ in **H**. That is, $[X,Y]$ is the set of homotopy classes of continuous functions from $X \to Y$.

PROPOSITION 5.10. *There is a functor **Top** → **H** that is the identity on objects and sends a morphism $f : X \to Y$ to the homotopy class of f. We will denote this class by $[f]$.*

PROOF. It suffices to verify compatibility with compositions: $[f \circ g] = [f] \circ [g]$, and that $[\mathrm{id}_X]$ is the identity in **H**. Both are routine. □

PROPOSITION 5.11. *A map $f : X \to Y$ is a homotopy equivalence if and only if $[f]$ is an isomorphism in **H**.*

PROOF. Suppose $[f]$ is an isomorphism. Then it has an inverse in **H**, which must be the class of some map $g : Y \to X$. We know that $[g] \circ [f] = [\mathrm{id}_X]$, and unwinding definitions this means $g \circ f \simeq \mathrm{id}_X$. The statement $f \circ g \simeq \mathrm{id}_Y$ is deduced similarly.

Now suppose that f is a homotopy equivalence. Let g be the homotopy inverse. Then $g \circ f \simeq \mathrm{id}_X$, which is to say that $[g \circ f] = [\mathrm{id}_X]$, and by functoriality, $[g] \circ [f] = [\mathrm{id}_X]$. The statement $[f] \circ [g] = [\mathrm{id}_Y]$ is similar. □

This view of homotopy equivalences allows us to prove some formal properties without much effort.

COROLLARY 5.12 (The 2-out-of-3 property). *Suppose $f : X \to Y$ and $g : Y \to Z$ are maps. Then, if two of the maps $\{f, g, g \circ f\}$ are homotopy equivalences, so is the third.*

COROLLARY 5.13 (The 2-out-of-6 property). *Suppose $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$ are maps. Then if $g \circ f$ and $h \circ g$ are homotopy equivalences, then f, g, h are homotopy equivalences.*

NOTATION 5.14. We write $X \simeq Y$ to indicate that X is homotopy equivalent to Y. Note that the symbol $\simeq$ is used in two related, but distinct, ways: $f_1 \simeq f_2$ means there is a homotopy from f_1 to f_2, whereas $X \simeq Y$ means that the spaces are homotopy equivalent, i.e., there are maps $f : X \to Y$ and $g : Y \to X$ that are homotopy inverse to one another.

Retractions and Deformation Retractions.

DEFINITION 5.15. A subspace $A \subset X$ is a *retract* if there exists a *retraction* map $r : X \to A$ such that $r(a) = a$ for all $a \in A$. More generally, an embedding $i : A \to X$ is a *retract* if $i(A)$ is a retract of X, or equivalently, if there is a map $r : X \to A$ such that $r \circ i = \mathrm{id}_A$.

DEFINITION 5.16. A *deformation retraction* of a space X onto a subspace A is a map $H : X \times I \to X$ such that:

(1) $H_0 : X \to X$ is the identity.
(2) The image of H_1 lies in A.
(3) $H_1(a) = a$ for all $a \in A$.

The deformation retraction is said to be a *strong deformation retraction* if $H_t(a) = a$ for all $t \in I$ and all $a \in A$. The space A will be said to be a (strong) *deformation retract* of A as appropriate.

That is, a deformation retraction includes both a retraction $H_1 : X \to A$ and a homotopy from id_X to H_1. The deformation retractions that arise in practice are usually strong. We may say that an embedding $i : A \to X$ is a deformation retract if $i(A)$ is a deformation retract of X.

LEMMA 5.17. *If $i : A \to X$ is a deformation retract, then i is a homotopy equivalence.*

PROOF. We know $i \circ r \simeq \mathrm{id}_X$ and $r \circ i = \mathrm{id}_A$, which establishes the homotopy equivalence. □

Showing that a map is a homotopy equivalence can be cumbersome, and there are pitfalls. Lemma 5.17 is very useful in calculations.

EXAMPLE 5.18. If X is any nonempty space at all, then any inclusion $\{x\} \to X$ is a retract, but in general this is not a deformation retract.

EXAMPLE 5.19. Recall that S^n denotes the subset of $\mathbf{R}^{n+1}$ consisting of elements of norm 1. There is a (strong) deformation retraction of $\mathbf{R}^{n+1} \setminus \{\mathbf{0}\}$ to S^n given by

$$H(\mathbf{v}, t) = \frac{1}{t(\|\mathbf{v}\| - 1) + 1}\mathbf{v}.$$

We verify this:

(1) H is continuous.
(2) $H(\mathbf{v}, 0) = \frac{1}{1}\mathbf{v} = \mathbf{v}$.
(3) $H(\mathbf{v}, 1) = \frac{1}{\|\mathbf{v}\|}\mathbf{v} \in S^n$.
(4) If $\|v\| = 1$, then $H(\mathbf{v}, t) = \frac{1}{1}\mathbf{v}$, as required.

This has the notable consequence that $S^n \simeq \mathbf{R}^{n+1} \setminus \{\mathbf{0}\}$.

2. The pointed homotopy category and homotopy groups

DEFINITION 5.20. Let the category $\mathbf{Top}_\bullet$ denote the category of *based topological spaces*. The objects are pairs (X, x_0) where $x_0 \in X$, and the morphisms $f : (X, x_0) \to (Y, y_0)$ are continuous functions such that $f(x_0) = y_0$.

REMARK 5.21. We frequently omit the basepoint x_0 from the notation.

NOTATION 5.22. Write $[(X, x_0), (Y, y_0)]_\bullet$ for the set of homotopy classes relative to x_0, of maps $f : X \to Y$ satisfying $f(x_0) = y_0$. That is, g is in the class of f if and only if there exists a homotopy

$$H : X \times I \to Y$$

such that

$$\begin{aligned} H(x,0) &= f(x) \quad \forall x \in X \\ H(x,1) &= g(x) \quad \forall x \in X \\ H(x_0,t) &= y_0 \quad \forall t \in I \end{aligned}$$

DEFINITION 5.23. We write $\mathrm{H}_\bullet$ for the *pointed homotopy category*: the objects are based spaces and the set of morphisms from X to Y is $[(X, x_0), (Y, y_0)]_\bullet$.

REMARK 5.24. As in the unpointed case, there is a functor $\mathbf{Top}_\bullet \to \mathbf{H}_\bullet$ that is the identity on objects and sends f to the (pointed) homotopy class of f.

REMARK 5.25. There can exist based spaces (X, x_0) and (Y, y_0), along with pointed maps $f, g : X \to Y$ such that $f \simeq g$ in the unpointed sense, but $f \not\simeq_{x_0} g$ in the pointed sense.

DEFINITION 5.26. Let (X, x_0) be a pointed space and let $n \geq 0$. Give S^n the basepoint $(1, 0, \ldots, 0)$. Then define the *n-th homotopy group of* (X, x_0) by

$$\pi_n(X, x_0) = [S^n, X]_\bullet$$

Note that the basepoint is dropped from the notation.

REMARK 5.27. If you prefer not to use the homotopy-category formulation, you can define $\pi_n(X, x_0)$ as follows:

$$\pi_n(X, x_0) = \{f : (S^n, s_0) \to (X, x_0) \mid f \text{ is a pointed map}\}/\simeq$$

where the relation $\simeq$ denotes pointed homotopy.

REMARK 5.28. The term "homotopy group" is a misnomer, since $\pi_0(X, x_0) = [S^0, X]_\bullet$ is not in general a group. For larger values of n, however, the sets $\pi_n(X, x_0)$ do have a natural group operation defined on them. We will prove this in the case of $\pi_1(X, x_0)$ later.

EXAMPLE 5.29. $\pi_0(X, x_0)$ is the set of based homotopy classes of based maps $S^0 \to (X, x_0)$. The space S^0 is the discrete space with two points. A based map $f : S^0 \to X$ corresponds exactly to a single element $f(-1) \in X$, since $f(1) = x_0$. Then two such maps f, g are homotopic if and only if there is a path in X from $f(-1)$ to $g(-1)$. Therefore we can identify $\pi_0(X, x_0)$ with the set of path components of X. There is a distinguished component, that of x_0.

In contrast to the case of the larger values of i, in the case where $i = 0$, one can define $\pi_0(X)$ without reference to the basepoint: $\pi_0(X) = [*, X]$. This is the set of path components of X without any distinguished choice of component.

REMARK 5.30. As defined, $\pi_n(X, x_0)$ is a composite of two functors: one being the functor from $\mathbf{Top}_\bullet \to \mathbf{H}_\bullet$ and the other being $[S^n, \cdot]_\bullet$. This implies that $\pi_n(X, x_0)$ is itself a functor, or, in less fancy language, if you have a based map $f : (X, x_0) \to (Y, y_0)$, then you get induced functions

$$f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$$

and these induced functions respect compositions and identities.

To define f_* explicitly with functions, do the following: suppose $g : (S^n, s_0) \to (X, x_0)$ is a map of pointed spaces. Then g represents a class $[g] \in \pi_n(X, x_0)$. The element $f_*([g])$ is the homotopy class of the map $f \circ g : (S^n, s_0) \to (Y, y_0)$.

REMARK 5.31. If $h : (X, x_0) \to (Y, y_0)$ is an isomorphism in $\mathbf{H}_\bullet$, then $[S^n, X]_\bullet \to [S^n, Y]_\bullet$ is an isomorphism. That is, if h is a pointed homotopy equivalence, then the induced map $h_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is a bijection.

This can be strengthened to the statement: If $h : X \to Y$ is an isomorphism in $\mathbf{H}$, and $x_0 \in X$, then the induced map $h_* : \pi_n(X, x_0) \to \pi_n(Y, h(y_0))$ is a bijection. We will not prove this for $n \geq 2$ in this course. For $n = 0$, it is easy to see because $\pi_0(X, x_0)$ is the set $\pi_0(X) = [*, X]$ equipped with a distinguished point, and $\pi_0(\cdot)$ does not depend on the basepoint. The result for π_1 will be proved later.

REMARK 5.32. The groups $\pi_i(X, x_0)$ are considered hard to compute when $i \geq 2$. For instance, I don't think $\pi_i(S^2, s_0)$ has been fully computed for values of i much beyond 30. We will concentrate in the rest of this course on the group about which a lot is known: $\pi_1(X, x_0)$. This is the *fundamental group* of (X, x_0).

CONSTRUCTION 5.33. Let (X, x_0) and (Y, y_0) be two pointed spaces. We define the *wedge sum* $(X, x_0) \vee (Y, y_0)$ to be the quotient of $X \coprod Y$ given by collapsing the subspace $\{x_0\} \coprod \{y_0\}$. The space $(X, x_0) \vee (Y, y_0)$ is naturally embedded as a subspace of $X \times Y$, by equating $x \in X$ with the pair (x, y_0) and $y \in Y$ with (x_0, y). The quotient $(X \times Y)/(X \vee Y) =: X \wedge Y$ is called the *smash product*.

If X is the one-point compactification of an LCH space U and Y is the one-point compactification of an LCH space V, and if X and Y are given the basepoints at ∞, then $X \wedge Y$ is the one-point compactification of $U \times V$.

In particular, $S^n \wedge S^m \approx S^{n+m}$.

3. Contractible spaces

DEFINITION 5.34. A map $f : X \to Y$ is *nullhomotopic* if there exists an element $y \in Y$ such that f is homotopic to the map with constant value y. If a map is not nullhomotopic, it is *essential*.

DEFINITION 5.35. A space X is *contractible* if the identity map $\mathrm{id}_X : X \to X$ is nullhomotopic.

REMARK 5.36. One can define nullhomotopy in the category of pointed spaces and based maps as well. In this case a map $f : (X, x_0) \to (Y, y_0)$ is *nullhomotopic* if there exists a basepoint-preserving homotopy of f to the constant map at y_0.

If a pointed map $f : (X, x_0) \to (Y, y_0)$ is nullhomotopic, then the underlying (unpointed) map f is nullhomotopic. In practice, it is rare to encounter a pointed map $f : (X, x_0) \to (Y, y_0)$ so that the underlying map f is nullhomotopic but where the pointed map is essential. An example is given by an inclusion of a point in the comb-space, so they do exist.

REMARK 5.37. Similarly to the unpointed case, one could define *pointed contractibility*: a pointed space (X, x_0) is pointed-contractible if the identity $\mathrm{id} : (X, x_0) \to (X, x_0)$ is homotopic to the constant map through a basepoint-preserving homotopy. Again, the comb-space can be used to produce a pointed space (X, x_0) so that (X, x_0) is not pointed-contractible, but X is contractible (forgetting the basepoint). Even worse, there exist examples of spaces Y that are contractible but such that (Y, y_0) is not pointed-contractible for any choice of $y_0 \in Y$.

For the classes of space that arise in practice, e.g., Manifolds or CW complexes (see later), one can often prove a result such as: (X, x_0) is pointed contractible if and only if X is contractible. Furthermore, it is the concept of contractibility, not pointed contractibility, that is most useful in applications. We do not really pursue pointed contractibility further in these notes.

PROPOSITION 5.38. *A space X is contractible if and only if it is homotopy equivalent to a one-point space.*

PROOF. Suppose X is contractible. In particular, there exists $x_0 \in X$ and a homotopy $H : X \times [0,1] \to X$ such that $H_0 : X \to X$ is id_X and $H_1 : X \to X$ is the constant map at $\{x_0\}$, i.e., there is a deformation retraction of X onto a one-point subspace.

Conversely, suppose there exists a homotopy equivalence $f : \{z\} \to X$, with homotopy inverse g. Then the fact that $f \circ g \simeq \mathrm{id}_X$ implies that there is a deformation retraction of X onto $\{f(z)\}$. □

PROPOSITION 5.39. *Suppose X is a contractible space and Y is any space. The set $[Y, X]$ of homotopy classes of maps from Y to X consists of one element.*

That is to say, every map from Y to X is homotopic to every other map.

PROOF. Since X is contractible, we can find an element $x_0 \in X$ so that $\mathrm{id}_X : X \to X$ is homotopic to $g : X \to X$, the constant map with value x_0. Then if $f : Y \to X$ is any map, there is a homotopy from $f = \mathrm{id}_X \circ f$ to $g \circ f$, and $g \circ f$ is the map $Y \to X$ with the constant value x_0. □

COROLLARY 5.40. *If X is contractible and $i : A \to X$ is a retract, then A is contractible.*

(Note: the hypothesis says “retract” and not “deformation retract”)

PROOF. We show that if $i : A \to X$ is a retract, then A is homotopy equivalent to X, and therefore to a point. Let $r : X \to A$ be a retraction, so that $r \circ i = \mathrm{id}_A$. But $i \circ r$ is a map from $X \to X$, and is therefore homotopic to id_X (since all maps $X \to X$ are homotopic by Proposition 5.39).

In summary, $r \circ i = \mathrm{id}_A$ and $i \circ r \simeq \mathrm{id}_X$, so that $A \simeq X$. Since X is contractible, so is A. □

Many commonly-occurring spaces are contractible. Here is a device for establishing this quickly.

PROPOSITION 5.41. *Suppose a space X has a multiplicative action by $[0,1]$; that is, there is a map $m : [0,1] \times X \to X$ for which*

(1) $m(t_1, m(t_2, x)) = m(t_1 t_2, x)$ *and*
(2) $m(1, x) = x$

for all $t_1, t_2 \in [0,1]$ and $x \in X$. Let X_0 denote the image of $m(0,\cdot)$. Then X_0 is a (strong) deformation retract of X.

PROOF. Certainly $i : X_0 \to X$ is an embedding. The map $m : [0,1] \times X \to X$ functions as the homotopy in a strong deformation retract. $\square$

COROLLARY 5.42. *Let V be a topological $\mathbf{R}$-vector space, and let X be a nonempty subspace of V that is closed under multiplication by $[0,1]$ (call such a region starshaped). Then X is contractible.*

REMARK 5.43. If V is as above and Y is a nonempty convex subset of V, then Y is homeomorphic to a translate of Y containing the origin. This translate is starshaped, and therefore Y is contractible.

Initial and terminal objects

REMARK 5.44. In a category $\mathbf{C}$, if an object $*$ has the property that all objects $c \in \mathbf{C}$ are have a unique map $c \to *$, then $*$ is called a *terminal object* of $\mathbf{C}$. You can prove many trivial things about terminal objects: for instance, any two terminal objects must be isomorphic and any map between terminal objects must be an isomorphism. Any object isomorphic to a terminal object must itself be terminal.

A space X is contractible if and only if X is a terminal object of $\mathbf{H}$. The trivialities mentioned above lead to abstract proofs of statements such as: any two maps $f, g : X \to Y$ between contractible spaces are homotopic.

REMARK 5.45. There is also the notion of an *initial object* $\emptyset$ in a category $\mathbf{C}$. An object is initial if all objects c are equipped with a unique morphism $\emptyset \to c$. Similarly to the case of final objects, initial objects are isomorphic, all maps between them are isomorphisms and all objects isomorphic to an initial object must be initial.

There is a unique initial object in $\mathbf{Top}$ and $\mathbf{H}$, the empty space. In $\mathbf{Top}_\bullet$ and $\mathbf{H}_\bullet$, the one-point spaces are initial, however. An object that is both initial and final is called a *zero object*.

CHAPTER 6

The fundamental groupoid and the fundamental group

1. The fundamental groupoid

CONSTRUCTION 6.1. Suppose $\gamma, \delta : I \to X$ are two paths, and suppose $\gamma(1) = \delta(0)$. We define a composite path $\gamma \cdot \delta : I \to X$ by

$$\gamma \cdot \delta(t) = \begin{cases} \gamma(2t) & \text{if } t \leq 1/2 \\ \delta(2t-1) & \text{if } t \geq 1/2 \end{cases} .$$

NOTATION 6.2. Let us say that two paths $\gamma, \gamma' : I \to X$ are *equivalent* and if $\gamma \simeq \gamma'$ relative to $\{0, 1\}$. In particular, γ and γ' have the same endpoints. Recall that homotopy (relative to a subspace) is an equivalence relation. We will write $[\gamma]$ for the equivalence class of γ.

REMARK 6.3. Warning: the notation $[\gamma]$ when $\gamma : I \to X$ is different from the notation $[f]$ used in the previous chapter. The homotopies used to define $[\gamma]$ are the homotopies relative to $\{0, 1\}$, and when $\gamma : I \to X$ is a path, the notation $[\gamma]$ will be used only in this sense.

What follows is some technical lemmas about path composition.

PROPOSITION 6.4. *If $[\gamma] = [\gamma']$ and $[\delta] = [\delta']$ and if $\gamma \cdot \delta$ is defined, then $[\gamma \cdot \delta] = [\gamma' \cdot \delta']$.*

REMARK 6.5. That is, the composition of paths descends to equivalence classes. After we have proved this result, we can define $[\gamma][\delta]$ by choosing representatives.

PROOF. Let H be a homotopy from γ to γ', relative to endpoints, and similarly, let E be a homotopy from δ to δ'. Then define

$$H \cdot E : I \times I \to X, \qquad E \circ H(t, s) = \begin{cases} H(2t, s) & \text{if } t \leq 1/2 \\ E(2t-1, s) & \text{if } t \geq 1/2 \end{cases}$$

This gives the required homotopy of $\gamma \cdot \delta$ to $\gamma' \cdot \delta'$ relative to $\{0, 1\}$. □

Composition of paths is not associative—you can check directly that $\gamma \cdot (\delta \cdot \epsilon) \neq (\gamma \cdot \delta) \cdot \epsilon$.

PROPOSITION 6.6. *Suppose γ, δ and ϵ are paths in X such that $\gamma \cdot (\delta \cdot \epsilon)$ is defined. Then* $[\gamma]([\delta][\epsilon]) = ([\gamma][\delta])[\epsilon]$.

That is, the composition is associative once we pass to homotopy classes.

PROOF. It is sufficient to write down a homotopy (relative to endpoints) between $\gamma \cdot (\delta \cdot \epsilon)$ and $(\gamma \cdot \delta) \cdot \epsilon$.

$$H(t,s) = \begin{cases} \gamma(4t/(2-s)) & \text{if } t \le 1/2 - s/4 \\ \delta(4t-2+s) & \text{if } 1/2 - s/4 \le t \le 3/4 - s/4 \\ \epsilon(1+4(t-1)/(1+s)) & \text{if } t \ge 3/4 - s/4 \end{cases}$$

□

DEFINITION 6.7. If X is a space and $x \in X$, define e_x to be the constant path at x, i.e., $e_x(t) = x$ for all t.

PROPOSITION 6.8. *Let X be a space and let γ be a path in X starting at x and ending at y. Then $[e_x] \cdot [\gamma] = [\gamma]$ and $[\gamma] \cdot [e_y] = [\gamma]$.*

PROOF. We'll show one of these. The other is similar.

Just write down a homotopy from $e_x \cdot \gamma$ to γ.

$$H(t,s) = \begin{cases} x & \text{if } 2t \le 1-s \\ \gamma(1+2(t-1)/(s+1)) & \text{if } 2t \ge 1-s \end{cases}$$

□

NOTATION 6.9. If $\gamma : I \to X$ is a path, write $\gamma^{\leftarrow}$ for the *reverse* of γ: $\gamma^{\leftarrow}(t) = \gamma(1-t)$. Clearly, $(\gamma^{\leftarrow})^{\leftarrow} = \gamma$.

PROPOSITION 6.10. *In the notation above, if γ is a path from x to y, then* $[\gamma] \cdot [\gamma^{\leftarrow}] = [e_x]$.

PROOF. We write down a homotopy:

$$H(t,s) = \begin{cases} \gamma(2t) & \text{if } 2t \le 1-s \\ \gamma(1-s) & \text{if } 1-s \le 2t \le 1+s \\ \gamma(2-2t) & \text{if } 1+s \le 2t \end{cases}$$

□

Recall that a *groupoid* $\mathcal{G}$ is a category having a set of objects and a set of morphisms and such that all morphisms are isomorphisms.

DEFINITION 6.11. Let X be a topological space and let $A \subset X$ be a subset of X. Define a *fundamental groupoid of X with endpoints in A*, denoted $\Pi(X,A)$, as the groupoid where

$$\operatorname{ob}\Pi(X,A) = A$$

and for $a_0, a_1 \in A$, the set of morphisms from a_0 to a_1, is the equivalence classes of paths in X starting at a_0 and ending at a_1. The previous propositions ensure that the composition law is well defined[1] and associative, that $[e_{a_0}]$ is the identity at a_0 and that inverses exist for all morphisms (just reverse the path). So this really is a groupoid.

[1]Warning: one defines the geometric composition of paths $[\gamma_1] \cdot [\gamma_2]$ to mean "first do γ_1 and then do γ_2". In order to match the conventions for compositions in a category, you have to declare that $[\gamma_2] \circ [\gamma_1] = [\gamma_1] \cdot [\gamma_2]$. We use the geometric presentation throughout.

REMARK 6.12. A special cases of the above is when $A = X$. In this case $\Pi(X, X)$ is often written simply $\Pi(X)$, and is called *the fundamental groupoid* of X.

CONSTRUCTION 6.13. If $f : X \to Y$ is a continuous map of spaces, and if $f(A) \subset B$, then there is an induced morphism of groupoids $f_* : \Pi(X, A) \to \Pi(Y, B)$, given by sending a point a to $f(a)$ and the class of a path $\gamma : I \to X$ to the class of $f \circ \gamma$. Proposition 5.6 assures us that f_* is well defined.

PROPOSITION 6.14. *There is a functor*

$$\mathbf{Top} \to \mathbf{Groupoids}$$

given by assigning to each space X its fundamental groupoid $\Pi(X)$ (i.e., $\Pi(X, X)$) and to each map of spaces $f : X \to Y$ the morphism of groupoids $f_* : \Pi(X) \to \Pi(Y)$.

PROOF. The statement of the proposition has specified what happens on objects and morphisms of **Top**. It remains to verify that the construction is actually a functor. That is, we must show that

(1) $(\mathrm{id}_X)_* = \mathrm{id}_{\Pi(X)}$, which is elementary, and
(2) If $f : X \to Y$ and $g : Y \to Z$ are composable continuous functions, then $(g \circ f)_* = g_* \circ f_*$.This is not difficult, but here are the details:
    For $x \in \mathrm{ob}\,\Pi(X)$, we have $(g \circ f)_*(x) = g(f(x)) = g_* \circ f_*(x)$. For $[\gamma] \in \mathrm{mor}\,\Pi(X)$, we may choose a representative path $\gamma : [0,1] \to X$. Then $(g \circ f)_*[\gamma]$ is the class of the path $g \circ f \circ \gamma : [0,1] \to Z$. For a similar reason, $f \circ \gamma : [0,1] \to Z$ is a representative of the class $f_*([\gamma])$, so that $g_*(f_*([\gamma]))$ is the class of $g \circ f \circ \gamma$ as well. This completes the proof.

□

2. The fundamental group

To avoid getting bogged down in category theory, we often restrict ourselves to the special case $A = \{x_0\}$.

DEFINITION 6.15. Let X be a space and $x_0 \in X$ a point. We define $\pi_1(X, x_0)$, the *fundamental group of X at the base point x_0*, to be the group $\Pi(X, \{x_0\})$. Here we identify a groupoid with one object, $\Pi(X, \{x_0\})$, with the group of morphisms in that groupoid.

REMARK 6.16. Here is an equivalent definition of $\pi_1(X, x_0)$ that does not mention the word "groupoid".

Consider the set S of loops in X that start and end at x_0: specifically, elements of S are maps $\gamma : [0,1] \to X$ such that $\gamma(0) = \gamma(1)$. There is an equivalence relation on S given by homotopy equivalence relative to $\{0,1\}$ (see Proposition 5.5), and $\pi_1(X, x_0)$ is the set of equivalence classes of S under this relation.

All the work that went into showing that $\Pi(X, A)$ is a groupoid now specializes to tell us $\pi_1(X, x_0)$ is a group. In brief: Construction 6.1 allows us to compose elements in S and Proposition 6.4 tells us that this endows $\pi_1(X, x_0)$ with a composition operation. Proposition 6.6 tells us that the composition in $\pi_1(X, x_0)$ is associative. There exists an identity element $[e_{x_0}] \in \pi_1(X, x_0)$, by virtue of Definition 6.7 and Proposition 6.8. There exist inverses in $\pi_1(X, x_0)$ given by reversing loops, just as in Notation 6.9 and Proposition 6.10.

CONSTRUCTION 6.17. Suppose $f : X \to Y$ is a map of spaces and $x_0 \in X$ is a point. There is an induced homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ given by $f_*([\gamma]) = [f \circ \gamma]$. This is a special case of Construction 6.13.

REMARK 6.18. The formation of $\pi_1(X, x_0)$ from the pointed space (X, x_0) is functorial. Specifically, there is a functor:

$$\pi_1 : \mathbf{Top}_\bullet \to \mathbf{Grp}$$

from the category of pointed spaces to the category of groups. On objects, $\pi_1(X, x_0)$ is the fundamental group. On morphisms $f : (X, x_0) \to (Y, y_0)$, we set $\pi_1(f) = f_*$ as in Construction 6.17. The verification that this is actually functorial, i.e., that it respects composition and preserves identities, is left to the reader.

The following are useful results.

PROPOSITION 6.19. *Let X be a topological space, let x_0 and x_1 be points in X and let α be a path from x_0 to x_1. There is an isomorphism $\phi_\alpha : \pi_1(X, x_0) \to \pi_1(X, x_1)$ given by $\gamma \mapsto [\alpha]^{-1}[\gamma][\alpha]$.*

PROOF. We consider $\pi_1(X, x_0)$ as a subgroupoid of $\Pi(X)$. The result we want is an instance of Proposition B.38. □

PROPOSITION 6.20. *Let X and Y be two spaces and let $f, g : X \to Y$ be two maps and let $H : X \times I \to Y$ be a homotopy between them. Let $x_0 \in X$. Let α be the path $t \mapsto H(x_0, t)$, from $f(x_0)$ to $g(x_0)$, and let ϕ_α be as above. Then the diagram*

$$\begin{tikzcd}
 & \pi_1(Y, f(x_0)) \arrow[dd, "\phi_\alpha", "\cong"'] \\
\pi_1(X, x_0) \arrow[ur, "f_*"] \arrow[dr, "g_*"'] & \\
 & \pi_1(Y, g(x_0))
\end{tikzcd}$$

commutes.

PROOF. Let γ be a loop in X based at x_0. There are two ways of producing a loop in Y, based at $g(x_0)$. First, one can produce $(\alpha^{\leftarrow} \cdot (f \circ \gamma)) \cdot \alpha$. Second, one can produce $g \circ \gamma$. The homotopy class (relative to $\{0, 1\}$) of the first is $\phi_\alpha(f_*[\gamma])$ and the class of the second is $g_*([\gamma])$. What we want to prove is that these two loops are homotopic (relative to $\{0, 1\}$).

Here is an explicit homotopy between them (relative to $\{0, 1\}$):

$$E(t, s) = \begin{cases} \alpha(1 - 4t) & 4t \leq 1 - s \\ H(\gamma((4t - 1 + s)/(1 + 3s)), s) & 1 - s \leq 4t \leq 2 + 2s \\ \alpha(2t - 1) & 1 + s \leq 2t \end{cases}.$$

To make sense of this, it helps to recall that $\alpha(s) = H(x_0, s)$ and $\gamma(0) = \gamma(1) = x_0$. □

COROLLARY 6.21. *Suppose $f : X \to Y$ is a homotopy equivalence and $x_0 \in X$. Then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.*

PROOF. Let $g : Y \to X$ be a homotopy inverse for f. The maps $g \circ f : X \to X$ and $\mathrm{id}_X : X \to X$ are homotopic. Note that we do not require $g(f(x_0)) = x_0$.

Using Proposition 6.20, we deduce that there is an isomorphism

$$\phi_\alpha : \pi_1(X, (g \circ f)(x_0)) \to \pi_1(X, x_0)$$

so that there is an equality of homomorphisms of group $\phi_\alpha \circ (g \circ f)_* = (\mathrm{id}_X)_*$. Unwinding a bit, this says that $\phi_\alpha \circ g_* \circ f_* = \mathrm{id}_{\pi_1(X,x_0)}$. Since ϕ_α is an isomorphism of groups, we deduce that $g_* \circ f_*$ is an isomorphism of groups so f_* has a right inverse: $(g_* \circ f_*)^{-1} \circ g_*$.

A symmetric argument says that $f_* \circ g_*$ is an isomorphism of groups as well, whereupon $g_* \circ (f_* \circ g_*)^{-1}$ is a left inverse for f_*.

Any morphism with both a left-inverse and a right-inverse is an isomorphism, so we conclude. □

DEFINITION 6.22. A space X is *simply connected* if X is nonempty, path connected and $\pi_1(X, x_0) = \{e_{x_0}\}$ for some basepoint $x_0 \in X$. Since X is path connected, Proposition 6.19 assures us that the fundamental groups of X at different basepoints are all isomorphic to each other, so if one is trivial, they all are.

Simple connectivity also admits a description in terms of fundamental groupoids.

PROPOSITION 6.23. *Suppose X is a nonempty topological space. Then the following are equivalent:*

- *The space X is simply connected;*
- *For all $x, y \in X$, there exists a unique morphism in $\Pi(X)$ from x to y.*

PROOF. Suppose X is simply connected and $x, y \in X$ are points. Since X is path connected, there is some path from x to y, so that $\mathrm{Mor}_{\Pi(X)}(x, y)$ is not empty.

Since $\Pi(X)$ is a groupoid, the set of morphisms $\mathrm{Mor}_{\Pi(X)}(x, y)$ has a free transitive action by $\mathrm{Mor}_{\Pi(X)}(x, x) = \pi_1(X, x)$ by composition. Since $\pi_1(X, x)$ is trivial, the transitivity of the action implies that $\mathrm{Mor}_{\Pi(X)}(x, y)$ is a singleton.

Conversely, suppose that Property 6.23 holds. For any two $x, y \in X$, we can find a path from x to y, so X is path connected. Furthermore, $\pi_1(X, x)$ consists of a single element, so must be the trivial group. This implies that X is simply connected. □

REMARK 6.24. Any contractible space is simply connected: for such a space X, we know that $\pi_0(X) = \pi_0(\mathrm{pt}) = [\mathrm{pt}]$ and $\pi_1(X, x_0) \cong \pi_1(\mathrm{pt}, \mathrm{pt}) = [e_{\mathrm{pt}}]$.

REMARK 6.25. We saw in Remark 6.18 that the fundamental group is functorial when viewed as a construction on pointed spaces. If we restrict Proposition 6.20 to basepoint-preserving maps $f : (X, x_0) \to (Y, y_0)$ and basepoint-preserving homotopies between them, then we deduce that π_1 can be viewed as a functor $\pi : \mathbf{H}_\bullet \to \mathbf{Grp}$, where the source category here is the "pointed homotopy category" whose objects are pointed spaces and where the morphisms are basepoint-preserving-homotopy-classes of basepoint-preserving maps. While this way of putting things may please people of a categorical turn of mind, it does not do justice to Proposition 6.20, which is a statement about all homotopies of maps, not only the basepoint preserving homotopies.

We cannot, however, say that π_1 is a functor on $\mathbf{H}$, because it is possible for $f, g : X \to Y$ to be homotopic as maps, so $[f] = [g]$ in $\mathbf{H}$, but for $f(x_0) \neq g(x_0)$, so the induced morphisms

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0)), \quad g_* : \pi_1(X, x_0) \to \pi_1(Y, g(y_0))$$

are not the same homomorphisms, having different codomains.

It is evident that π_1 has some sort of property rather like 'functoriality on $\mathbf{H}$', but articulating exactly what seems to require dealing with multiple basepoints at once: that is, the functoriality that we should consider is actually that of Π, not π_1. Doing so would require the development of more category theory, however.

PROPOSITION 6.26. *Let $\{(X_j, x_j)\}_{j\in J}$ be a family of based topological spaces. Let (X, x) denote the product, based at the point which projects onto x_j for all j. Then there is an isomorphism $\pi_1(X, x) \to \prod_{i\in J} \pi_1(X_j, x_j)$ given by $(\mathrm{proj}_j)_*$ in the j-th component.*

PROOF. In fact, the universal property of products says that a family of maps $\{g_i : [0,1] \to X_j\}_{j\in J}$ is equivalent to a single pointed map $g = \prod_{i\in J} g_i : [0,1] \to X$. It is routine to verify that if g_j is actually a loop based at x_j, i.e., $g_j(0) = g_j(1) = x_j$, for all j, then $g : [0,1] \to X$ is a loop based at x.

Similarly, a family of homotopies $\{H_i : [0,1] \times [0,1] \to X\}_{j\in J}$ (each of which is relative to $\{0,1\}$) is equivalent to a single homotopy $\prod_{j\in J} H_j : [0,1] \times [0,1] \to X$, (relative to $\{0,1\}$). Using these two observations, it is routine to show that the homomorphism in the proposition is a bijection. $\square$

3. Functoriality for the Fundamental Groupoid

Homotopy gives us a notion of equivalence for maps between spaces. There is also a notion of equivalence of morphisms between groupoids, given by natural transformations of functors.

DEFINITION 6.27. Suppose $F_0, F_1 : G \to H$ are morphisms of groupoids. An *equivalence* $\nu : F_0 \Rightarrow F_1$ consists of a set of morphisms in H, indexed by the objects of G,

$$\nu = \{\nu_g : F_0(g) \to F_1(g)\}_{g\in \mathrm{ob}\, G}$$

such that for all morphisms $\gamma : g \to g' \in G$, the diagram

$$\begin{tikzcd} F_0(g) \arrow[r, "\nu_g"] \arrow[d, "F_0(\gamma)"] & F_1(g) \arrow[d, "F_1(\gamma)"] \\ F_0(g') \arrow[r, "\nu_{g'}"] & F_1(g') \end{tikzcd}$$

commutes.

NOTATION 6.28. If there exists $\nu : F_0 \to F_1$, we say F_0 is equivalent to F_1. It is routine to verify that equivalence is reflexive, symmetric and transitive.

EXAMPLE 6.29. When G, H are groups, F_0 and F_1 are group homomorphisms, and $\mathrm{ob}\, G$ consists of only one object. In this case, an equivalence consists of a single element $\nu \in H$ such that $\nu F_1 = F_0 \nu$. Two group homomorphisms are equivalent if and only if they differ by conjugation in the target group.

There is a functor

$$\Pi : \mathbf{Top} \to \mathbf{Groupoid}.$$

PROPOSITION 6.30. *If $f, g : X \to Y$ are homotopic maps of spaces, then f_*, g_* are equivalent homomorphisms of groupoids.*

CHAPTER 7

The Van Kampen theorem

1. Van Kampen for Groupoids

The basic problem is as follows. Suppose $X = U \cup V$ where U and V are open sets such that $U \cap V$, U and V are connected. Let $x_0 \in U \cap V$ be a point. Can we determine $\pi_1(X, x_0)$ from $\pi_1(U, x_0)$, $\pi_1(V, x_0)$ and $\pi_1(U \cap V, x_0)$?

The answer is yes, and in fact, we can do better. We can do the calculation for fundamental groupoids, of which the fundamental groups are a special case.

PROPOSITION 7.1. *Let X be a topological space and let U, V be open subspaces with $X = U \cup V$. In the diagram below, the morphisms that are not labelled are all induced by inclusions of spaces. Let $\mathcal{G}$ be a groupoid and suppose that in the diagram below, the outer square commutes. Then there exists a unique map of groupoids indicated by f making the whole diagram commute.*

$$\begin{tikzcd}
\Pi(U \cap V) \arrow[r] \arrow[d] & \Pi(U) \arrow[d] \arrow[ddr, bend left, "g"] & \\
\Pi(V) \arrow[r] \arrow[drr, bend right, "h"'] & \Pi(X) \arrow[dr, dashed, "f"] & \\
 & & \mathcal{G}
\end{tikzcd}$$

PROOF. A story about groupoids is a story in two parts. The first part is about the objects. The set of objects of $\Pi(X)$ is the underlying set of X, and similarly for the other fundamental groupoids.

By hypothesis, every point $x \in X$ lies in at least one of U or V. Define $f(x) = g(x)$ if $x \in U$ and $f(x) = h(x)$ if $x \in V$. Since $g|_{U \cap V} = h|_{U \cap V}$, the function $f : \operatorname{ob}\Pi(X) \to \operatorname{ob}\mathcal{G}$ is well defined even for points in $U \cap V$, where we apparently had a choice. A moments thought tells us that this definition of f is the only one that will make the diagram below commute:

$$\begin{tikzcd}
\operatorname{ob}\Pi(U \cap V) \arrow[r] \arrow[d] & \operatorname{ob}\Pi(U) \arrow[d] \arrow[ddr, bend left, "g"] & \\
\operatorname{ob}\Pi(V) \arrow[r] \arrow[drr, bend right, "h"'] & \operatorname{ob}\Pi(X) \arrow[dr, dashed, "f"] & \\
 & & \operatorname{ob}\mathcal{G}.
\end{tikzcd}$$

Now we have to worry about morphisms, i.e., homotopy classes of paths in the fundemantal groupoids. Suppose $\gamma : I \to X$ is a path in X, from $a = \gamma(0)$ to $b = \gamma(1)$. We can find some decomposition of $[0,1]$ into closed subintervals $[t_0 = 0, t_1]$, $[t_1, t_2]$, ..., $[t_{r-1}, t_r = 1]$ such that for any i, the path $\gamma([t_i, t_{i+1}]))$ lies either in U or in V. Let γ_i denote a reparametrization of $\gamma|_{[t_i, t_{i-1}]}$. This allows us to define a candidate $f([\gamma])$ as the composite of the images in $\mathscr{G}$ of $\gamma_0, \gamma_1, \ldots, \gamma_{r-1}$. We have not shown that $f([\gamma])$ is well defined, but observe that if a map $f : \Pi(X) \to \mathscr{G}$ of groupoids exists making the diagram commute, then it must be this one, since we have factored $[\gamma] = [\gamma_0][\gamma_1]\ldots[\gamma_{r-1}]$, and what f does to $[\gamma_0]$, $[\gamma_1]$, ..., $[\gamma_{r-1}]$ is forced on us by h and g.

Now let us prove that $f([\gamma])$ is an invariant of the homotopy class (relative to $\{0,1\}$) of $\gamma : I \to X$. That is, if we choose a possibly different path $\gamma' : I \to X$ such that $[\gamma] = [\gamma']$ and a decomposition of γ', we obtain the same definition of $f([\gamma])$.

Suppose we have two paths γ and γ', a homotopy $H : I \times I \to X$ relative to $\{0,1\}$ and two decompositions of I as above. Using the Lebesgue covering lemma, we can find a tessellation of $I \times I$ into small rectangles R_{ij} with sides parallel to the sides of $I \times I$ and with disjoint interiors so that for each such rectangle $H|_{R_{ij}}$ lies either in U or in V, and so that the restrictions of the tessellation to $I \times \{0\}$ and $I \times \{1\}$ refine the two decompositions of I. Now consider $H|_{R_{ij}}$ in each R_{ij}. They each give a relation in the fundamental groupoid either of U or of V: namely $[H|_{\text{bottom}}] + [H|_{\text{right}}] = [H|_{\text{left}}] + [H|_{\text{top}}]$ where $H|_{\text{bottom}}$ denotes the restriction of H to the bottom edge of the rectangle, and similarly for the other four restrictions. Applying either h or g, as required, each R_{ij} gives us a relation in the groupoid $\mathscr{G}$. Integrating these relations together over the whole square shows that $f([\gamma]) = f([\gamma'])$.

The proof that f is really a map of groupoids is not difficult. One simply has to verify that it preserves composition. The statement that it preserves composition follows by taking a composite $[\gamma][\delta]$ and decomposing each into short paths lying either in U or V: $[\gamma_0]\ldots[\gamma_{r-1}][\delta_0]\ldots[\delta_{s-1}]$. By construction $f([\gamma][\delta])$ is the product in $\mathscr{G}$ of $f([\gamma_0])\ldots f([\delta_{s-1}]) = f([\gamma])f([\delta])$. □

COROLLARY 7.2. *Let X and U, V be as above. Let $A \subset U \cap V$ be a set of points such that each of path component of U, V and $U \cap V$ contains at least one point of A. In the diagram below, morphisms that are not labelled are induced by inclusions of spaces. Let $\mathscr{G}$ be a groupoid, and again suppose the outer diagram commutes:*

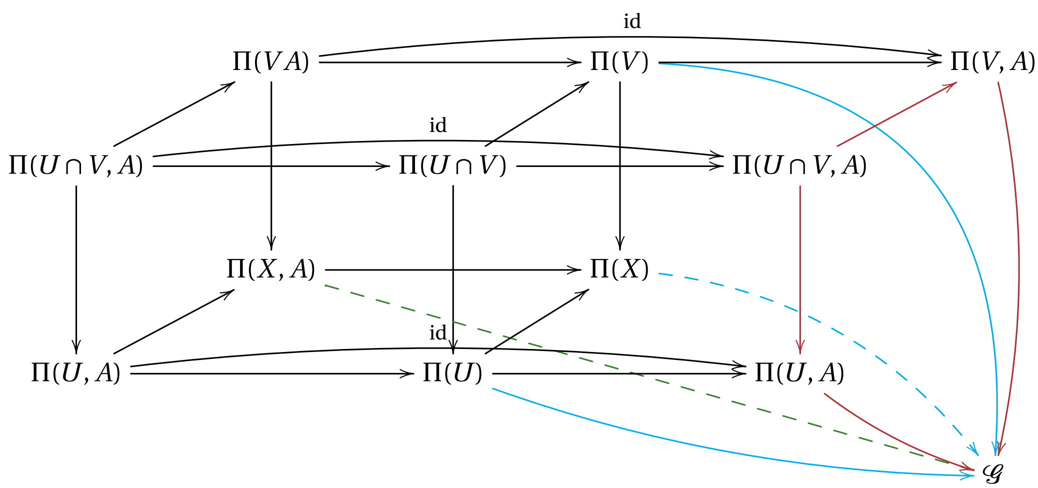
$$\begin{tikzcd}
\Pi(U \cap V, A) \arrow[r] \arrow[d] & \Pi(U, A) \arrow[d] \arrow[ddr, bend left, "g"] & \\
\Pi(V, A) \arrow[r] \arrow[drr, bend right, "h"'] & \Pi(X, A) \arrow[dr, dashed, "f"] & \\
& & \mathscr{G}
\end{tikzcd}$$

Then there exists a unique map of groupoids making the diagram commute.

PROOF. For each $x \in (U \cap V) \setminus A$, choose a fixed isomorphism in $\Pi(U \cap V, A)$ from x to some point $a \in A$, $\gamma : x \to a$. By means of this isomorphism, we construct a map of groupoids $\Pi(U \cap$

$V) \to \Pi(U \cap V, A)$. We can do the same for U and V and X, and where applicable choose the same isomorphisms.

The rest of the argument follows by showing that the diagram in the corollary is a retract of the diagram in the proposition. This is a diagram chase that is best done live. We give the diagram here and advise the reader to do the chase.



Suppose the diagram with red arrows is given, then one can construct the solid cyan arrows, then using Proposition 7.1, one constructs the dashed cyan arrow. By composing, one obtains the dashed green arrow. One then verifies that this is the map f asked for in the corollary. □

REMARK 7.3. The corollary determines $\Pi(X, A)$ up to unique isomorphism. It is the groupoid one obtains generated by the maps in $\Pi(U, A)$ and in $\Pi(V, A)$ subject to the relations $\Pi(U \cap V, A)$.

EXAMPLE 7.4. Cover $S^1 \subset \mathbf{C}$ by two open sets, $S^1 \setminus \{(0,\pm 1)\}$. Let A be the set of points $\{(\pm 1, 0)\}$. Since we will refer to these often, write $p = (1,0)$ and $q = (-1,0)$. While $U = S^1 \setminus \{(0,1)\}$ and $V = S^1 \setminus \{(0,-1)\}$ are both contractible, $U \cap V = S^1 \setminus \{(0,1),(0,-1)\}$ is a disjoint union of two contractible sets. In the case of contractible spaces, we understand fundamental groupoids by virtue of Remark **??**

The fundamental groupoids are

$$\Pi(U,A) = \{\phi : q \leftrightarrow p : \phi^{-1}\} \quad \Pi(V,A) = \{\psi : q \leftrightarrow p : \psi^{-1}\} \quad \Pi(U \cap V, A) = \{q \quad p\}$$

The remark following the corollary suggests that the fundamental groupoid of S^1 on the points $\{p, q\}$ has two objects and is generated by two different morphisms $q \to p$, with no relations between them other than what is forced by the groupoid axioms.

To make this precise, let $\mathscr{H}$ denote the groupoid whose objects are p and q, and where there are two different morphisms $\phi, \psi : q \to p$ and no relations between them, i.e., the set of morphisms $q \to p$ consists of all strings of the forms

- $\phi\psi^{-1}\phi \cdots \psi^{-1}\phi$;
- $\psi\phi^{-1}\psi \cdots \phi^{-1}\psi$.

One can give similar enumerations of the sets of morphisms $p \to p$, $q \to p$ and $q \to q$ in this groupoid based on this description.

We can define a functor $\Pi(U, A) \to \mathscr{H}$ by the identity on objects and by $\phi \mapsto \phi$ on morphisms Similarly, we can define a functor $\Pi(V, A) \to \mathscr{H}$ by the identity on objects and by $\psi \mapsto \psi$ on morphisms. The diagram

$$\begin{tikzcd}
\Pi(U \cap V, A) \arrow[r] \arrow[d] & \Pi(U, A) \arrow[d] \\
\Pi(V, A) \arrow[r] & \mathscr{H}
\end{tikzcd}$$

commutes—there is virtually nothing to check here.

Next, we verify that the groupoid $\mathscr{H}$ has the universal property we would like. The diagram encapsulating this property is given:

$$\begin{tikzcd}
\Pi(U \cap V, A) \arrow[r] \arrow[d] & \Pi(U, A) \arrow[d] \arrow[ddr, bend left] & \\
\Pi(V, A) \arrow[r] \arrow[drr, bend right] & \mathscr{H} \arrow[dr, dashed] & \\
& & \mathscr{G}.
\end{tikzcd}$$

Again, there is very little to check: we know where p, q must go to in $\operatorname{ob}\mathscr{G}$, and we know where ϕ, ψ must go to in $\operatorname{Mor}\mathscr{G}$. The fact that no relations were imposed between ϕ, ψ in $\mathscr{H}$ means that nothing can go wrong, and the dashed arrow really does exist.

Therefore $\mathscr{H}$ has the required universal property, and for the usual universal property reasons, it is the fundamental groupoid $\Pi(S^1, A)$, at least up to unique isomorphism. Based on our specific construction, where $\operatorname{ob}\mathscr{H} \subset S^1$ and where ϕ, ψ really are classes of paths in S^1, the groupoid $\mathscr{H}$ actually is $\Pi(S^1, A)$, rather than being 'merely' uniquely isomorphic to it.

We can calculate the fundamental group $\pi_1(S^1, p)$ by simply remembering all the maps $p \to p$ in $\Pi(S^1, p)$. This group is infinite cyclic, generated by $\psi\phi^{-1}$. That is to say, it is generated by a loop that goes around the circle once counterclockwise.

We have proved:

PROPOSITION 7.5. *Let $p = (1, 0)$. Then $\pi_1(S^1, p)$ is an infinite cyclic group generated by the class of the loop $\gamma : [0, 1] \to S^1$ given by $\gamma(t) = (\cos 2\pi t, \sin 2\pi t)$.*

CONSTRUCTION 7.6. Let G and H be two groups. A *word* in G and H is a string of elements $s_1 \dots s_n$, each one either in G or H. They are subject to reduction, i.e., removing an identity element or replacing a pair $g_1 g_2$ by its product in G, or similarly in H. A *reduced word* is a word that cannot be reduced further.

The *free product* $G * H$ is the group of reduced words, with concatenation-followed-by-reduction as an operation.

There are obvious homomorphisms $G \to G * H$ and $H \to G * H$.

REMARK 7.7. Strictly speaking, the definition above presupposes that G and H are disjoint groups. If not, then we replace G and H by disjoint isomorphic groups before forming the free product.

DEFINITION 7.8. Let G, H and K be three groups and $\phi : K \to G$, $\psi : K \to H$ be homomorphisms. The notation $G *_K H$ denotes the *pushout* of G and H over K. This is the quotient of $G * H$ by the normal subgroup generated by elements $\phi(k)\psi(k)^{-1}$ as k ranges over the elements in K. This is sometimes called the *amalgamated product* of G and H over K, although some people prefer to reserve the term "amalgamated product" for when ϕ, ψ are both inclusions.

REMARK 7.9. Strictly, the homomorphisms ϕ and ψ should appear in the notation, but they do not.

As in the free case, there are homomorphisms $\iota_G : G \to G *_K H$ and $\iota_H : H \to G *_K H$. In this case, these homomorphisms make the diagram

$$\begin{tikzcd} K \arrow[r, "\psi"] \arrow[d, "\phi"'] & H \arrow[d] \\ G \arrow[r] & G *_K H \end{tikzcd}$$

commute. If $G *_K H$ were replaced by $G * H$, then this diagram would not generally commute. In what follows below, we will omit the homomorphisms ι_G and ι_H from the notation: if an element $g \in G$ is to be considered as an element of $G * H$, it is understood that $\iota_G(g)$ is meant.

LEMMA 7.10. *Suppose there is a groupoid M and there are homomorphisms $\alpha : G \to M$ and $\beta : H \to M$ making the diagram below commute (without the dashed arrow)*

(1)
$$\begin{tikzcd} K \arrow[r, "\phi"] \arrow[d, "\psi"'] & G \arrow[d, "\iota_G"] \arrow[ddr, bend left, "\alpha"] & \\ H \arrow[r, "\iota_H"] \arrow[drr, bend right, "\beta"'] & G *_K H \arrow[dr, dashed, "\delta"] & \\ & & M. \end{tikzcd}$$

Then there exists a unique homomorphism δ making the diagram commute.

PROOF. It is actually sufficient to consider the case where M is a group, since only one object of the groupoid M is ever considered. So suppose M is a group.

Consider the case where ϕ, ψ are trivial, so that $G *_K H = G * H$. Then each element of $G * H$ can be written as a word $w = g_1 h_1 \ldots g_r h_r$ (it is possible that g_1 or h_r is the identity element). In this case, we define $\delta'(w) = \alpha(g_1)\beta(h_1)\ldots\alpha(g_r)\beta(h_r)$. It is routine to verify that this is a group homomorphism. Once that is verified, it is immediate that δ' is the unique homomorphism such that $G \to G * H \xrightarrow{\delta'} M$ agrees with α and $G \to G * H \xrightarrow{\delta'} M$ agrees with β.

In the general case, when ϕ and ψ are not trivial, we construct δ as a quotient of δ'. Let N denote the normal subgroup of $G * H$ generated by words of the form $\phi(k)\psi(k)^{-1}$. Observe that

$\delta'(\phi(k)\psi(k)^{-1}) = e \in M$ for all $k \in K$, so that $N \subset \ker\delta'$. Therefore there is an induced homomorphism: $\delta : G *_K H \to M$.

Verifying that this makes the diagram of (1) commute and is unique with this property is routine and is left as an exercise. □

COROLLARY 7.11 (The Van Kampen theorem). *Let X be a topological space with basepoint x_0, and U, V path connected open subsets that cover X and such that $U \cap V$ is path connected and contains x_0. Then $\pi_1(X, x_0)$ can be identified with $\pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0)$, where the morphisms $\pi_1(U\cap V, x_0) \to \pi_1(U, x_0)$ and $\pi_1(U\cap V, x_0) \to \pi_1(V, x_0)$ are those induced by the inclusions of subspaces.*

PROOF. Under the hypothesis that $U, V, U\cap V$ are all path connected and $x_0 \in U \cap V$, we can apply Corollary 7.2 with $A = \{x_0\}$, i.e, the Van Kampen theorem for groupoids. That tells us that $\pi_0(X, x_0)$ has a universal property that is also possessed by $\pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0)$, according to Lemma 7.11. A standard argument now says that $\pi_1(X, x_0)$ is isomorphic to $\pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0)$, and the isomorphism is uniquely specified if we require it to be compatible with the already-given homomorphisms from $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ to each of $\pi_1(X, x_0)$ and $\pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0)$. This is the result. □

2. Examples

EXAMPLE 7.12. Suppose X is a space that can be written as the union of two open sets U and V where U and V are both simply connected and where $U \cap V$ is path connected. Let $x_0 \in U \cap V$ be a basepoint. Then by use of the Van Kampen theorem:

$$\pi_1(X, x_0) \cong \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) = \{e\} *_{\pi_1(U\cap V, x_0)} \{e\} \cong \{e\}.$$

This implies that $\pi_1(X, x_0) = \{e\}$.

EXAMPLE 7.13. A special case of the previous example is the case of $\pi_1(S^n, s_0)$ whenever $n \geq 2$. To see that this case applies, consider

$$s_0 = (1, 0, 0, \dots, 0), \qquad s_0' = (-1, 0, 0, \dots, 0)$$

both of which are points on S^n. We know that $S^n \setminus \{s_0\} \approx \mathbf{R}^n$, since S^n is a one-point compactification of $\mathbf{R}^n$. Similarly, $S^n \setminus \{s_0'\} \approx \mathbf{R}^n$.

For all n, we deduce $S^n \setminus \{s_0, s_0'\} \approx \mathbf{R}^n \setminus \{0\}$, and if $n \geq 2$, then this is path connected (it is disconnected for $n = 1$). We may then apply Example 7.12 to deduce that $\pi_1(S^n, s_0') \cong \{e\}$ when $n \geq 2$.

We will not prove it in this course, but the spaces $S^2, S^3, \dots$ give examples of spaces that are simply connected but not contractible.

EXAMPLE 7.14. Suppose (X, x_0) and (Y, y_0) are two based spaces, and suppose $x_0 \in X$ has an open neighbourhood $U \ni x_0$ such that $\{x_0\}$ is a (strong) deformation retract of U—i.e., a deformation retract in which the homotopy is relative to $\{x_0\}$—, and similarly y_0 has an open neighbourhood V such that $\{y_0\}$ is a strong deformation retract of V. A schematic picture is given in Figure 1.

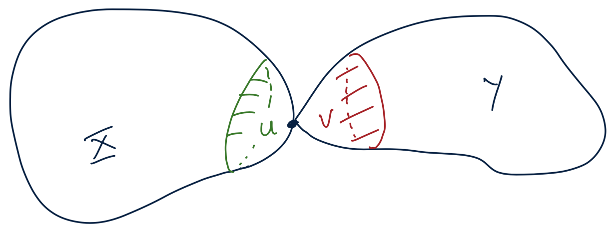


FIGURE 1. The open sets U and V in $X \vee Y$.

Form the space $X \vee Y$, and denote the basepoint by z. Consider the two open subspaces $X \vee V$ and $U \vee Y$. We observe that $X \subset X \vee V$ is a deformation retract and $Y \subset U \vee Y$ is a deformation retract as well. Moreover,

$$U \vee V = (X \vee V) \cap (U \vee Y)$$

is contractible (it has the basepoint as a deformation retract).

We can apply the Van Kampen theorem to deduce $\pi_1(X \vee Y, z) \cong \pi_1(X, x_0) * \pi_1(Y, y_0)$.

REMARK 7.15. The conditions that Example 7.14 requires of the based spaces (X, x_0) and (Y, y_0) are frequently satisfied in practice: for instance, if X and Y are topological manifolds, so that every point of X and Y has an open neighbourhood homeomorphic to $\mathbf{R}^n$, then these conditions are satisfied.

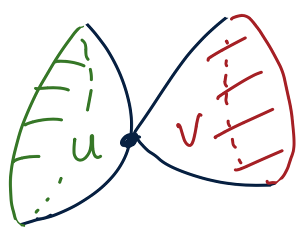


FIGURE 2. The set $U \vee V$ admits the basepoint as a deformation retract

We also remark that under the same hypotheses as Example 7.14, the wedge sum $X \vee Y$ has an open neighbourhood $U \vee V$ that admits the basepoint z as a deformation retract. Therefore, the idea in Example 7.14 can be iterated.

EXAMPLE 7.16. As a special case of Example 7.14, we can calculate π_1 of any iterated wedge sum of circles:

$$X = \overbrace{S^1 \vee S^1 \vee \cdots \vee S^1}^{k\text{-copies}}.$$

Write F_n for the free group generated by n symbols: $x_1, x_2, \ldots, x_n$ (and no relations, so that in particular, for $n \geq 2$, the group F_n is not abelian).

By a direct induction argument

$$\pi_1(X,z) = \pi_1(\overbrace{S^1 \vee \cdots \vee S^1}^{k-1\text{-copies}}, z) * \pi_1(S^1, s_0) \cong F_{k-1} * F_1 \cong F_k.$$

EXAMPLE 7.17. Let us show explicitly a presentation of $\pi_1(S^1 \vee S^1, z)$. In order to make the presentation more obvious, we replace $S^1 \vee S^1$ with a slightly larger space X, that is homotopy equivalent to it (there is an obvious deformation retract).

Two generators for $\pi_1(X,z)$ are illustrated in Figure: α and β.

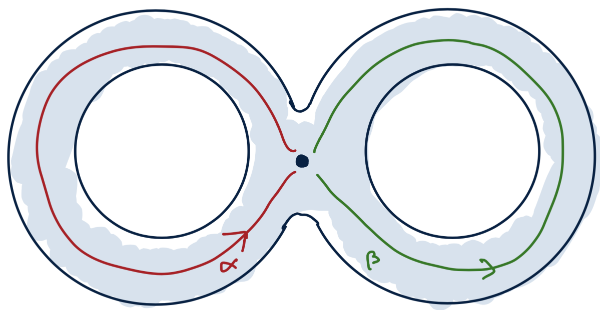


FIGURE 3. A space that is homotopy equivalent to $S^1 \vee S^1$, and chosen generators for $\pi_1(X,z)$.

The fundamental group in this case is not abelian, and you can verify that a loop in the class of $\alpha\beta^{-1}$ (first go around the loop on the left, then on the right, illustrated in Figure 4) is not

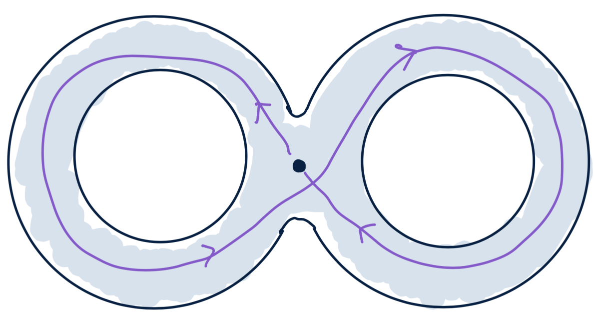


FIGURE 4. A loop in the class of $\alpha\beta^{-1}$.

homotopic (relative to endpoints) to a loop in the class of $\alpha\beta^{-1}$ (first go around the loop on the right, then on the left, illustrated in Figure 5)

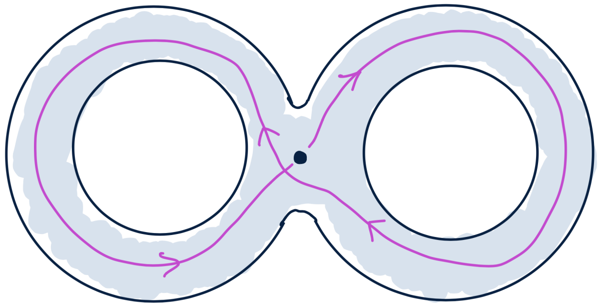


FIGURE 5. A loop in the class of $\beta^{-1}\alpha$.

EXAMPLE 7.18. Here we consider the complement of two points, p, q in $\mathbf{R}^2$. Pick any basepoint $x_0 \in \mathbf{R}^2 \setminus \{p, q\}$. The complement $\mathbf{R}^2 \setminus \{p, q\}$ is homotopy equivalent to $S^1 \vee S^1$, and so $\pi_1(\mathbf{R}^2 \setminus \{p, q\}, x_0)$ is a free group on two generators. One generator, α, may be taken to be a loop around p in the anticlockwise direction and the other generator may be taken to be β, a loop around q in the anticlockwise direction.

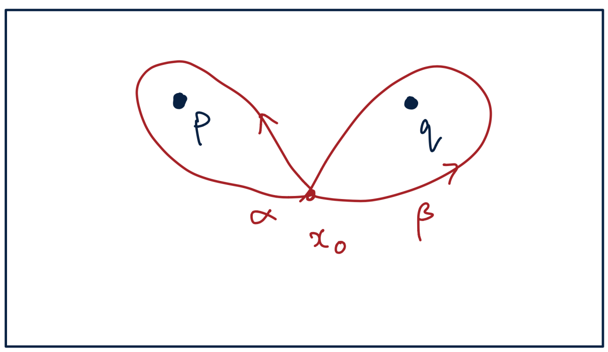


FIGURE 6. $\mathbf{R}^2 \setminus \{p, q\}$ and chosen generators for π_1.

We may consider an element in $\pi_1(\mathbf{R}^2 \setminus \{p, q\}, x_0)$ such as $w = \alpha^{-1}\beta\alpha\beta^{-1}$, but that becomes trivial if we apply any homomorphism $\pi_1(\mathbf{R}^2 \setminus \{p, q\}, x_0) \to G$ where either $\alpha \mapsto e$ or $\beta \mapsto e$. A loop in the class of $\alpha^{-1}\beta\alpha\beta^{-1}$ is illustrated in Figure 7. This picture (and ones like it) are often referred to as solutions to the “two nails problem". The idea is to view the plane, $\mathbf{R}^2$, as a wall on which one wishes to hang a picture. Then two nails are hammered into the wall, at p and q, then a piece of string is tied in a loop and wound around p and q as illustrated in Figure 7. A picture can be hung from this piece of string, and it will not fall to the floor, because the element $\alpha^{-1}\beta\alpha\beta^{-1}$ is not trivial, and therefore the string cannot be detached from the nails—if you could detach it from the nails, you would then be able to contract the loop to a point.

If one or other nail is removed, however, i.e., if either p or q is returned to the plane, then the loop will become contractible again. For instance, if p is filled in, so the loop is viewed as

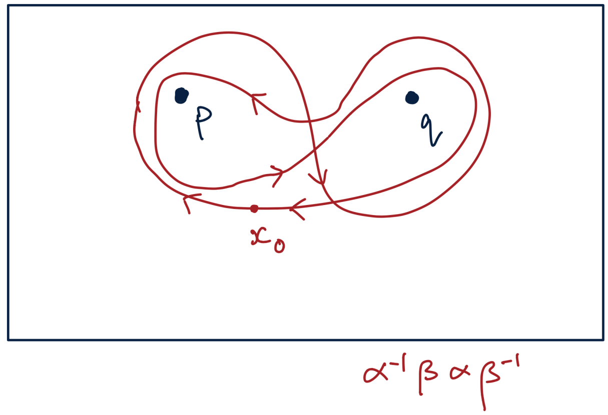


FIGURE 7. A loop in the class of $\alpha^{-1}\beta\alpha\beta^{-1}$.

representing an element in $\pi_1(\mathbf{R}^2 \setminus \{q\}, x_0) = \mathbf{Z}\beta$, by applying the homomorphism

$$\phi : \pi_1(\mathbf{R}^2 \setminus \{p, q\}, x_0) \to \pi_1(\mathbf{R}^2 \setminus \{q\}, x_0)$$

given by $\phi(\alpha) = e$ and $\phi(\beta) = \beta$. We observe that $\phi(\alpha^{-1}\beta\alpha\beta^{-1}) = \beta\beta^{-1} = e$. You can verify this yourself by visualizing a contraction of the loop to the basepoint, provided p is added back to $\mathbf{R}^2$. An intermediate stage of the contraction is illustrated in Figure 8.

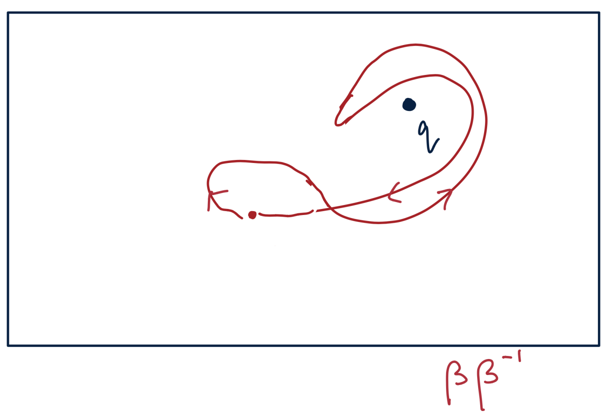


FIGURE 8. A loop representing the trivial element in $\pi_1(\mathbf{R}^2 \setminus \{q\}, x_0)$.

The effect of removing the nail at q is similar, again the class of the loop becomes trivial.

EXAMPLE 7.19. In this example, we calculate $\pi_1(S^1 \times S^1, x_0)$. From Proposition 6.26, we already know that $\pi_1(S^1 \times S^1, x) \cong \pi_1(S^1, s_0) \times \pi_1(S^1, s_0) \cong \mathbf{Z} \times \mathbf{Z}$, the free abelian group of rank 2. Here, we give another proof of this fact.

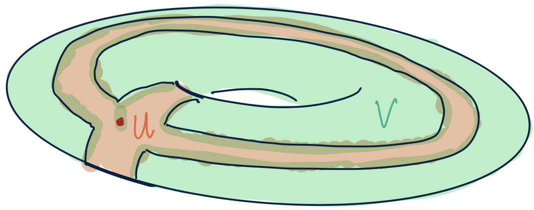


FIGURE 9. Two open subsets covering the torus.

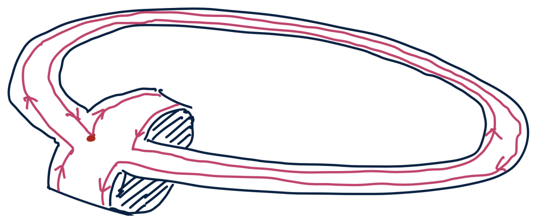


FIGURE 10. A loop γ in $U \cap V$.

Cover $S^1 \times S^1$ by two open sets U and V, as indicated in Figure 9.

The open set U is homotopy equivalent to $S^1 \vee S^1$ (it admits $S^1 \vee S^1$ as a deformation retract). The open set V is contractible, since it is homeomorphic to an open ball in $\mathbf{R}^2$. The intersection $U \cap V$ is homotopy equivalent to S^1. This gives us the following list of homotopy groups:

$$\pi_1(U, x_0) = \langle \alpha, \beta \rangle$$
$$\pi_1(V, x_0) = \{e\}$$
$$\pi_1(U \cap V, x_0) = \langle \gamma \rangle$$

We can now use the Van Kampen theorem to present

$$\pi_1(S^1 \times S^1, x_0) \cong \pi_1(U, x_0) *_{\pi_1(S^1, x_0)} \{e\}.$$

In order to finish the calculation, we must determine the homomorphisms $\phi : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ and $\psi : \pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$. The latter of these is trivial. We concentrate on the former. We see from Figure 10 that $\phi(\gamma) = \alpha\beta\alpha^{-1}\beta^{-1}$. Therefore the Van Kampen Theorem says that

$$\pi_1(S^1 \times S^1, x) \cong \langle \alpha, \beta | \alpha\beta\alpha^{-1}\beta^{-1} \rangle,$$

that is, the quotient of the free group by the normal subgroup generated by $\alpha\beta\alpha^{-1}\beta^{-1}$. This is a presentation of the free abelian group on two generators, which is what we expected.

CHAPTER 8

Covering spaces

1. Covering Spaces

DEFINITION 8.1. A map of topological spaces $f: Y \to X$ is a *covering space map* if, for all $x \in X$, there is some open $U \ni x$ such that the inverse image $f^{-1}(U)$ is homeomorphic to a disjoint union $\coprod_{j \in J} V_j$ such that each induced map $f|_{V_j}: V_j \to U$ is a homeomorphism.

REMARK 8.2. Some people might require the map f to be surjective, but we do not.

EXAMPLE 8.3. The prototypical examples are $f_n: S^1 \to S^1$ given by $z \mapsto z^n$, and $f_\infty: \mathbf{R} \to S^1$ given by $f_\infty(t) = (\cos 2\pi t, \sin 2\pi t)$.

EXAMPLE 8.4. Other, more trivial, examples, include $X \coprod X \to X$ or the inclusion of open component into a disconnected but locally connected space. These are sort of silly, so we generally concentrate in examples on the cases where both X and Y are connected.

DEFINITION 8.5. A map of topological spaces $f: Y \to X$ is an *étale map* if it is locally an open embedding: that is, if each $y \in Y$ has an open neighbourhood U_y such that $f|_{U_y}: U_y \to X$ is an open embedding.

EXAMPLE 8.6. Both covering space maps and open embeddings are étale.

EXAMPLE 8.7. In general, open embeddings are not covering space maps. For instance, the inclusion of an open set $U \hookrightarrow X$ that is not also closed is an étale map but not a covering space map (consider neighbourhoods of a point $x \in \partial U$).

PROPOSITION 8.8. *A covering space map is open.*

DEFINITION 8.9. Let $f: Y \to X$ be a map and let $x \in X$. Define the *fibre* of f at x to be $f^{-1}(x)$, and denote it F_x.

NOTATION 8.10. Let $f: Y \to X$ be a covering space and let $W \subset X$ be an open set. We say that f *trivializes* over W if $f^{-1}(W)$ is a disjoint union of open sets mapping homeomorphically to W.

Here comes a technical and very important proposition.

PROPOSITION 8.11. *Suppose $f: Y \to X$ is a covering map and that Z is a space. Write i_0 : $Z \times \{0\} \to Z \times [0,1]$ for the inclusion at* 0. *Suppose that that there are maps as indicated making* *the diagram (without the dashed arrow) commute:*

(2)
$$\begin{tikzcd} Z \times \{0\} \arrow[r, "\tilde{g}_0"] \arrow[d, "i_0"] & Y \arrow[d, "f"] \\ Z \times [0,1] \arrow[r, "g"] \arrow[ur, dashed, "\tilde{g}"] & X. \end{tikzcd}$$

Then there is a unique map $\tilde{g}$ making both triangles commute.

A continuous function such as $\tilde{g}$ making the lower triangle commute is called a *lift* of g.

LEMMA 8.12. *Assume the hypotheses of Proposition 8.11. Assume that Z has an open cover $\{U_m\}_{m\in M}$ with the following property: Whenever $V \subset U_m$ for some $m \in M$, there exists a unique lift in the diagram*

$$\begin{tikzcd} V \times \{0\} \arrow[r, "\tilde{g}_0|_{V\times\{0\}}"] \arrow[d, "i_0|_{V\times\{0\}}"] & Y \arrow[d, "f"] \\ V \times [0,1] \arrow[r, "g|_{V\times[0,1]]}"] \arrow[ur, dashed, "\tilde{g}|_{V\times[0,1]}"] & X. \end{tikzcd}$$

Then the conclusion of Proposition 8.11 holds.

PROOF. We must define a continuous lift $\tilde{g} : Z \times [0,1] \to Y$, and then show it is the unique lift.

To define $\tilde{g}$, we need only observe that it has already been defined on each set in an open cover $\{U_m \times [0,1]\}_{m\in M}$ of $Z \times [0,1]$, and if $z \in U_m \cap U_{m'}$, then the restrictions of $\tilde{g}|_{U_m\times[0,1]}$ and $\tilde{g}|_{U_{m'}\times[0,1]}$ to $\{z\} \times [0,1]$ agree by uniqueness of the lifts. Therefore our various definitions of $\tilde{g}|_{U_m\times[0,1]}$ glue to give a lift $\tilde{g} : Z \times [0,1] \to Y$.

To establish uniqueness of $\tilde{g}$, suppose for the sake of contradiction that there are two lifts, $\tilde{g}$ and $\tilde{g}'$ and that they do not agree. Then there is some $z \in Z$ for which $\tilde{g}|_{\{z\}\times[0,1]} \neq \tilde{g}'|_{\{z\}\times[0,1]}$, but since $z \in U_m$ for some m, this contradicts the hypotheses of the lemma. □

LEMMA 8.13. *Assume all the hypotheses of Proposition 8.11, and also suppose that $f : Y \to X$ is a trivial covering space, in that Y is homeomorphic to $\coprod_{j\in K} X_j$ where $f|_{X_j} X_j \to X$ is a homeomorphism. Then the conclusion of Proposition 8.11 holds.*

PROOF. Identify $Y = \coprod_{j\in J} X_j$, so that each $X_j \subset Y$ is an open subset. Write $Z_j = g|_0^{-1}(X_j)$. The Z_j form an open cover of Z, so by Lemma 8.12, it suffices to prove the lemma under the assumption that there exists some unique j for which $g|_0(Z) \subset X_j$.

To define $\tilde{g}$, we set $\tilde{g} = f|_{X_j}^{-1} \circ g$. To verify uniqueness, suppose for the sake of contradiction that there is some lift $\tilde{g}' : Z \times [0,1] \to Y$ different from $f|_{X_j}^{-1} \circ g$. We deduce that there must be some (z, t_1) such that $\tilde{g}'(z, t_1) \notin X_j$, or else

$$\tilde{g}'(z, t_1) = f|_{X_j}^{-1} \circ f \circ \tilde{g}'(z, t_1) = \tilde{g}(z, t_1).$$

There is some j' so that $\tilde{g}'(z, t_1) \in X_{j'}$. But now $\tilde{g}'(z, t)$ gives a path joining $g_0(z, 0) \in X_J$ and $\tilde{g}'(z, t_1) \in X_{j'}$, which is impossible. This establishes uniqueness. □

LEMMA 8.14. *Assume all the hypotheses of Proposition 8.11. Let $\{W_j\}_{j\in J}$ be an open cover of X trivializing f, and let $V \subset Z$ be a subset such that there exists a partition*

$$[t_0 = 0, t_1] \cup [t_1, t_2] \cup \cdots \cup [t_{r-1}, t_r = 1]$$

of $[0,1]$ such that for all $l \in \{1,\ldots,r\}$ there exists some j for which $g^{-1}(V \times [t_{l-1}, t_l]) \subset W_j$. Then the conclusion of Proposition 8.11 holds for all subsets $S \subset V$, in that there exists a unique map $\tilde{g}|_{S\times[0,1]}$ making the diagram

$$\begin{tikzcd} S \times \{0\} \arrow[r, "\tilde{g}_0|_{S\times\{0\}}"] \arrow[d, "i_0"] & Y \arrow[d, "f"] \\ S \times [0,1] \arrow[r, "g|_{S\times[0,1]}"] \arrow[ur, dashed, "\tilde{g}|_{S\times[0,1]}"] & X. \end{tikzcd}$$

commute.

PROOF. There is nothing to be lost in assuming $V = S$ throughout.

To simplify the notation, we omit many notations that signify "restriction to S" in this proof. Nothing is defined on any subset of Z larger than S in this proof.

We produce the lift $\tilde{g}$ and prove it is unique by induction on l. The base case and the general case proceed similarly. Suppose that a map $\tilde{g}$ has been defined on $S \times [0, t_{l-1}]$—in the base case, this is a point and in general it is a compact interval of positive length. In particular $\tilde{g}(s, t_{l-1})$ has been defined for all $s \in S$. The additional hypothesis of the lemma is that $g(S \times [t_{l-1}, t_l]) \subset W_j$ for some j, and commutativity of the diagram implies that $f(\tilde{g}(s, t_{l-1})) = g(s, t_l)$ for all $s \in S$.

The cover f trivializes over W_j, so we may apply Lemma 8.13 to get a lift in the diagram (following reparametrization):

$$\begin{tikzcd} S \times \{t_{l-1}\} \arrow[r, "\tilde{g}|_{S\times\{t_{l-1}\}}"] \arrow[d, "i_0"'] & f^{-1}(W_j) \arrow[d, "f|_{f^{-1}(W_j)}"] \\ S \times [t_{l-1}, t_l] \arrow[r, "g|_{S\times[t_{l-1},t_l]}"] \arrow[ur, dashed, "\tilde{g}|_{S\times[t_{l-1},t_l]}"] & W_j. \end{tikzcd}$$

This furnishes a unique extension of $\tilde{g}$ from $S \times [0, t_{l-1}]$ to $S \times [0, t_l]$, completing the induction step. ☐

PROOF OF PROPOSITION 8.11. Fix an open cover $\{W_j\}$ of X that trivializes f.

It suffices to show that Z has an open cover $\{V_j\}$ such that each V_j satisfies the hypotheses of Lemma 8.14. The result will then follow by applying Lemma 8.12 to the open cover $\{V_j\}$.

For each $z \in Z$, the space $\{z\} \times [0,1]$ is compact and $\{g^{-1}(W_j)\}$ is an open cover. By the Lebesgue Covering Lemma (3.14), we may find some partition $[t_0 = 0, t_1] \cup [t_1, t_2] \cup \cdots \cup [t_{r-1}, t_r = 1]$ such that for all $l \in \{1,\ldots,r\}$ there exists some j_l for which $\{z\} \times [t_{l-1}, t_l] \subset g^{-1}(W_{j_l})$. By the Tube Lemma (3.11), for each $l \in \{1,\ldots,r\}$, we can find an open set $U_l \ni z$ so that $U_l \times [t_{l-1}, t_l] \subset g^{-1}(W_{j_l})$. Then set $V_z = \bigcap_{l=1}^{r} U_l$. This V_z is an open set containing z for which $V_z \times [t_{l-1}, t_l] \subset g^{-1}(W_{j_l})$ for all $l \in \{1,\ldots,l\}$. That is, V_z satisfies the hypotheses of Lemma 8.14. Since z was arbitrary, we can cover Z by open sets V_z, proving the proposition. ☐

1.1. Path lifting and the fundamental groupoid. Proposition 8.11 allows us to make the following construction.

CONSTRUCTION 8.15. Let X be a topological space. Let $f : Y \to X$ be a covering space. For each $a \in X$, define $F_a = f^{-1}(a)$, which is a discrete topological space, i.e., a set. For each path γ in X from a to b, define a function $\tilde{\gamma} : F_a \to F_b$ as follows.

Given an element $y \in F_a$, there exists a unique lift Γ in the diagram

by constructing the unique lift in

$$\begin{tikzcd} \{0\} \arrow[r, "0 \mapsto y"] \arrow[d] & Y \arrow[d] \\ {[0,1]} \arrow[r, "\gamma"] \arrow[ur, "\Gamma"] & X. \end{tikzcd}$$

Now define $\tilde{\gamma}(y) = \Gamma(1) \in F_y$.

PROPOSITION 8.16. *The map $\tilde{\gamma}$ defined above depends only on $[\gamma]$, the homotopy class of γ (rel. endpoints).*

PROOF. Continue to write $a = \gamma(0)$ and $b = \gamma(1)$, and let $y \in F_a$ be some element. Give Γ the same definition as above.

Suppose $\delta : [0,1] \to X$ is another path and $H : [0,1] \times [0,1] \to X$ happens to be a homotopy from γ to δ relative to $\{0,1\}$. Specifically,

$$\begin{aligned} H(t,0) &= \gamma(t) \quad \forall t \in [0,1], \\ H(t,1) &= \delta(t) \quad \forall t \in [0,1], \\ H(0,s) &= a \quad \forall s \in [0,1], \\ H(1,s) &= b \quad \forall s \in [0,1]. \end{aligned}$$

Using Proposition 8.11, we see that there is a unique lift making this diagram commute:

$$\begin{tikzcd} {[0,1] \times \{0\}} \arrow[r, "\Gamma"] \arrow[d, "i_0"'] & Y \arrow[d, "f"] \\ {[0,1] \times [0,1]} \arrow[r, "H"] \arrow[ur, "\tilde{H}"] & X. \end{tikzcd}$$

We now investigate the lift $\tilde{H} : [0,1] \times [0,1] \to Y$, exploiting the uniqueness part of 8.11.

(1) Along the segment $[0,1] \times \{0\}$, the map $\tilde{H}$ is a lift of γ starting at $\Gamma(0) = x$, and therefore is Γ.
(2) Along the segment $\{0\} \times [0,1]$, the map $\tilde{H}$ is a lift of e_a starting at y. Since e_x is a possible lift of this path, uniqueness of lifts tells us that $\tilde{H}(0,s) = \Gamma(0) = y$ for all $s \in [0,1]$;
(3) A similar argument says that $\tilde{H}(1,s) = \Gamma(1) = \tilde{\gamma}(x)$;
(4) Along the segment $[0,1] \times \{1\}$, the map $\tilde{H}$ is a lift of δ starting at y. Let us call this path $\Delta : [0,1] \to Y$.

Observe that $\Delta(1)$ is $\tilde{\delta}(y)$ by definition.

On the one hand, $\tilde{H}(1,1) = e_{\Gamma(1}(1) = \tilde{\gamma}(y)$, while on the other hand $\tilde{H}(1,1) = \Delta(1) = \tilde{\delta}(y)$. Therefore $\tilde{\gamma}(y) = \tilde{\delta}(y)$ as desired. □

In order to keep the notation from becoming littered with $[\cdot]$ symbols, we will write γ instead of $[\gamma]$ from now on. We will have to rely on context to determine when γ (the path) is meant and when $[\gamma]$ (the homotopy class of the path) is meant, but this will turn out not to be difficult to do.

Now come two small, but very helpful, results about lifting compositions of paths.

PROPOSITION 8.17. *Suppose $f : Y \to X$ is a covering space and γ, δ are two paths in X such that $\gamma(1) = \delta(0)$. Then the two functions*

$$\tilde{\delta} \circ \tilde{\gamma} : F_{\gamma(0)} \to F_{\delta(1)}$$

and

$$\widetilde{(\gamma \cdot \delta)} : F_{\gamma(0)} \to F_{\delta(1)}$$

agree.

PROOF. Choose a point $y \in F_{\gamma(0)}$. Let $\Gamma : I \to Y$ denote the unique lift of γ to Y that starts at y. Let $\Delta : I \to Y$ be the unique lift of δ to Y that starts at $\Gamma(1)$. Then $\Gamma \cdot \Delta$ is defined and is in fact a lift of $\gamma \cdot \delta$ that starts at y. It is therefore the unique such lift. We now can compare:

$$\tilde{\delta} \circ \tilde{\gamma}(y) = \Delta(1) = \Delta \cdot \Gamma(1) = \widetilde{\gamma \cdot \delta}(y),$$

which is what we wanted. $\square$

PROPOSITION 8.18. *Suppose $f : Y \to X$ is a covering space, and $e_x : I \to X$ is the constant path at $x \in X$. Then $\tilde{e}_x : F_x \to F_x$ is the identity map.*

PROOF. If $y \in F_x$, then the unique lift of e_x starting at y is e_y. $\square$

PROPOSITION 8.19. *Suppose $f : Y \to X$ is a covering space. The following construction:*

$$\begin{aligned} x &\mapsto F_x, \\ [\gamma : x \to y] &\mapsto \tilde{\gamma}. \end{aligned}$$

*is a functor $F : \Pi(X) \to$ **Set**.*

PROOF. Proposition 8.16 ensures that this is well defined on morphisms. Functoriality is given by Propositions 8.17 and 8.18. $\square$

COROLLARY 8.20. *Suppose $f : Y \to X$ is a covering space and $\gamma : [0,1] \to X$ is a path, then $\tilde{\gamma} : F_{\gamma(0)} \to F_{\gamma(1)}$ is a bijection.*

In particular, if X is path-connected, then every fibre is in bijective correspondence with every other fibre.

PROOF. The path γ is an isomorphism in $\Pi(X)$, and therefore induces an isomorphism of sets, $\tilde{\gamma} : F_{\gamma(0)} \to F_{\gamma(1)}$, i.e., a bijection. $\square$

2. Fundamental group action

In the previous section, we saw that if $f : Y \to X$ is a covering space, then a path γ in X induces a bijection $F_{\gamma(0)} \to F_{\gamma(1)}$. In particular, when γ is a loop based at $x \in X$, we obtain an automorphism of F_x. These automorphisms are compatible with composition of paths, according to Proposition 8.17, although there is a reversal of the order.

CONSTRUCTION 8.21. Suppose $f : Y \to X$ is a covering space and $x \in X$ is a point. For each $\gamma \in \pi(X, x)$ and each $y \in F_x$, define $y \cdot \gamma$ to be $\tilde{\gamma}(y) \in F_x$, i.e., the endpoint of the path in Y obtained by lifting γ, starting at y.

The assignment $(y, \gamma) \mapsto y \cdot \gamma$ is a function $F_x \times \pi_1(X, x) \to F_x$.

PROPOSITION 8.22. *The function $F_x \times \pi_1(X, x) \to F_x$ defined above is a right action of $\pi_1(X, x)$ on F_x.*

PROOF. To prove this is a right action, we have to prove two things:

First, we want to show $y \cdot e_x = y$, where e_x denotes the constant loop at x. This is Proposition 8.18

Second, $(y \cdot \gamma) \cdot \delta = y \cdot (\gamma\delta)$. This is a special case of Proposition 8.17. □

NOTATION 8.23. This action of $\pi_1(X, x)$ on the fibre over x is called the *monodromy action.*

REMARK 8.24. There is a category of *covering spaces of X* where the objects are maps $f : Y \to X$ and the maps are maps $h : Y \to Y'$

$$\begin{tikzcd} Y \arrow[rr, "h"] \arrow[dr, "f"'] & & Y' \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

making the diagram commute. Write this category as $\mathbf{Cov}_X$.

There is an obvious category of right $\pi_1(X, x_0)$-sets, written $\mathbf{Set}$-$\pi_1(X, x_0)$.

We now discuss the relationship, for a covering space $f : X \to Y$, between $\pi_1(Y, y)$, $\pi_1(X, x)$ and the fibre F_x.

PROPOSITION 8.25. *Let $f : Y \to X$ be a covering space. Fix a basepoint $x \in X$. Two points $y, y' \in F_x$ lie in the same orbit of the $\pi_1(X, x)$-action if and only if y and y' lie in the same path component of Y.*

PROOF. Suppose y, y' are in the same path component of Y. Then there is a path Γ from y to y' in Y. Define $\gamma = f \circ \Gamma : [0, 1] \to X$ from x to x. This gives us a class $\gamma \in \pi_1(X, x)$. To determine $y \cdot \gamma$, take the lift of γ starting at y, which is Γ, and evaluate it at 1. This gives us $y \cdot \gamma = y'$.

Suppose conversely that $y \cdot \gamma = y'$. If we lift γ to a path $\Gamma : [0, 1] \to Y$ that starts at y, we must get a path ending at $\Gamma(1) = y'$. Therefore y and y' are in the same path component of Y. □

Next, we investigate path components of Y.

PROPOSITION 8.26. *Let $f : Y \to X$ be a covering space, and choose basepoints $y \in Y$ and $x = f(y) \in X$. The map $f_* : \pi_1(Y, y) \to \pi_1(X, f(x))$ is injective, and the image is the stabilizer subgroup of the action of $\pi_1(X, x)$ on y.*

PROOF. First we verify injectivity. Suppose $f_*(\Gamma) = e_x$ in $\pi_1(X, x)$. That is, $f \circ \Gamma \simeq e_x$ rel. $\{0,1\}$. Thefore, the lifts of these two paths in X to paths in Y starting at Y are homotopic rel. $\{0,1\}$. One lift is Γ and the other is e_y. This implies Γ is trivial in $\pi_1(Y, y)$, as required.

Next we verify the image is as claimed. For an arbitrary $\gamma \in \pi_1(X, x)$, the element $y \cdot \gamma \in F_x$ is defined by lifting γ to a path $\Gamma : [0,1] \to Y$ starting at y, then evaluating at 1. We see that $y \cdot \gamma = y$ if and only if γ has a lift to some $\Gamma \in \pi_1(Y, y)$, i.e., if and only if $\gamma = f_*(\Gamma)$ for some $\Gamma \in \pi_1(Y, y)$. □

COROLLARY 8.27. *Suppose $f : Y \to X$ is a covering space and $x \in X$ is a basepoint. Consider the action of $\pi_1(X, x)$ on F_x. If Y is path connected, then this action is transitive. If Y is simply connected and F_x is not empty, then the action is also free.*

PROOF. The first statement, about transitivity of the action, is immediate from Proposition 8.25.

The second statement follows from Proposition 8.26: a transitive action is free if and only if the stabilizer of one point (and therefore of every point) is trivial. □

DEFINITION 8.28. Let X be a path-connected space. For reasons that may be clarified later, a simply-connected covering space $f : \tilde{X} \to X$ is called a *universal covering space* of X.

Let X be a space and $x \in X$ a basepoint. We can use a universal covering space $f : \tilde{X} \to X$ to calculate $\pi_1(X, x)$ by exploiting the free transitive action of $\pi_1(X, x)$ on F_x. Choose a basepoint $y_0 \in F_x$. This sets up a bijection of sets $F_x \to \pi_1(X, x)$ where each $y \in F_x$ corresponds to the unique $\gamma \in \pi_1(X, x)$ such that $y_0 \cdot \gamma = y$. To figure out what γ is from the knowledge of y, we do the following: find a path $\Gamma : [0,1] \to Y$ from y_0 to y. Since Y is simply connected, this Γ is uniquely determined up to homotopy rel. $\{0,1\}$. Then $\gamma = f \circ \Gamma$.

This sets up a bijection of sets: $F_x \to \pi_1(X, x)$. Determining the group structure from this is a little clunky, because F_x is not itself a group. You can argue as follows: Suppose $y, y' \in F_x$ are two elements, corresponding to $\gamma, \gamma' \in \pi_1(X, x)$. Then $\gamma\gamma'$ corresponds to the element of F_x obtained by taking an explicit representative $\gamma' : [0,1] \to X$, then lifting this to a path starting at y. The endpoint of the lift corresponds to $\gamma\gamma'$.

EXAMPLE 8.29. Consider the covering space map $f_\infty : \mathbf{R} \to S^1$ given by $f_\infty(r) = (\cos(2\pi r), \sin(2\pi r))$. Observe that $\mathbf{R}$ is contractible and therefore is simply connected, so is a universal cover of S^1.

Endow S^1 with the basepoint $p = (1,0)$. Then $F_p = \mathbf{Z} \subset \mathbf{R}$ by elementary trigonometry. For all $x, y \in \mathbf{Z}$, let us define

$$\Gamma_{[x,y]} : [0,1] \to \mathbf{R}, \quad \Gamma_{[x,y]}(t) = x + (y - x)t.$$

This is a path starting at x and ending at y. By the same elementary trigonometry as before, the loops $f \circ \Gamma_{[x,y]}$ depend only on the difference $y - x$.

Endow F_p with the basepoint 0. This gives us a bijection $\pi_1(X, x) \leftrightarrow \mathbf{Z}$, given explicitly by $m \mapsto f_\infty \circ \Gamma_{[0,m]}$. That is, m is assigned to the loop

$$\gamma_m : [0,1] \to S^1, \quad \gamma_m(t) = (\cos(2\pi m t), \sin(2\pi m t)).$$

To compose $\gamma_m, \gamma_{m'}$, we must lift $\gamma_{m'}$ to a path starting at $m \in Z$. Such a lift is given by $\Gamma_{[m,m+m']}$. Now we calculate the endpoint of the composite $\Gamma_{[0,m]} \cdot \Gamma_{[m,m+m']}$, which is $m+m'$. This is the value in $\mathbf{Z} = F_p$ corresponding to $\gamma_m \gamma_{m'}$.

We have deduced, in a different way from Example 7.4, that $\pi_1(S^1, p) \cong \mathbf{Z}$, generated by the loop $t \mapsto (\cos(2\pi t), \sin(2\pi t))$.

3. Lifting of general maps to covering spaces

PROPOSITION 8.30. *Let $f : Y \to X$ be a covering space and let $y_0 \in Y$ be a basepoint and $x_0 = f(y_0)$. Let (Z, z_0) be a pointed space where Z is path connected and locally path connected. Let $g : (Z, z_0) \to (X, x_0)$ be a map. Suppose* $\operatorname{im}(g_* : \pi_1(Z, z_0) \to \pi_1(X, x_0)) \subset \operatorname{im}(f_* : \pi_1(Y, y_0) \to \pi_1(X, x_0))$. *Then there exists a unique continuous lift h making the diagram*

$$\begin{tikzcd} & Y \arrow[d, "f"] \\ Z \arrow[ur, dashed, "h"] \arrow[r, "g"] & X \end{tikzcd}$$

commute.

PROOF. Define h as a function as follows: if $z \in Z$, then let γ be a path in Z from z_0 to z. The image under g of γ is a path in X from x_0 to $g(z)$. This path has a unique lift to a path $\tilde{\gamma}$ from y_0 to $\tilde{\gamma}(1)$. Define $h(z) = \tilde{\gamma}(1)$.

This definition seems to depend on the choice of γ. Let us show that it is really independent of that choice. Suppose δ is a path in Z from z_0 to z, possibly different from γ. Then $\delta\gamma^{\leftarrow}$ is a loop in Z, and so $g(\delta\gamma^{\leftarrow})$ is a loop in X. The hypothesis on the fundamental groups implies that $g(\delta\gamma^{\leftarrow}) \in \pi_1(X, x_0)$ acts trivially on y_0, so that $(\delta\gamma^{\leftarrow})$ lifts to a loop, based at y_0. By uniqueness of lifting, this loop must be $\tilde{\delta}\tilde{\gamma}^{\leftarrow}$, so that in particular, $\tilde{\delta}(1) = \tilde{\gamma}(1)$. This implies that the definition of $h(z)$ was independent of the choice of path.

We must show that the function h that we defined is continuous. Suppose $W \subset Y$ is an open set and $h(z) \in W$. We can find an open neighbourhood $U \ni h(z)$ and $U \subset W$ such that $f|_U$ is a homeomorphism onto an open set. Note that $g(z) \in f(U)$. Since g is continuous, there is an open set $V \ni z$ such that V is path connected and such that $g(V) \subset f(U)$. Take any $z' \in V$. There exists a path ϵ from z to z' in V, and by composing $\gamma\epsilon$, we obtain a path from z_0 to z'. We may use this path to calculate $h(z') = \tilde{\gamma}\tilde{\epsilon}(1)$. By uniqueness of lifting, $\tilde{\epsilon}$ lies entirely within U, and therefore $h(z') = \tilde{\gamma}\tilde{\epsilon}(1)$ lies in U. It follows that $z \in U \subset h^{-1}(W)$. Since z was arbitrary in W, the function h is continuous.

Finally, we show that h is unique. Suppose h' is another lift and $h(z) \neq h'(z)$. Let $\gamma : I \to Z$ be a path from z_0 to z', then $h \circ \gamma$ and $h' \circ \gamma$ give two different lifts of the path $g \circ \gamma$ both starting at y_0. This is impossible. $\square$

4. The classification of covering spaces

We have seen that if $f : Y \to X$ is a covering space and $x \in X$, then the fibre F_X is a (discrete) set with a right $\pi_1(X, x)$-action.

DEFINITION 8.31. Let X be a topological space and $f_1 : Y_1 \to X$, $f_2 : Y_2 \to X$ be two covering spaces. A *map of covering spaces* $h : Y_1 \to Y_2$ is a map of spaces that also satisfies $f_1 = f_2 \circ h$, i.e., it makes

$$\begin{tikzcd} Y_1 \arrow[rr, "h"] \arrow[dr, "f_1"'] & & Y_2 \arrow[dl, "f_2"] \\ & X & \end{tikzcd}$$

commute.

PROPOSITION 8.32. *If $h : Y_1 \to Y_2$ is a map of covering spaces and $x \in X$ is a point, then there is an induced function $h_x : f_1^{-1}(x) \to f_2^{-1}(x)$ given by $h_x(y) = h(y)$. This function is compatible with the right $\pi_1(X,x)$-actions on the fibres in the sense that*

$$h(y \cdot \gamma) = h(y) \cdot \gamma$$

PROOF. To form $y \cdot \gamma$, we lift γ to a path Γ in Y_1 starting at y. Then $h \circ \Gamma$ is the unique lift of γ in Y_2 starting at $h(y)$.

$$h(y) \cdot \gamma = h \circ \Gamma(1) = h(y \cdot \gamma).$$

□

REMARK 8.33. That is, a map of covering spaces yields an induced morphism of fibres as sets with right $\pi_1(X,x)$ action. It is easy to verify that;

(1) The morphism on fibres induced by the identity is the identity.
(2) If h_1 and h_2 are two composable maps of covering spaces, then $(h_2)_x \circ (h_1)_x = (h_2 \circ h_1)_x$.

Taken together, these imply that we have a functor

$$\text{Fibre at } x\text{: Covering spaces of } X \to \text{Sets with right } \pi_1(X,x)\text{-action.}$$

Under mild hypotheses on X (including path connectedness), this functor tells you everything about the covering spaces of X. Specifically:

(1) All sets with a right $\pi_1(X,x)$-action arise as the fibre of some covering space (Proposition 8.38), provided X is connected, locally path connected and semilocally simply connected.
(2) All morphisms of sets with right $\pi_1(X,x)$-action arise from maps of covering spaces (Proposition 8.36, provided X is connected and locally path connected.
(3) If $h_1, h_2 : Y_1 \to Y_2$ are two maps of covering spaces that give rise to the same map of fibres, then $h_1 = h_2$ (Proposition 8.35), provided X is path connected.

REMARK 8.34. The term for the situation above, when X is connected, locally path connected and semilocally simply connected, is that there is an *equivalence of categories* between the category of covering spaces of X and the category of sets with a right $\pi_1(X,x)$-action.

The equivalence yields a table:

PROPOSITION 8.35. *Suppose X is a path connected space with basepoint x. Suppose $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ are two covering spaces and write F_1 and F_2 for the fibres of each over x. Suppose*

$f : Y \to X$	F_x
Map of covering spaces	$\pi_1(X,x)$-equivariant function
Path component	Orbit
$\pi_1(Y,y)$	Stabilizer of $y \in F_x$.

TABLE 1. The equivalence of categories between covering spaces and sets with right $\pi_1(X,x)$-action.

$h_1, h_2 : Y_1 \to Y_2$ are two maps of covering spaces of X such that the two induced functions on fibres $(h_i)_x : F_1 \to F_2$ agree. Then $h_1 = h_2$.

PROOF. Let $y \in Y_1$ be a point (not necessarily in the fibre), and let γ be a path in X from x to $f_1(y)$. We can lift γ uniquely to a path Γ in Y_1 ending at y, and taking the starting point of this path gives us $y_0 \in Y_1$, a basepoint for Y in the fibre F_1. By hypothesis, $h_1(y_0) = h_2(y_0)$, and now $h_1 \circ \Gamma$ and $h_2 \circ \Gamma$ are two lifts of γ to Y_2, both starting at the same point, $h_1(y_0)$. Uniqueness of lifts now implies that $h_1 \circ \Gamma = h_2 \circ \Gamma$, so that in particular their endpoints agree, i.e. $h_1(y) = h_2(y)$. Since $y \in Y$ was an arbitrary point,we have proved the proposition. □

PROPOSITION 8.36. *Suppose X is a path connected and locally path connected space with basepoint x. Suppose $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ are two covering spaces and write F_1 and F_2 for the fibres of each over x. Suppose $\phi : F_1 \to F_2$ is a function between the fibres that is compatible with the right $\pi_1(X,x)$-actions. Then there is a map $h : Y_1 \to Y_2$ of covering spaces inducing ϕ.*

PROOF. The bulk of this has been done already. Note that the previous proposition states that h is unique if it exists.

Choose a point $y \in F_1$. We define h on the path component of y in Y_1. There is no loss of generality in supposing Y_1 is path-connected, since the $\pi_1(X,x)$-action restricts to an action on the intersection of F_1 with the path component of y. Similarly, we may replace Y_2 by the path component of $\phi(y)$.

In particular, once we assume Y_1 and Y_2 are path connected, we know that F_1 and F_2 each consist of a single orbit for the $\pi_1(X,x)$-action.

Use y as a basepoint for Y_1 and $\phi(y)$ as a basepoint for Y_2. Then we can define an injective homomorphism $(f_1)_* : \pi_1(Y_1, y) \to \pi_1(X,x)$ and similarly for $(f_2)_*$.

There is an isomorphism of sets with a right $\pi_1(X,x)$-action $F_1 \cong \pi_1(X,x)/(\mathrm{im}(f_1)_*$ and similarly there is an isomorphism of sets with a right $\pi_1(X,x)$-action $F_2 \cong \pi_1(X,x)/(\mathrm{im}(f_2)_*)$. The existence of a morphism ϕ of sets with $\pi_1(X,x)$-action implies that any element of $\pi_1(X,x)$ that acts trivially on y must also act trivially on $\phi(y)$. Therefore, $\mathrm{im}(f_2)_* \subset \mathrm{im}(f_1)_*$. We may then use Proposition 8.30 to produce a unique map h making the diagram

$$\begin{tikzcd} & Y_2 \arrow[d, "f_2"] \\ Y_1 \arrow[ur, dashed, "h"] \arrow[r, "f_1"'] & X \end{tikzcd}$$

commute and satisfying $h(y) = \phi(y)$.

It remains to show that h really does yield ϕ when restricted to the fibres F_1 and F_2. To show this, suppose $\gamma \in \pi_1(X, x)$. Since F_1 consists of a single orbit, it will suffice to show that $h(y \cdot \gamma) = h(y) \cdot \gamma$. First consider $y \cdot \gamma$. This is the element $\Gamma(1)$, where Γ is a lift of γ to Y_1 starting at y. Then consider $h \cdot \Gamma$, which is a lift of γ to a path starting at $\phi(y)$. Therefore $h \cdot \Gamma(1) = \phi(y) \cdot \gamma = h(y) \cdot \gamma$, which is what we wanted. □

COROLLARY 8.37. *Let X be a path connected and locally path connected space with basepoint x. Suppose $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ are covering spaces such that the fibres F_1 and F_2 are isomorphic as sets with right $\pi_1(X, x)$-action. Then Y_1 and Y_2 are isomorphic as covering spaces of X.*

PROOF. There exists an abstract isomorphism $\phi : F_1 \to F_2$ and an inverse $\phi^{-1} : F_2 \to F_1$. Using Proposition 8.35, form the maps of covering spaces associated to each of these: ϕ is associated to h and ϕ^{-1} is associated to a map h' that satisfies $h \circ h' = \mathrm{id}_{F_2}$ and $h' \circ h = \mathrm{id}_{F_1}$ by virtue of Proposition 8.35. This implies that $h' = h^{-1}$, so h is an isomorphism of covering spaces, as required. □

PROPOSITION 8.38. *Suppose X is a path connected and locally simply connected space and let $x \in X$ be a basepoint. Let S be a set with a right $\pi_1(X, x)$-action. There is a covering space $f : Y \to X$ where the fibre $f^{-1}(x)$ is isomorphic to S with the given right $\pi_1(X, x)$-action.*

PROOF. If S is empty, there is nothing to do, so assume S is not empty.

As before, it suffices to consider the case where S consists of a single orbit, since a covering space $f : Y \to X$ is made up of disjoint path components, each corresponding to a different orbit in the fibre.

Let $y \in F$ be a basepoint. Consider K, the stabilizer of the $\pi_1(X, x)$ action on y. Impose an equivalence relation on paths $\gamma : [0, 1] \to X$ such that $\gamma(0) = x$; we say $\gamma \sim \gamma'$ if $[\gamma'] \circ [\gamma^{\leftarrow}] \in K$. In particular $\gamma \sim \gamma'$ implies that $\gamma(1) = \gamma'(1)$. Let Y denote the set of equivalence classes of such paths and denote the class of γ by $[[\gamma]]$.

We generate a topology on Y as follows. Choose a point $[[\gamma]]$. Write $x' = \gamma(1)$. Find $W \ni x'$ that is simply connected—any $x' \in X$ has a local base consisting of such sets. For each z in W, there exists a path δ from x' to z. The class of $[[\gamma\delta]]$ is independent of the choice of δ by virtue of the local simple-connectivity of X: any alternative choice of δ' would give us $\delta'\delta^{\leftarrow} \simeq e_{x'}$. Define an open set $W_\gamma \subset Y$ to consist of

$$\{[[\gamma\delta]] \in Y \mid \delta : [0, 1] \to W\}.$$

Sets of the form W_γ make up a base for a topology on Y: the verification of this is routine and tedious so we omit it.

With this topology, by restricting to open sets W that are locally simply connected we see that f has the property that it is a covering space map (in particular, it is continuous). Finally, $f^{-1}(x)$ is isomorphic to $K \backslash \pi_1(X, x_0) \cong S$ as a right $\pi_1(X, x)$-set, as required. □

REMARK 8.39. The hypotheses of Proposition 8.38 are slightly more restrictive than necessary. It is possible to prove it while assuming only that each $x \in X$ has a local base of neighbourhoods U such that each loop in U based at x can be contracted to a point via a homotopy in X—our assumption is that the homotopy is defined in U. An X satisfying the weaker condition is called *semilocally simply connected*. In practice, there are few examples we care about of spaces that are semilocally simply connected but not locally simply connected.

REMARK 8.40. Suppose X is a path connected and locally simply connected space with basepoint x. Among all the covering spaces of X, there is one covering space (up to isomorphism) $f : \tilde{X} \to X$ in particular that is very special: the one that has fibre isomorphic to $\pi_1(X, x)$ itself as a set with right $\pi_1(X, x)$-action. This has the properties that:

(1) $\tilde{X}$ is path connected because $\pi_1(X, x)$ consists of exactly one $\pi_1(X, x)$-orbit
(2) The inclusion $f_* : \pi_1(\tilde{X}, x) \to \pi_1(X, x)$ is the inclusion of the trivial group. This is because the fibre is isomorphic to $\operatorname{im}(f_*)\backslash\pi_1(X, x)$ as a set with right $\pi_1(X, x)$-action. In this case, this implies that $\operatorname{im}(f_*)$ is trivial.

Recall that a covering space $f : \tilde{X} \to X$ with the property that $\tilde{X}$ is simply connected is called the *universal cover* of X. We have proved that if X is a path connected and locally simply connected space, then X has a universal cover. What is more, Corollary 8.37 shows that the universal cover is unique up to isomorphism of covering spaces.

REMARK 8.41. What is universal about the universal cover? One might as well ask what is universal about the set $\pi_1(X, x)$ as a set with right $\pi_1(X, x)$-action.

Suppose X is connected and locally path connected. Fix a universal cover $f : \tilde{X} \to X$ and choose a basepoint $\tilde{x} \in \tilde{X}$ (set $x = f(\tilde{x})$). Given any covering space $h : Y \to X$ and any point y in the fibre of h over x, there exists a unique map of covering spaces $\bar{f} : \tilde{X} \to Y$ taking $\tilde{x}$ to y.

Showing this is an exercise in using Proposition 8.36 (to construct $\bar{f}$) and Proposition 8.35 (to show it is unique).

We have restricted our analysis to the case of a single basepoint $x \in X$. One could develop a whole theory that is independent of the basepoint by using fundamental groupoids. We will not do that here. Instead, we have the following embellishment of Proposition 6.19.

PROPOSITION 8.42. *Let $f : Y \to X$ be a map, and let $\alpha : [0,1] \to Y$ be a path in Y. Write $y = \alpha(0)$ and $y' = \alpha(1)$. Write $x = f(y)$ and $x' = f(y')$. Then the following change-of-basepoint diagram commutes:*

$$\begin{array}{ccc} \pi_1(Y, y) & \xrightarrow{\phi_\alpha} & \pi_1(Y, y') \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \pi_1(X, x) & \xrightarrow{\phi_{f\circ\alpha}} & \pi_1(X, x'). \end{array}$$

PROOF. As you might expect, this is really a consequence of the functoriality of the fundamental groupoid.

The isomorphism ϕ_α is given by $[\gamma] \mapsto [\alpha^{\leftarrow}] \cdot [\gamma] \cdot [\alpha]$, and $\phi_{f\circ\alpha}$ is defined by a similar formula. The result follows by inspection. □

5. Deck transformations

DEFINITION 8.43. Let $f : Y \to X$ be a covering space. An automorphism $h : Y \to Y$ of Y over X is called a *deck transformation*. The set of all such transformations forms a group, $\operatorname{Aut}_X(Y)$.

REMARK 8.44. The deck transformations are functions $h: Y \to Y$, and therefore compose as functions do: $h_2 \circ h_1$ means "do h_1 and then do h_2". In particular, the group $\mathrm{Aut}_X(Y)$ is acting on Y on the left. This is notationally different from the right actions we have seen up until now, but practically very similar.

PROPOSITION 8.45. *Let $f: Y \to X$ be a covering space where X is connected and locally path connected and Y is connected. Then the group $\mathrm{Aut}_X(Y)$ of deck transformations acts freely on Y.*

PROOF. If Y is empty, there is nothing to show. Let $y \in Y$ and suppose $g \in \mathrm{Aut}_X(Y)$ is a deck transformation such that $g(y) = y$. We must show $g = \mathrm{id}_Y$.

Use y as a basepoint for Y, and $x = f(y)$ as a basepoint for X. Write F_x for the fibre. The deck transformation $g: Y \to Y$ over X corresponds, using Propositions 8.35, 8.36 to a unique automorphism $g_x: F_x \to F_x$ as a right $\pi_1(X,x)$-set. Since $g(y) = y$, it is the case that $g_x(y) = y$, and so g_x is the identity on the entire $\pi_1(X,x)$-orbit of y. Since Y is connected, this orbit is all of F_x, so g_x is the identity, and therefore $g = \mathrm{id}_Y$. □

If G is a group and K is a subgroup, then we will write $K\backslash G$ for the set of right cosets $\{Kg \mid g \in G\}$.

REMARK 8.46. If X is connected and locally path connected with basepoint x, then covering spaces $f: Y \to X$ are determined by their fibres F_x as $\pi_1(X,x)$-sets, and automorphisms of Y over X are determined by self-maps $F_x \to F_x$ that are compatible with the group action. This is all by virtue of Propositions 8.35, 8.36. Explicitly, there is an isomorphism of automorphism groups:

$$\mathrm{Aut}_X(Y) \cong \mathrm{Aut}_{\pi_1(X,x)\text{-Sets}}(F_x).$$

Therefore, understanding the deck transformations of covering spaces over such an X is equivalent to understanding the automorphisms of the right $\pi_1(X,x)$-sets that arise as fibres. This is easiest to do when F_x consists of one orbit, i.e., when Y is path-connected and nonempty.

5.1. Automorphisms of one-orbit right G-sets. For a little while, we will work in pure algebra to describe the automorphisms (in the category of right G-sets) of right G-sets consisting of one orbit.

NOTATION 8.47. If K is a subgroup of G, we will write $K\backslash G$ for the set of right cosets of G by K, i.e.,

$$K\backslash G = \{Kg \mid g \in G\}.$$

Suppose G is a group that acts on a nonempty set F on the right in such a way that F consists of one orbit only.

Let $y \in F$ be an element, and write G_y for the stabilizer. The orbit–stabilizer theorem says that there is a bijection $G_y\backslash G \to F$ given by sending the coset $G_y g$ to yg. Although $G_y\backslash G$ may not be a group, there is a right G-action on it by multiplication. The orbit–stabilizer bijection is actually a bijection of right G-sets, since

$$G_y g g' \mapsto y g g' = (yg)g'.$$

Therefore, the orbit–stabilizer theorem tells us that every nonempty one-orbit right G-set is isomorphic to $K\backslash G$ for some K, the isomorphism being in the category of right G-sets.

In summary, for our purposes it suffices to determine the groups $\mathrm{Aut}_{G\text{-}\mathbf{Set}}(K\backslash G)$.

DEFINITION 8.48. Let $K \subset G$ be a subgroup of a group. Then $N_G(K)$, the *normalizer* of K in G is the subgroup of elements $g \in G$ such that $gKg^{-1} = K$.

In other words $N_G(K)$ is the largest subgroup of G such that K is a normal subgroup of $N_G(K)$.

PROPOSITION 8.49. *The following formula*

$$n \cdot Kg = Kng, \quad \text{when } n \in N_G(K) \text{ and } g \in G$$

defines a left action of $N_G(K)$ on $K\backslash G$ as a right G-set.

PROOF. First, we must show that $n \cdot Kg$ above does not depend on the specific choice of g. Suppose $Kg = Kg'$, so that there exists some $k \in K$ for which $g' = kg$. Since $n \in N_G(K)$, the element n normalizes K, and we can find some $k' \in K$ such that $nk = k'n$. Now we can write $Kng' = Knkg = Kk'ng = Kng$, as required.

Second, we can verify that the formula, once it is known to be well defined, gives us a left action of $N_G(K)$ on $K\backslash G$ as a set. The verification is routine and we omit it.

Third, we check that the action is compatible with the right G-action on $K\backslash G$. This is not difficult. The calculation is

$$[n(Kg)] \cdot g' = (Kng) \cdot g' = Kngg' = n(Kgg').$$

This shows that we do indeed have an action as stated. □

PROPOSITION 8.50. *The action of $N_G(K)$ on $K\backslash G$ in Proposition 8.49 induces an isomorphism $N_G(K)/K \xrightarrow{\cong} \mathrm{Aut}_{G\text{-}\mathbf{Set}}(K\backslash G)$.*

PROOF. The action gives a group homomorphism $N_G(K) \to \mathrm{Aut}_{G\text{-}\mathbf{Set}}(K\backslash G)$.

We first show that every automorphism of $K\backslash G$ as a right G-set is given by action by some $n \in N_G(K)$. Suppose therefore that $\alpha \in \mathrm{Aut}_{G\text{-}\mathbf{Set}}(K\backslash G)$. The automorphism α is completely determined by $\alpha(Ke)$, since compatibility with the right G-action implies that $\alpha(Kg) = \alpha(Ke) \cdot g$ for general g. Now suppose $\alpha(Ke) = Kg$. For any $k \in K$, we must have $\alpha(Ke) \cdot k = \alpha(Ke)$, so that $Kgk = Kg$, which implies that $g \in N_G(K)$ and therefore $\alpha = \phi(g)$. This shows that ϕ is surjective.

We now show that the kernel of ϕ is $K \subset N_G(K)$. That is we show that $n \in N_G(K)$ acts trivially on all cosets Kg if and only if $n \in K$. This is a direct verification: $n \cdot Kg = Kng$ and $Kng = Kg$ for all g if and only if $n \in K$.

This implies that the surjective homomorphism ϕ factors through a bijective homomorphism $N_G(K)/K \to G$. □

REMARK 8.51. The action of $N_G(K)$ on $K\backslash G$ is transitive if and only if K is a normal subgroup of G.

To see this, argue as follows. The action is transitive if and only if the following is true: for all $g \in G$, there is some $n \in N_G(K)$ such that $Kn = n(Ke) = Kg$. Therefore $g \in K \cdot N_G(K)$ and since $K \subset N_G(K)$, we see that $g \in N_G(K)$.

COROLLARY 8.52. *Let F be a nonempty right G-set consisting of* 1 *orbit. Let $y \in F$ be some element. Then there is an isomorphism*

$$\mathrm{Aut}_{G\text{-}\mathbf{Set}}(F) \cong N_G(G_y).$$

This automorphism group acts transitively on F if and only if G_y is a normal subgroup of G.

PROOF. One may replace F by the right G-set $G_y \backslash G$, which is isomorphic to it as a right G-set. The results then follow from the rest of this subsection. □

5.2. Back to covering spaces.

PROPOSITION 8.53. *Let X be a connected and locally path connected space. Let $x \in X$ be a basepoint and write $G = \pi_1(X, x)$. Suppose $f : Y \to X$ is a covering space such that Y is nonempty and connected.*

The group of deck transformations of Y over X is $N_G(K)/K$ where $K = f_*(\pi_1(Y, y))$ for some choice of $y \in F_x$.

PROOF. We have done the work for this already. Write F_x for the fibre. The group of automorphisms of Y as a covering space of X is isomorphic to $\mathrm{Aut}_{G\text{-}\mathbf{Set}}(F_x)$, by Remark 8.46. In Proposition 8.26, we showed that this stabilizer is $f_*(\pi_1(Y, y))$. Then in Corollary 8.52, we showed that $\mathrm{Aut}_{G\text{-}\mathbf{Set}}(F_x) = N_K(G)$. □

The next corollary is philosophically important. As a rule, nonabelian groups arise as groups of symmetries of something, but $\pi_1(X, x)$ was not defined to be the group of symmetries of anything. We see that really it is the group of symmetries of the universal cover.

COROLLARY 8.54. *Let $f : \tilde{X} \to X$ be a universal cover of a connected and locally path connected space. Then $\mathrm{Aut}_X(\tilde{X}) \cong \pi_1(X, x)$.*

PROOF. Applying the proposition to the case of a universal cover, we discover that $\mathrm{Aut}_X(\tilde{X}) = N_{\pi_1(X,x)}(\{e\})/\{e\}$. Since everything in $\pi_1(X, x)$, normalizes the trivial subgroup, the result follows. □

A more careful development of the theory would allow us to say slightly more. Under the isomorphism $\mathrm{Aut}_X(\tilde{X}) \to \pi_1(X, x)$ that we have constructed, an automorphism $\alpha : \tilde{X} \to \tilde{X}$ corresponds to an element of $\pi_1(X, x)$ in the following way: choose any element $y \in F_x$, then $\alpha(y) = y \cdot g$ for some $g \in \pi_1(X, x)$. One can determine g from the geometry by drawing a path γ from y to $y \cdot g$ in $\tilde{X}$ (since $\tilde{X}$ is simply connected, this is possible and there is a unique class of such paths up to homotopy rel. $\{0, 1\}$). Then map γ down to X to get a loop based at X. This is the element of $\pi_1(X, x)$ corresponding to α.

EXAMPLE 8.55. Let $p = (1, 0) \in S^1$. Let us calculate $\pi_1(S^1, p)$ for the third time. The first two calculations were in Example 7.4 and Example 8.29.

There is a covering space map $f : \mathbf{R} \to S^1$ given by $f(x) = (\cos(2\pi x), \sin(2\pi x))$, and since $\mathbf{R}$ is contractible, this is a universal covering space. The group $\mathrm{Aut}_{S^1}(\mathbf{R})$ is the group of homeomorphisms $\alpha : \mathbf{R} \to \mathbf{R}$ that satisfy $f \circ \alpha = \alpha$, i.e., the set of homeomorphisms such that

$$\forall x \in \mathbf{R}, \quad x - \alpha(x) \in \mathbf{Z} \subset \mathbf{R}.$$

Since $x-\alpha(x)$ is continuous, $\mathbf{R}$ is connected and $\mathbf{Z}$ is discrete, it follows that $x-\alpha(x) \in \mathbf{Z}$ is a constant independent of x. Therefore, there exists some integer m for which the function α is

$$\alpha(x) = x + m.$$

It is easy to verify that such αs really do give deck transformations. Function composition coincides with addition of constants.

Therefore, we have established an isomorphism $\pi_1(X, x) \cong \mathbf{Z}$. The generators are classes of the loops in S^1 that correspond to paths from an integer m to $m \pm 1$.

REMARK 8.56. If we allow the covering space Y to be disconnected, the group of deck transformations can become larger than $\pi_1(X, x_0)$. For instance, the split covering $* \coprod * \to *$ has nontrivial deck transformation group.

DEFINITION 8.57. Let $f: Y \to X$ be a covering space where X is connected and locally path connected, and Y is connected. We say $f: Y \to X$ is a *normal* covering space if the group of deck transformations $\mathrm{Aut}_X(Y)$ acts transitively on F_x for all choices of $x \in X$.

PROPOSITION 8.58. *Assume $f: Y \to X$ is a covering space where X is connected and locally path connected and Y is connected. Let $y \in Y$ be a point and $x = f(y)$. The following are equivalent:*

(1) $\mathrm{Aut}_X(Y)$ *acts transitively on* F_x.
(2) $f_*: \pi_1(Y, y) \to \pi_1(X, x)$ *is the inclusion of a normal subgroup.*
(3) *For all choices of point $y' \in Y$, the map $f_*: \pi_1(Y, y') \to \pi_1(X, f(y'))$ is the inclusion of a normal subgroup.*
(4) *The covering f is normal.*

PROOF. Let us start by showing the equivalence of 1 and 2.

We use the equivalence between covering spaces $f: Y \to X$ and sets with right $\pi_1(X, x)$-action. For brevity, write $G = \pi_1(X, x)$ and $K = \mathrm{im}\, f_*$. Under this equivalence, to say $\mathrm{Aut}_X(Y)$ acts transitively on F_x is to say that $N_G(K)/K$ acts transitively on $K \backslash G$, which it does if and only if K is normal in G as we remarked earlier. This shows that 1 and 2 are equivalent.

Now we can use the change-of-basepoint result, Proposition 8.42, to say that if $f_*: \pi_1(Y, y) \to \pi_1(X, x)$ is the inclusion of a normal subgroup, then so too is the homomorphism $f_*: \pi_1(Y, y') \to \pi_1(X, x')$ where y' is any other point in Y. Therefore 2 implies 3. Applying the equivalence of 2 and 1 to an arbitrary pair of basepoints y', x' shows that 3 implies 4, and certainly 4 implies 1. $\square$

EXAMPLE 8.59. Not all covering spaces are normal covering spaces. An example is given in Figure 1. In the figure, a (topological) graph is depicted. The edges labelled α map to the edge labelled α and similarly for β. The vertices map to the unique vertex in the target. The target X is not the quotient of Y by a group action.

6. Fundamental groups of quotients

Remember that a (left) action of a group G on a space Y is free if, for all $g \in G \setminus \{e\}$ and $y \in Y$, we have $gy \neq y$. Here is a strengthening of this condition.

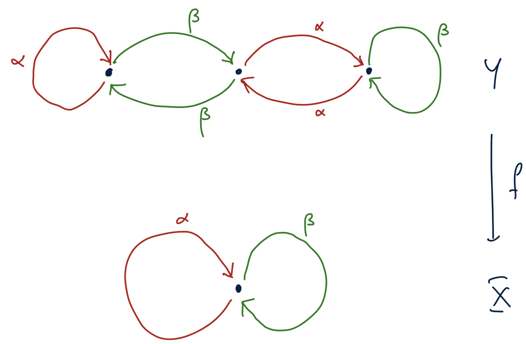


FIGURE 1. A covering space that is not a normal covering.

DEFINITION 8.60. A left action of a (discrete) group G on a space Y is a *covering space action* if, for all $y \in Y$ there exists an open set $U \ni y$ such that $U \cap gU \neq \emptyset$ implies $g = e$.

This is sometimes called a "properly discontinuous action", which may be misleading. The term "discontinuous" here means that the action by two different elements $g, h \in G$ are not "close", but the maps $Y \to Y$ given by left-multiplication by $g \in G$ are certainly continuous.

PROPOSITION 8.61. *Let G be a finite group acting freely on the left on a Hausdorff space Y. Then the action is a covering space action.*

PROOF. Let $y \in Y$ and $g \in G \setminus \{e\}$. We want to find an open set U such that $U \ni y$ and $U \cap gU = \emptyset$.

Since Y is Hausdorff, we can find disjoint open sets $W_g \ni y$ and $V_g \ni gy$. Define $U_g = W_g \cap g^{-1} V_g$, and form $U = \bigcap_{g \in G \setminus \{e\}} U_g$, which is an open neighbourhood of y. We claim U is the desired neighbourhood. Suppose for the sake of contradiction that $z \in U \cap gU$ for some g. It is the case that $z \in U_g$, so that $z \in W_g$ and $z \in g^{-1} V_g$, which implies $z \in W_g \cap V_g = \emptyset$, a contradiction. □

PROPOSITION 8.62. *If Y is a topological space and G is a group acting on Y by means of a covering space action, then the quotient map $q : Y \to Y/G$ is a covering space map.*

PROOF. For all $x \in Y/G$, we can choose a preimage $y \in q^{-1}(Y)$ and an open neighbourhood $U \ni y$ meeting the condition that $U \cap gU = \emptyset$ unless $g = e$. Then $q(U) = W$ is an open neighbourhood of x and $q^{-1}(W) \approx \coprod_{g \in G} gU$. Furthermore, for all $g \in G$, the map $q|_{Ug} : gU \to W$ is an open bijection, i.e., a homeomorphism, as required. □

REMARK 8.63. If a finite group G acts freely on a Hausdorff space Y, then the quotient is Hausdorff. Here is a short argument: Let $q : Y \to X$ be the quotient map, and let x, z be distinct points in X. Let $\tilde{x} \in q^{-1}(x)$ and $\tilde{z} \in q^{-1}(z)$ be points in the inverse images. For each pair of elements $g, h \in G$, we can find disjoint open sets $\tilde{U}_{g,h} \ni g\tilde{x}$ and $\tilde{V}_{g,h} \ni h\tilde{z}$. Now let $\tilde{U} = \bigcap_{g,h \in G} g^{-1} \tilde{U}_{g,h}$ and $\tilde{V} = \bigcap_{g,h \in G} h^{-1} \tilde{V}_{g,h}$. The sets $\tilde{U}$, $\tilde{V}$ are open neighbourhoods of $\tilde{x}$, $\tilde{z}$.

Let $U = q(\tilde{U})$ and $V = q(\tilde{V})$. Since q is a covering space map, it is open and U, V are open neighbourhoods of x, z. Furthermore $q^{-1}(U) = \bigcup_{g\in G} g\tilde{U}$ and $q^{-1}(V) = \bigcup_{h\in G} h\tilde{V}$. It is routine to verify that these are disjoint, so that U, V are disjoint as well.

EXAMPLE 8.64. Let C_2 denote the group $\{\pm 1\}$. This groups acts on the spheres S^n for all n by multiplication of coordinates. This is a free action of a finite group on a Hausdorff space, and so the quotient map is a covering map $q : S^n \to \mathbf{R}\mathrm{P}^n$ and $\mathbf{R}\mathrm{P}^n$ is Hausdorff.

EXAMPLE 8.65. Here is a variation on the previous example. Let r be a natural number and let C_r denote the group $\{1, \zeta, \ldots, \zeta^{r-1}\}$ where ζ is a primitive r-th root of unity. Then C_r acts on $\mathbf{C}^n \setminus \mathbf{0}$ by multiplication, and the action restricts to an action on the submanifold S^{2n-1} consisting of n-tuples $(z_1, \ldots, z_n)$ such that $|z_1|^2 + \cdots + |z_n|^2 = 1$.

EXAMPLE 8.66. There are covering space actions by infinite groups, of course. For instance, $\mathbf{Z}$ acts on $\mathbf{R}$ by translation $(m, x) \mapsto m + x$. The quotient map is $q : \mathbf{R} \to S^1$ (up to homeomorphism).

EXAMPLE 8.67. More strikingly, $\mathbf{Z}$ acts on $\mathbf{R}^2 \setminus \mathbf{0}$ by $m \cdot (x, y) = (2^m x, 2^{-m} y)$. This is a covering space map, but the quotient space X is not Hausdorff.

The space X, like $\mathbf{R}^2 \setminus \mathbf{0}$, is locally homeomorphic to $\mathbf{R}^2$ since the quotient map is a local homeomorphism, but since X is not Hausdorff, it is not a manifold.

The space Y might not be connected, and elements of the group G can permute the components of Y. If we restrict our attention, however, to the case of connected (and locally path connected) Y, however, we see that the theory of covering space actions is really just the theory of deck transformations of normal covering spaces.

PROPOSITION 8.68. *Suppose Y is a nonempty connected, locally path connected space, and G acts on Y by a covering space action. Write $q : Y \to X$ for the quotient map. Then $q : Y \to X$ is a normal covering space. The group of deck transformations,* $\mathrm{Aut}_X(Y)$*, is precisely G.*

PROOF. We have already seen that q is a covering space. Since Y has these properties, X is connected and locally path connected.

The action of G on Y is certainly by automorphisms of Y over $X = Y/G$, essentially by construction. That is, G is a subgroup of the group $\mathrm{Aut}_X(Y)$ of deck transformations.

To show $q : Y \to X$ is a normal covering space, it suffices by Proposition 8.58 to show that the deck transformations act transitively on one fibre. Let $x \in X$ and write $F_x = q^{-1}(x)$ and choose $y \in Y$. By construction of the quotient, $F_x = \{gy\}_{g\in G}$, and clearly G acts transitively on this fibre, and therefore so does $\mathrm{Aut}_X(Y)$ which is a group containing G.

Now to show that $\mathrm{Aut}_X(Y)$ is not bigger than G, suppose we have some deck transformation $h : Y \to Y$. Let $h(y) = g(y)$ for some $g \in G$–here we write $g(y)$ for gy—, then consider $g^{-1}h(y)$, which is a deck transformation sending y to y. Since the deck transformations act freely on Y by Proposition 8.45, it must be the case that $g^{-1}h = \mathrm{id}_Y$, so that $h \in G$. □

REMARK 8.69. We are somewhat hamstrung in Proposition 8.68 by our decision to constrain normal covering spaces to the case where Y is connected. One could say a covering space $f : Y \to X$ is normal if and only if the deck transformations act transitively on all the fibres (whether or not Y is connected). Normal transformations in this sense will not correspond to normal subgroups of $\pi_1(X, x)$, but Proposition 8.68 still applies to them.

THEOREM 8.70. *Let G be a group acting on a connected and locally path connected space Y by means of a left covering space action and let $q : Y \to X$ be the quotient map. Let $y \in Y$ and let $x = q(y)$. Then there is a short exact sequence of groups*

$$\{e\} \to \pi_1(Y, y) \xrightarrow{q_*} \pi_1(X, x) \to G \to \{e\}.$$

In other words, q_* is injective, the image of q_* is a normal subgroup, and $G \cong \pi_1(X, x)/\operatorname{im} q_*$.

PROOF. Everything here has been proved already: q_* is injective because q is a covering space map. The image of q_* is normal in $\pi_1(X, x)$ because $q : Y \to X$ is a normal covering space. The quotient $\pi_1(X, x)/\operatorname{im} q_*$ is the group of deck transformations, which is also G. □

EXAMPLE 8.71. One very useful case of the theorem is when Y is simply connected. Then $\pi_1(X, x) \cong G$, and Y is the universal cover of X.

For instance, we see immediately that if $n > 1$, then $q : S^n \to \mathbf{R}\mathrm{P}^n$ is a universal covering space map and $\pi_1(\mathbf{R}\mathrm{P}^n, x) \cong \mathbf{Z}/(2)$.

EXAMPLE 8.72. More exotic calculations are quite possible. For instance, let $\alpha : \mathbf{R}^2 \to \mathbf{R}^2$ be the translation $\alpha(x, y) = (x, y+1)$, and let $\beta : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation $\beta(x, y) = (x+1, 1-y)$. The group G generated by these two transformations acts on $\mathbf{R}^2$ by means of a covering space action. We can determine the structure of G precisely

$$G = \langle \alpha, \beta | \alpha\beta\alpha = \beta \rangle.$$

The quotient $\mathbf{R}^2/G$ is the Klein bottle, K. Therefore $\pi_1(K, x) \cong G$ (for any choice of basepoint $x \in K$).

We've seen that quotient constructions give us normal covering spaces. The next proposition says that conversely, every normal covering space is a quotient by a covering space action by the deck transformations.

PROPOSITION 8.73. *Let X be a space that is connected and locally path connected. Let $f : Y \to X$ be a normal covering space, so that in particular, Y is connected and nonempty. Write G for the group of deck transformations* $\mathrm{Aut}_X(Y)$*. Then the action of G on Y is a covering space action and $f : Y \to X$ is a quotient for this action.*

PROOF. Let $y \in Y$ be a point, and write $x = f(y)$. Since f is a covering space map, there exists an open set $U \ni x$ trivializing f. We may suppose, for convenience, that U is path connected. Write F_x for the fibre $f^{-1}(x)$ and for each $y' \in F_x$, let $V_{y'}$ denote the path component of $f^{-1}(U)$ containing y'. The open sets $\{V_{y'}\}_{y' \in F_x}$ are disjoint. Each $V_{y'}$ is mapped homeomorphically to U by f.

Then the group G of deck transformations acts on $f^{-1}(U)$, and consequently it must act by permuting the $V_{y'}$s. Since the covering space is normal, the action of G on F_x is free and transitive (Propositions 8.58, 8.45) and therefore the action on the set $\{V_{y'}\}_{y' \in F_x}$s is free and transitive. Therefore, for all $g \in G$, if $gV_y \cap V_y \neq \emptyset$, we see $g = e$. In summary, the action is a covering space action.

Finally, let us verify that indeed $f : Y \to X$ is a quotient map. There is no real surprise here. Certainly the points in X correspond to G-orbits of points in Y, so the underlying set is correct.

The map f is continuous. It suffices to prove that $U \subset X$ is open whenever $f^{-1}(U)$ is open in Y, but f is a covering space map and $f(f^{-1}(U)) = U$, so the argument is complete. □

REMARK 8.74. This argument shows that the action by deck transformations is a covering space action whether or not the covering space is normal. In the non-normal case, however, the quotient is not X.

6.1. Quotients by subgroups. Suppose G acts on Y by a covering space action, and H is a subgroup of G. Then H acts on Y, also by a covering space action, and we may form the quotient space Y/H. One may wonder whether Y/H is a covering space of Y/X. At least in good circumstances, this turns out to be the case.

PROPOSITION 8.75. *Suppose*

$$\begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr, "f\circ g"'] & & Z \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

is a commutative diagram of spaces in which g and $f \circ g$ are covering space maps and where g is surjective, and in which X is locally path connected. Then f is a covering space map.

PROOF. Suppose $x \in X$. Let U be a path connected open neighbourhood of x over which $f \circ g$ trivializes. We wish to show that $f^{-1}(U)$ is also a disjoint union of spaces mapping to U by homeomorphisms.

Write $(f \circ g)^{-1}(U) = \coprod_{j \in J} V_j$, where $f \circ g|_{V_j} V_j \to U$ is a homeomorphism.

For each $j \in J$, the map $g|_{V_j}$ is open, being a restriction of g to the open set V_J, continuous and injective, since $f \circ g|_{V_j}$ is injective. Therefore $g|_{V_j}$ is a homeomorphism $V_j \to g(V_j)$.

It is elementary to check that $f^{-1}(U) = \bigcup_{j \in J} g(V_j)$. It will be enough to show that $\bigcup_{j \in J} g(V_j)$ is a disjoint union of some of the $g(V_j)$s, since each $g(V_j)$ is homeomorphic to $V_j \approx U$, and $f|_{g(V_j)} : V_j \to U$ is a homeomorphism by a diagram chase.

Therefore it suffices to show that for all $j, j' \in J$, either $g(V_j) = g(V'_j)$ or $g(V_j) \cap g(V_{j'}) = \emptyset$. Let j, j' be a pair of indices and suppose $z_0 \in g(V_j) \cap g(V_{j'})$. Let $z \in g(V_j)$ be a point. We wish to show that $z \in g(V_{j'})$.

There is a path γ in $g(V_j)$ from z_0 to z. Let $x_0 = f(z_0)$ and $x = f(z)$. Using the homeomorphisms over U, we may map γ to a path γ' in $g(V_{j'})$, which also begins at z_0 and ends at some point, z'. We have two paths, γ and γ', each of which starts at z_0, and each of which lies over the same path $f(\gamma)$ in U.

Choose some y_0 in Y lying over z_0, and lift γ, γ' to paths $\tilde{\gamma}, \tilde{\gamma}'$ starting at y_0. By applying uniqueness of path-lifting to $f \circ g$ and $f(\gamma) = f(\gamma')$, we see that $\tilde{\gamma} = \tilde{\gamma}'$, so $\gamma = \gamma'$ and in particular, $z = z'$. □

PROPOSITION 8.76. *Let G act on a connected, locally path connected space Y by a covering space action. Write $X = Y/G$. Choose a basepoint $x \in X$, and let H be a subgroup of G. Then $Y/H \to Y/G$ is a covering space whose fibre is isomorphic to $H \backslash G$ as a set with right $\pi_1(X, x)$-action.*

6.2. Proper actions. Here is a small note about when a quotient of a Hausdorff space is Hausdorff.

DEFINITION 8.77. Let Y be a locally compact Hausdorff space and suppose $\alpha : Y \times G \to Y$ is a right G action. We say that the action of G on Y is *proper* if the following holds: for all compact subsets K of Y, the set of $g \in G$ such that $K \cap Kg \neq \emptyset$ is finite.

REMARK 8.78. A map $f : A \to B$ of locally compact Hausdorff spaces is said to be *proper* if $f^{-1}(K)$ is compact whenever K is a compact subset of B. Associated to the action map of spaces above, there is a map $\Phi : Y \times G \to Y \times Y$ given by $\Phi(y, g) = (y, yg)$. It is an exercise to verify that the action is proper in the sense above if and only if Φ is proper in the sense defined here.

REMARK 8.79. An action by a finite group is always proper.

PROPOSITION 8.80. *Suppose $\alpha : Y \times G \to Y$ is a free proper action by a group G on a locally compact Hausdorff space Y. Then α is a covering space action and the quotient space $X = Y/G$ is locally compact and Hausdorff.*

Write $q : Y \to X$ for the quotient map.

PROOF. First we show the action is a covering space action. Let $y \in Y$ be a point and choose an open $U \ni y$ with compact closure. By virtue of the proper hypothesis on the action, there are only finitely many $g \in G$ such that U and gU intersect. Denote the set of such g by A. For each $g \in A$, it is possible to find disjoint open neighbourhoods $W_g \ni y$, $V_g \ni yg$. Then form $U_g = U \cap W_g \cap V_g g^{-1}$, which is open. Note that $U_g \cap U_g g = \emptyset$.

Form $U' = \cap_{g \in A} U_g$. We leave it as an exercise to establish that $U' \cap U' g = \emptyset$ for all $g \in G$.

Being locally compact is a local condition on a space: i.e., X is locally compact if and only if every point has a locally compact neighbourhood. Therefore, if $q : Y \to X$ is a surjective covering space, then Y is locally compact if and only if X is locally compact.

It remains to show that the quotient is Hausdorff. Suppose x_1 and x_2 are two distinct points in X and let y_1 and y_2 be points in Y such that $q(y_i) = x_i$. We can choose an open U_1 containing y_1 and U_2 containing y_2 so that each has compact closure. In particular, only finitely many of the translates gU_2 meet U_1. As before, we can replace U_1 and U_2 by smaller open neighbourhoods of y_1 and y_2, now having the property that U_1 and gU_2 are disjoint for all g in G. We observe that this implies that $gU_1 \cap g'U_2 = \emptyset$ for all $g, g' \in G$, since $x \in gU_1 \cap g'U_2$ implies $xg^{-1} \in U_1 \cap g'g^{-1}U_2$.

Let $V_1 = \bigcup_{g \in G} gU_1$ and $V_2 = \bigcup_{g \in G} gU_2$. These are two open sets, and they are *saturated* in that $q^{-1}q(V_1) = V_1$ and similarly for V_2. In particular, $q(V_1)$ is an open neighbourhood of x_1 and $q(V_2)$ is an open neighbourhood of x_2 and their intersection is empty, as required. $\square$

CHAPTER 9

Quotients and group actions

1. More about quotients

Recall that a function $q : X \to Y$ between topological spaces is a *quotient map* if q is surjective and Y has the finest topology such that q is continuous. Equivalently, the topology on Y is is such that $V \subset Y$ is open if and only if $q^{-1}(V)$ is open.

PROPOSITION 9.1. *Suppose $q_1 : X \to Y$ and $q_2 : Y \to Z$ are quotient maps, then $q_2 \circ q_1$ is a quotient map.*

DEFINITION 9.2. Let $q : X \to Y$ be a function. Say that a subset $U \subset X$ is *saturated for f* if $U = f^{-1}(f(U))$.

REMARK 9.3. It is elementary to verify that if $V \subset Y$ is a subset, then $q^{-1}(V)$ is saturated. This is because $qq^{-1}(V) = V$ for any subset V of Y.

Given any subset $U \subset X$, we can *saturate* U by taking $q^{-1}q(U)$. If U is not saturated, then $q^{-1}q(U)$ is strictly larger than U.

PROPOSITION 9.4. *Let $q : X \to Y$ be a surjective (continuous) map. Then the following are equivalent:*

(1) $q : X \to Y$ is a quotient map;
(2) For all saturated open sets $U \subset X$, the image $q(U)$ is open.

PROOF. In one direction, suppose q is a quotient map. Suppose $U \subset X$ is saturated and open. Then $q^{-1}q(U) = U$ is open in X, so $q(U)$ is open in Y.

In the other direction, we assume the images of saturated opens are open and wish to prove q is a quotient map. Let $V \subset Y$ be a subset such that $q^{-1}(V)$ is open. The set $q^{-1}(V)$ is saturated. By hypothesis, $qq^{-1}(V)$ is open, but $qq^{-1}(V) = V$. We have proved that V is open if $q^{-1}(V)$ is open. This proves that the map q is a quotient map. $\square$

What follows is a surprisingly difficult theorem. It is due to J. H. C. Whitehead, from 1948.[1]

THEOREM 9.5. *Suppose $q : X \to Y$ be a quotient map and Z is a locally compact Hausdorff space. Then $q \times \mathrm{id}_Z : X \times Z \to Y \times Z$ is a quotient map.*

The hypothesis of local compactness cannot be dropped.

[1]The version he proved is slightly stronger

PROOF. The map $q \times \mathrm{id}_Z$ is continuous and surjective. It remains to show that $(q \times \mathrm{id}_Z)(W)$ is open whenever W is a saturated open set.

Let W be a saturated open subset of $X \times Z$ and let $(y_0, z_0) \in (q \times \mathrm{id}_Z)(W)$. We will prove that $(q \times \mathrm{id}_Z)(W)$ is open in $Y \times Z$ by constructing an open neighbourhood around (y_0, z_0) contained in $(q \times \mathrm{id}_Z)(W)$.

Choose an arbitrary $x_0 \in q^{-1}(y_0)$. Then we can find a neighbourhood of (x_0, z_0) of the form $A \times B$, where $A \subset X$ is open and $B \subset Z$ is open. Because Z is locally compact and Hausdorff, we can further restrict to an open neighbourhood V of z_0 such that $\bar{V} \subset B$ and $\bar{V}$ is compact. We concentrate on $\bar{V}$.

Let U denote the set

$$U = \{x \in X \mid \{x\} \times \bar{V} \subset W\}.$$

First, we observe that $x_0 \in U$: in fact $A \times \bar{V} \subset A \times B \subset W$.

Second, we claim that U is saturated for the map $q : X \to Y$. Suppose $x \in U$ and $x' \in X$ satisfy $q(x) = q(x')$. For all $z \in \bar{V}$, it is the case that $(x, z) \in W$. Since W itself is saturated, it is also the case that $(x', z) \in W$. Therefore $x' \in U$, establishing the claim.

Third, we show U is open. Suppose $x \in U$. Then $\{x\} \times \bar{V}$ is a compact subset of $X \times Z$ and W is an open neighbourhood of this compact subset. By the generalized Tube Lemma, we can find some open sets $D \subset X$ and $E \subset Z$ such that $\{x\} \times \bar{V} \subset D \times E \subset W$. Choose any $x' \in D$ and any $z \in \bar{V}$, then $(x', z) \subset W$. This shows that $x' \in U$, so that D is an open neighbourhood of x in U. This shows that U is an open set.

We have therefore found two open sets $U \subset X$ and $V \subset Z$ such that

- $(x_0, z_0) \in U \times V$;
- U is saturated for q;
- $U \times V \subset W$.

Apply $(q \times \mathrm{id}_Z)(U \times V)$ to get $q(U) \times Z$, which is an open subset of $Y \times Z$ containing (y_0, z_0) and contained in $(q \times \mathrm{id}_Z)(W)$. Since (y_0, z_0) was arbitrary, this shows that $(q \times \mathrm{id}_Z)(W)$ is open, as required. $\square$

2. Topological Groups

DEFINITION 9.6. A *topological group* is a space G equipped with a continuous multiplication map $m : G \times G \to G$, an identity element $e \in G$, and a continuous inversion $i : G \to G$, all satisfying the usual axioms for a group:

(1) The operation m is associative: for all $a, b, c \in G$, the equation $m(a, m(b, c)) = m(m(a, b), c)$ holds.
(2) The element e is an identity: for all $a \in G$, the equations $m(a, e) = a = m(e, a)$ hold.
(3) The map i is an inversion: for all $a \in G$, the equations $m(a, i(a)) = e = m(i(a), a)$ hold.

EXAMPLE 9.7. Any group (not topological) can be given the discrete topology. This makes it a *discrete group*. This is the usual topology to put on a finite group.

EXAMPLE 9.8. The groups $\mathrm{GL}_n(\mathbf{R})$ and $\mathrm{GL}_n(\mathbf{C})$ are subsets of $\mathbf{R}^{n^2}$ and $\mathbf{C}^{n^2}$ respectively. If we give $\mathrm{GL}_n(\mathbf{R})$ and $\mathrm{GL}_n(\mathbf{C})$ the subspace topology, then they become topological groups.

REMARK 9.9. If G is a topological group and H is a subgroup of G, then H inherits a topological group structure. The most common instances of this are when H is an open subgroup of G and, especially, when H is a closed subgroup of G.

EXAMPLE 9.10. There are many closed subgroups of $\mathrm{GL}_n(\mathbf{R})$ and $\mathrm{GL}_n(\mathbf{C})$. They are all topological groups with the induced topology. Write A^T for the transpose of a matrix and A^* for the hermitian conjugate (i.e., the complex conjugate of the transpose of A). Each of the following determines a subgroup of $\mathrm{GL}_n(\mathbf{R})$ or $\mathrm{GL}_n(\mathbf{C})$ as appropriate:

(1) $O_n = \{A \in \mathrm{GL}_n(\mathbf{R}) \mid AA^T = I_n\}$;
(2) $SO_n = \{A \in \mathrm{GL}_n(\mathbf{R}) \mid AA^T = I_n, \det(A) = 1\}$;
(3) $U_n = \{A \in \mathrm{GL}_n(\mathbf{C}) \mid AA^* = I_n\}$;
(4) $SU_n = \{A \in \mathrm{GL}_n(\mathbf{C}) \mid AA^* = I_n, \det(A) = 1\}$.

They include O_n (the orthogonal groups), SO_n (the special orthogonal groups), U_n (unitary) and SU_n (special unitary

EXAMPLE 9.11. The examples above include S^1 in two different ways: S^1 may be identified with SO_2 (rotations of $\mathbf{R}^2$) and also with U_1 (complex numbers of unit length).

3. Group actions

DEFINITION 9.12. Given a topological group G and a space X, a *left G-action* on X is a map $\alpha : G \times X \to X$ such that:

(1) $\alpha(e, x) = x$ for all $x \in X$.
(2) $\alpha(h, \alpha(g, x)) = \alpha(hg, x)$ for all $g, h \in G$ and $x \in X$.

NOTATION 9.13. As well as $\alpha(g, x)$, different notations are used for group actions. For instance, $g \cdot x$ is common notation for $\alpha(g, x)$, or even just gx.

REMARK 9.14. A *right G-action* can be defined similarly. The key difference is in the composition rule. For a right action, $\alpha(h, \alpha(g, x)) = \alpha(gh, x)$. It is common to write right actions as $x \cdot g$ or x^g. Then you have the formulas $(x \cdot g) \cdot h = x \cdot (gh)$ or $(x^g)^h = x^{gh}$.

REMARK 9.15. A left G-action on X can be converted into a right-G-action by letting $x \cdot g = g^{-1} \cdot x$. This means that theorems and definitions for left G-actions always have counterparts for right G-actions. As a matter of practice, it is best to be clear and unambiguous about whether one is using a left- or a right-G action.

REMARK 9.16. All ordinary groups may be viewed as topological groups, by giving them the discrete topology. This means that everything we say here applies to ordinary (discrete) groups acting on topological spaces.

DEFINITION 9.17. If $\alpha : G \times X \to X$ is a left G-action and $x \in X$, then the *stabilizer* of $x \in X$ is the subgroup $G_x = \{g \in G \mid g \cdot x = x\}$. The *orbit* of x is $\{y \in X \mid \exists g \in G, gx = y\}$.

A group action is called:

- *faithful* if the intersection of the stabilizers is trivial (in which case, every $g \in G \setminus \{e\}$ acts nontrivially on some element of X). For instance, $\mathrm{GL}_n(\mathbf{R})$ acts faithfully on $\mathbf{R}^n$.

- *free* if the stabilizer of each $x \in X$ is trivial (in which case, every $g \in G \setminus \{e\}$ acts non-trivially on every element of X). For instance, a subgroup H acts freely on G by left-multiplication.
- *transitive* if, for every pair (x, y) of elements in X, there is some $g \in G$ such that $gx = y$. In this case, X consists of a single G-orbit.

The orbit–stabilizer theorem says that the continuous map $G/G_x \to \operatorname{orbit}(x)$ given by sending gG_x to gx is a bijection.

4. Quotients by group actions

CONSTRUCTION 9.18. Suppose $G \times X \to X$ is a left G-action. The notation X/G denotes the set of equivalence classes of elements of X under the relation $x \sim g \cdot x$. There is a surjective map $q : X \to X/G$. Give X/G the quotient topology. We will call the resulting map $q : X \to X/G$ the *quotient map* and X/G the *quotient space*.

Of course, a similar construction applies to right G-actions. In fact, the notation X/G for quotients by left G-actions is unsatisfactory. It ought to be $G\backslash X$.

PROPOSITION 9.19. *Suppose G is topological group acting on a space X. Then the quotient map $q : X \to X/G$ is open.*

PROOF. Let U be an open subset of X. Then $q^{-1}q(U) = \bigcup_{g \in G} g \cdot U$. Since $x \mapsto g \cdot x$ is a homeomorphism of X with itself, the sets $g \cdot U$ are all open, hence $q^{-1}q(U)$ is open. This implies that $q(U)$ is open. □

REMARK 9.20. This contrasts with the case of general quotients in topology, which are not always open. For instance, the quotient map $q : [0,1] \to S^1$ given by collapsing $\{0,1\}$ is not an open map, as you can see by considering the image of $[0, 1/2)$.

EXAMPLE 9.21. Even if G acts freely on a Hausdorff space X, the quotient space X/G may not be Hausdorff. For example, consider the case of $\mathbf{Z}$ (with the discrete topology) acting on $\mathbf{R}^2 \setminus \{(0,0)\}$ by $n \cdot (x, y) = (2^n x, 2^{-n} y)$.

EXAMPLE 9.22. A similar example to the above is that of $\mathbf{R}^\times$ acting on $\mathbf{R}^2 \setminus \{(0,0)\}$ by $\lambda \cdot (x, y) = (\lambda x, \lambda^{-1} y)$.

DEFINITION 9.23. A map $f : X \to Y$ of topological spaces is *proper* if, for all compact sets $K \subset Y$, the subset $f^{-1}(K)$ is compact in X.

APPENDIX A

The compact–open topology

1. Definition

The fundamental problem in this section is how to put a topology on the set of all continuous functions $f : X \to Y$. In good circumstances, which is to say when X is locally compact and Hausdorff, there is a standard default choice, which is the subject of this chapter.

The subject of functional analysis is the study of topologies and metrics on spaces of functions, and we know that functional analysis is not a small topic, so this chapter is not the last word on spaces of functions, by any means.

NOTATION A.1. Let X and Y be topological spaces. Let $K \subset X$ be compact and $U \subset Y$ be open. Write $o(K,U)$ for the set of continuous functions $f : X \to Y$ such that $f(K) \subset U$.

DEFINITION A.2. Let X and Y be topological spaces. Let $\mathscr{C}(X,Y)$ denote the set of continuous functions from X to Y, endowed with the topology generated by open sets of the form $o(K,U)$ as above. This topology is called the *compact-open* topology.

EXAMPLE A.3. Suppose $K \subset X$ is a singleton $\{x\}$. This set is guaranteed to be compact. Let U be any open set in Y. Then the set $o(\{x\},U)$ is the set of continuous f such that $f(x) \in U$.

This has the following consequence for the compact-open topology: if $(f_n) \to f$ is a convergent sequence in $\mathscr{C}(X,Y)$, and if $f(x) \in U$, then each open set $o(\{x\},U)$ has to contain a tail of (f_n). This is equivalent to saying that some tail $f_n(x), f_{n+1}(x), \dots$ is contained in U, or in other other words, that $(f_n(x)) \to f(x)$.

EXAMPLE A.4. Suppose Y is a metric space, and X is Hausdorff. In this case the compact-open topology is also called the *topology of uniform convergence on compact subsets* for the following reason.

A sequence of function (f_n) in $\mathscr{C}(X,Y)$ converges to f in the compact-open topology if and only if, for all compact $K \subset X$ and all $\epsilon > 0$, there exists some $N \in \mathbf{N}$ such that $d(f_n(k), f(k)) < \epsilon$ for all $k \in K$.

The "only if" direction is mostly left as an exercise. Start by proving that if K is a compact Hausdorff space that the compact-open topology on $\mathscr{C}(K,Y)$ is induced by the *uniform metric*: $d(f,g) = \max_{k\in K} d(f(k), g(k))$. Below we will also show that the map determined by restriction of functions $\mathscr{C}(X,Y) \to \mathscr{C}(K,Y)$ is continuous. This suffices to complete the "only if" direction.

Let's do the "if" direction. Suppose $(f_n) \to f$ uniformly on all compact sets. Consider a general open neighbourhood of f. This contains an open neighbourhood of the form

$$W = \bigcap_{i=1}^{n} o(K_i, U_i) \ni f$$

Still to do

because opens like this form a basis.

PROPOSITION A.5. *The construction of* $\mathscr{C}(X,Y)$ *is contravariantly functorial in* X *and covariantly functorial in* Y. *In less technical language, if* $g : X' \to X$ *and* $h : Y \to Y'$ *are continuous function, then precomposition with g and postcomposition with h yields a function*

$$\Phi_{g,h} : \mathscr{C}(X,Y) \to \mathscr{C}(X',Y') \quad f \mapsto h \circ f \circ g$$

and this function is continuous.

PROOF. It is sufficient to show that $\Phi^{-1}_{g,h}(o(K,U))$ is open when K is compact in X' and U is open in Y'. This is $o(g(K'),h^{-1}(U))$, which is open. □

PROPOSITION A.6. *Let* Y *be a topological space, then the map* $Y \to \mathscr{C}(*,Y)$ *sending* y *to the constant function with value* y *is a homeomorphism.*

2. Currying and Uncurrying

Suppose we have a continuous function $f : X \times Y \to Z$. We can *curry* this function to produce a function $\alpha_f : X \to \mathscr{C}(Y,Z)$ defined by

$$\alpha_f(x)(y) = f(x,y).$$

The function $\alpha_f(x) : Y \to Z$ is the composite

$$Y \xrightarrow{i_x} X \times Y \xrightarrow{f} Z$$

and since both functions here are continuous, so is $\alpha_f(x)$.

PROPOSITION A.7. *The function* $\alpha_f : X \to \mathscr{C}(Y,Z)$ *is continuous.*

PROOF. It suffices to prove that $\alpha_f^{-1}(o(K,U)) \subset X$ is open, where $K \subset Y$ is compact and $U \subset Z$ is open.

Explicitly: $\alpha_f^{-1}(o(K,U))$ is the set of all $x \in X$ such that for all $y \in K$, the value $f(x,y) \in U$.

Fix K,U and suppose $x \in \alpha_f^{-1}(o(K,U))$. This implies that $f(\{x\} \times K) \subset U$ in Z, or equivalently $\{x\} \times K \subset f^{-1}(U)$. Then by the generalized tube lemma, there are some open $V \ni x$ and $W \supseteq K$ such that $f(V \times W) \subset U$. In particular, $f(V \times K) \subset U$, which implies that $x \in V \subset \alpha_f^{-1}(o(K,U))$. Since x was arbitrary, $\alpha_f^{-1}(o(K,U))$ is open. □

We can also *uncurry* functions. Suppose $f : X \to \mathscr{C}(Y,Z)$ is a continuous function, then we can define $\beta_f : X \times Y \to Z$ by the formula $\beta_f(x,y) = f(x)(y)$.

PROPOSITION A.8. *With the definition as above, if* Y *is locally compact and Hausdorff, then* β_f *is continuous.*

PROOF. Let U be an open set in Z. We want to show that $\beta_f^{-1}(U)$ is open. To do this, we take $(x,y) \in \beta_f^{-1}(U)$ and show that it has some neighbourhood $W \times V$ contained in $\beta_f^{-1}(U)$.

The element $f(x) \in \mathscr{C}(Y,Z)$ is a continuous function, so that $f(x)^{-1}(U)$ is an open set of Y containing y. Since Y is locally compact and Hausdorff, there is some open V satisfying $y \in$

$V \subset \bar{V} \subset f(x)^{-1}(U)$ such that $\bar{V}$ is compact. Now consider the open set $o(\bar{V}, U)$ in $\mathscr{C}(Y, Z)$. It contains f. Furthermore, the set $W = f^{-1}(o(\bar{V}, U))$ is open, because f is continuous, and it contains x because $f(x)(\bar{V}) \subset f(x)(f(x)^{-1}(U)) = U$. Now if we apply $\beta_f(W \times V)$, we get $f(x')(y')$ where $x' \in W$ and $v' \in V$. Note that $f(x') \in o(\bar{V}, U)$, so that $f(x')(y') \in U$, as required. □

COROLLARY A.9. *Let X, Z be topological spaces and Y a locally compact Hausdorff space. The two constructions of currying and uncurrying yield a natural bijective correspondence*

$$\mathscr{C}(X \times Y, Z) \leftrightarrow \mathscr{C}(X, \mathscr{C}(Y, Z))$$

REMARK A.10. If X and Y are both locally compact Hausdorff, then this is actually a homeomorphism.

COROLLARY A.11. *Let X be a locally compact Hausdorff space. Then the evaluation function*

$$ev\colon \mathscr{C}(X, Y) \times X \to Y$$

given by $ev(f, x) = f(x)$ is continuous.

PROOF. Apply the previous corollary to the bijection

$$\mathscr{C}(\mathscr{C}(X, Y) \times X, Y) \leftrightarrow \mathscr{C}(\mathscr{C}(X, Y), \mathscr{C}(X, Y)).$$

Take the identity function on the right. This corresponds to the evaluation function on the left. □

COROLLARY A.12. *Let X and Z be topological spaces and let Y be a locally compact Hausdorff space. Then the composition function*

$$\mathscr{C}(Y, Z) \times \mathscr{C}(X, Y) \xrightarrow{\circ} \mathscr{C}(X, Z)$$

is continuous.

PROOF. It is equivalent to show the adjoint

$$\mathscr{C}(Y, Z) \times \mathscr{C}(X, Y) \times X \to Z,$$

the map that sends (g, f, x) to $g \circ f(x)$, is continuous.

We can factor this as two evaluation maps

$$\mathscr{C}(Y, Z) \times \mathscr{C}(X, Y) \times X \xrightarrow{\mathrm{id} \times ev} \mathscr{C}(Y, Z) \times Y$$

and

$$\mathscr{C}(Y, Z) \times Y \xrightarrow{ev} Z.$$

Both of these are continuous. □

3. The pointed case

APPENDIX B

Category Theory

1. Objects and morphisms

We generally disregard problems of size, viz. whether or not something is a set.

DEFINITION B.1. A category $\mathbf{C}$ consists of a collection of objects, $\mathrm{ob}\,\mathbf{C}$ and a collection of morphisms $\mathrm{Mor}\,\mathbf{C}$, such that

(1) Every morphism has a *source* in $\mathrm{ob}\,\mathbf{C}$ and a *target* in $\mathrm{ob}\,\mathbf{C}$. A morphism f is often written $f : X \to Y$ or $X \xrightarrow{f} Y$, where X is the source and Y is the target.
(2) For any two objects X and Y, there is a set $\mathrm{Mor}_{\mathbf{C}}(X,Y)$ or $\mathbf{C}(X,Y)$, consisting of precisely those morphisms of $\mathbf{C}$ having source X and target Y.
(3) For any three objects X,Y,Z of $\mathbf{C}$, there is a composition of morphisms

$$\circ : \mathbf{C}(X,Y) \times \mathbf{C}(Y,Z) \to \mathbf{C}(X,Z)$$

and this composition is associative in that $f \circ (g \circ h) = (f \circ g) \circ h$ whenever these composites are defined.
(4) For each object X of $\mathbf{C}$, there exists an *identity morphism* $\mathrm{id}_X \in \mathbf{C}(X,X)$ such that $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_X \circ g = g$ whenever these composites are defined.

REMARK B.2. An easy and standard argument proves that id_X is the unique morphism $X \to X$ with the stated property.

NOTATION B.3. There are categories $\mathbf{Set}, \mathbf{Gr}, \mathbf{Ab}$, of sets, groups, abelian groups, and many other similar categories of objects commonly studied in mathematics. These are generally *large* categories, in that the collection of objects does not form a set.

EXAMPLE B.4. There are also *small* categories, where the collection of objects forms a set, and therefore the collection of morphisms also forms a set (under our hypotheses). For instance, given any partially ordered set S, one can construct a category, also called S, where one regards 'element of' and 'object of' as synonymous, and then declares that $S(a,b) = \emptyset$ if $b < a$ and that $S(a,b)$ consists of one morphism if $a \leq b$.

It is often possible to depict such small categories diagrammatically. It is customary to draw only a subset of all morphisms, and to leave out morphisms that can be inferred from the morphisms and objects drawn. In particular, identity morphisms are seldom drawn.

(1) The standard span:

$$\bullet \longleftarrow \bullet \longrightarrow \bullet$$

(2) The standard cospan:

$$\bullet \longrightarrow \bullet \longleftarrow \bullet$$

(3) The category $\mathbb{N}$ (with the usual order)

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots$$

EXAMPLE B.5. There is a category **Top** of topological spaces where the objects are topological spaces and morphisms are continuous functions. There is also a category of *pointed spaces*, $\mathbf{Top}_\bullet$, where the objects are pairs (X, x_0) where X is a topological space and $x_0 \in X$. The morphisms are the based maps, i.e., $\mathbf{Top}_\bullet((X, x_0), (Y, y_0))$ is the set of continuous $f : X \to Y$ such that $f(x_0) = y_0$.

DEFINITION B.6. Given a category $\mathbf{C}$, a *subcategory*, $\mathbf{D}$ of $\mathbf{C}$ consists of a subcollection $\operatorname{ob}\mathbf{D}$ of $\operatorname{ob}\mathbf{C}$ and a subcollection $\operatorname{Mor}\mathbf{D}$ of $\operatorname{Mor}\mathbf{C}$, containing id_X for all objects X in $\operatorname{ob}\mathbf{D}$, such that $\operatorname{Mor}\mathbf{D}$ is closed under composition.

EXAMPLE B.7. There are many examples of subcategories that arise by restricting the class of objects, but not restricting the morphisms between the objects. For instance, **Ab** is the subcategory of **Gr** where the groups considered are required to be abelian, but given any two abelian groups G, H, one has $\mathbf{Gr}(G, H) = \mathbf{Ab}(G, H)$. In this situation, **Ab** is a *full* subcategory of **Gr**.

EXAMPLE B.8. At the other extreme, it is possible to form subcategories where one considers all the objects, but strictly fewer morphisms. For instance, given a field k, one might consider the category having as objects the collection of finite-dimensional k vector spaces, but where the morphisms are restricted to be isomorphisms. This is a subcategory of the usual category of finite-dimensional k vector spaces and all k linear maps, and it appears in some definitions of algebraic K-theory.

DEFINITION B.9. Given two categories, $\mathbf{C}$ and $\mathbf{D}$, it is possible to form a *product category* $\mathbf{C} \times \mathbf{D}$. The objects in this category are ordered pairs (X, Y) where X is an object of $\mathbf{C}$ and Y is an object of $\mathbf{D}$. The morphisms are also ordered pairs, $(f, g) : (X, Y) \to (Z, W)$ is a morphism in the product category if $f : X \to Z$ is a morphism in $\mathbf{C}$ and $g : Y \to W$ is a morphism in $\mathbf{D}$.

DEFINITION B.10. If $\mathbf{C}$ is a category, and $f : X \to Y$ is a morphism in this category, then we say that f is an *isomorphism* if there exists a morphism $f^{-1} : Y \to X$ such that $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$. It is immediate that id_X is an isomorphism.

REMARK B.11. An isomorphism in **Top** or a related category is generally called a *homeomorphism*

DEFINITION B.12. If $\mathbf{C}$ is a category, and $f : X \to Y$ is a morphism in this category, then we say that f is

(1) a *monomorphism* if, whenever $g, h : Z \to X$ are morphisms, the statement $f \circ g = f \circ h$ implies $g = h$. That is, f is *left cancellable*,
(2) an *epimorphism* if, whenever $g, h : Y \to Z$ are morphisms, the statement $g \circ f = h \circ f$ implies $g = h$. That is, f is *right cancellable*,
(3) a *bimorphism* if it is both a monomorphism and an epimorphism.

DEFINITION B.13. If $\mathbf{C}$ is a category, and $f : X \to Y$ is a morphism in this category, then we say that f is

(1) a *split monomorphism* if there exists a morphism $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$.
(2) a *split epimorphism* if there exists a morphism $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$.

Exercises.

(1) Suppose $f : X \to Y$ is an isomorphism. Prove that f^{-1} is uniquely determined by f.
(2) Prove that the class of isomorphisms in a category has the *two-out-of-three* property, namely: if
$$A \xrightarrow{f} B \xrightarrow{g} C$$
are composable morphisms such that two of f, g and $g \circ f$ are isomorphisms, then so too is the third.
(3) Prove that the class of isomorphisms in a category has the *two-out-of-six* property, namely: if
$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} F$$
are composable morphisms such that $g \circ f$ and $h \circ g$ are isomorphisms, then so too are f, g, h and $h \circ g \circ f$.
(4) Determine the monomorphisms, epimorphisms and bimorphisms in the category of sets.
(5) Give an example in **Top** of a bimorphism that is not an isomorphism.
(6) Let **Haus** denote the full subcategory of Hausdorff topological spaces. Give an example in **Haus** of an epimorphism that is not surjective.

2. Functors and Natural Transformations

DEFINITION B.14. Given two categories $\mathbf{C}$ and $\mathbf{D}$, a *(covariant) functor* $F : \mathbf{C} \to \mathbf{D}$ consists of an assignment
$$F : \mathrm{ob}\,\mathbf{C} \to \mathrm{ob}\,\mathbf{D}$$
and for every pair of objects X, Y in $\mathrm{ob}\,\mathbf{C}$, a function
$$F : \mathbf{C}(X, Y) \to \mathbf{D}(F(X), F(Y))$$
such that

(1) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all object X of $\mathbf{C}$ and
(2) $F(f \circ g) = F(f) \circ F(g)$ wherever $f \circ g$ is defined.

EXAMPLE B.15. Given any category $\mathbf{C}$, there is an identity functor $\mathrm{id}_{\mathbf{C}}$.

DEFINITION B.16. Given two categories $\mathbf{C}$ and $\mathbf{D}$, a *contravariant functor* $F : \mathbf{C} \to \mathbf{D}$ consists of an assignment
$$F : \mathrm{ob}\,\mathbf{C} \to \mathrm{ob}\,\mathbf{D}$$
and for every pair of objects X, Y in $\mathrm{ob}\,\mathbf{C}$, a function
$$F : \mathbf{C}(X, Y) \to \mathbf{D}(F(Y), F(X))$$
such that

(1) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all object X of $\mathbf{C}$ and
(2) $F(f \circ g) = F(g) \circ F(f)$ wherever $f \circ g$ is defined.

REMARK B.17. Warning: contravariant functors reverse the direction of morphisms. Failure to keep adequate track of the variance of functors is the category-theoretical analogue of a sign error in arithmetic. These errors are minor, frustrating and common.

NOTATION B.18. Given a category $\mathbf{C}$, there is an *opposite category*, $\mathbf{C}^{\mathrm{op}}$ having the same collection of objects, but where

$$\mathbf{C}^{\mathrm{op}}(X, Y) = \mathbf{C}(Y, X).$$

One may view a contravariant functor $F : \mathbf{C} \to \mathbf{D}$ as a covariant functor $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$.

EXAMPLE B.19. There are many functors in mathematics that consist largely of forgetting structures. Such functors are often called “forgetful”, but it is difficult to give a precise definition of what this means. Common examples include:

(1) $V : \mathbf{Top}_\bullet \to \mathbf{Top}$, forgetting the basepoint.
(2) $V : \mathbf{Top} \to \mathbf{Set}$, forgetting the topology.
(3) $V : \mathbf{Ab} \to \mathbf{Grp}$, forgetting that the group is abelian.

EXAMPLE B.20. There is a canonical functor $\eta : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Set}$ given by $\eta(X, Y) = \mathbf{C}(X, Y)$. Fixing either X or Y gives rise to functors

(1) $\eta_X : \mathbf{C} \to \mathbf{Set}$,
(2) $\eta^Y : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$.

DEFINITION B.21. Let $F : \mathbf{C} \to \mathbf{D}$ be a functor. We say F is

(1) *full* if, for any two objects X, Y of $\mathbf{C}$, the function $F : \mathbf{C}(X, Y) \to \mathbf{D}(F(X), F(Y))$ is surjective.
(2) *faithful* if, for any two objects X, Y of $\mathbf{C}$, the function $F : \mathbf{C}(X, Y) \to \mathbf{D}(F(X), F(Y))$ is injective.
(3) *essentially surjective* if, for any object Z of $\mathbf{D}$, one can find an object X of $\mathbf{C}$ such that there exists an isomorphism $Z \to F(X)$.

DEFINITION B.22. Given two (covariant) functors $F, G : \mathbf{C} \to \mathbf{D}$, a *natural transformation* $\Psi : F \to G$ consists of a collection of morphisms $\Psi_X : F(X) \to G(X)$, one for each object X of $\mathbf{C}$, such that for any morphism $h : X \to Y$ in the category $\mathbf{C}$, the square

$$\begin{tikzcd} F(X) \arrow[r, "\Psi_X"] \arrow[d, "F(h)"] & G(X) \arrow[d, "G(h)"] \\ F(Y) \arrow[r, "\Psi_Y"] & G(Y) \end{tikzcd}$$

commutes, which is to say: $G(h) \circ \Psi_X = \Psi_Y \circ F(h)$.

REMARK B.23. A similar definition of *natural tranformation* can be made if F and G are both contravariant. The details are left to the reader.

REMARK B.24. The word "natural" is often applied to morphisms between objects in categories. It should be used only to apply to a morphism that is part of a, possibly implicit, natural transformation. If the morphisms Ψ_X are all of a certain type, for instance all isomorphisms or all inclusions or all homotopy equivalences, then Ψ_X may be said to be a natural isomorphism or a natural inclusion or natural homotopy equivalence as appropriate.

EXAMPLE B.25. Fix a field k. Let $k\mathbf{Vect}$ denote the category of k vector spaces and all linear maps between them. Then there is a contravariant functor sending $f : V \to W$ to $f^* : W^* \to V^*$, where $V^* = \mathrm{Hom}_k(V, k)$ and f^* is the evident map $\mathrm{Hom}_k(W, k) \to \mathrm{Hom}_k(V, k)$ given by post-composing with f.

There is a covariant functor sending $f : V \to W$ to $f^{**} : V^{**} \to W^{**}$ given by applying V^* twice. That is, V^{**} is the k vector space of linear functionals on the k vector space of linear functionals on V. There is a natural transformation $e : \mathrm{id}_{\mathbf{Vect}} \to (\cdot)^{**}$ given by a collection of k-linear maps $e_V : V \to V^{**}$ given by defining $e_V(x)$, where $x \in V$, to be the functional sending $\psi \in V^*$ to $\psi(x)$.

At least if one assumes the Axiom of Choice, the map $e_V : V \to V^{**}$ defined above is a natural inclusion. If one restricts to the full subcategory of finite dimensional k vector spaces, then e is a natural isomorphism, but if V is not finite dimensional, then $e_V : V \to V^{**}$ is not an isomorphism.

DEFINITION B.26. If $F : \mathbf{C} \to \mathbf{D}$ is a functor, then we say F is an *equivalence of categories* if there exists a functor $G : \mathbf{D} \to \mathbf{C}$ and natural isomorphisms $\Phi : G \circ F \to \mathrm{id}_{\mathbf{C}}$ and $\Psi : F \circ G \to \mathrm{id}_{\mathbf{D}}$.

REMARK B.27. In contrast to the case of isomorphisms, the functor F is not sufficient to determine G, Ψ and Φ uniquely. The notion of "isomorphism of categories", where $G \circ F$ and $F \circ G$ are required to be identity functors, is not particularly common.

REMARK B.28. In the presence of a sufficiently strong version of the Axiom of Choice, a functor is an equivalence of categories if an and only if it is full, faithful, and essentially surjective.

EXAMPLE B.29. Let **Fin** denote the category of finite sets. This category is not small. Let **N** denote the full subcategory of sets $\{\emptyset, \{1\}, \{1, 2\}, \ldots\}$. Then $\mathbf{N} \to \mathbf{Fin}$ is an equivalence of categories. In this situation, one says that **N** is a *small skeleton* for **Fin**.

REMARK B.30. If one restricts attention to small categories, then one can define a "category of categories", but as we have remarked, the notion of isomorphism one gets is not generally useful. It is better to incorporate the natural transformations and form a "2-category" of small categories, a structure having objects (categories), morphisms (functors), and morphisms of morphisms (natural tranformations). We will not pursue this further here.

Exercises.

(1) Let $F : \mathbf{C} \to \mathbf{D}$ be a functor. Show that F preserves isomorphisms and split mono- and epimorphisms. Show by example that it need not preserve monomorphisms or epimorphisms that are not split.

3. Groupoids

DEFINITION B.31. A *groupoid* is a small category $\mathscr{G}$ in which every morphism is an isomorphism.

NOTATION B.32. If a, b are two objects in a groupoid $\mathscr{G}$, we will write $\mathscr{G}(a, b)$ to denote the set $\mathrm{Mor}_{\mathscr{G}}(a, b)$.

EXAMPLE B.33. You should think of a groupoid $\mathscr{G}$ as an algebraic thing, rather like a group. The starring role is played by the morphisms in $\mathscr{G}$. The main difference between groups and groupoids is that if g_1, g_2 are morphisms in a groupoid $\mathscr{G}$, their composition $g_2 \circ g_1$ may not be defined.

PROPOSITION B.34. *Let $\mathscr{G}$ be a groupoid and a an object of $\mathscr{G}$. Composition in the groupoid endows $\mathscr{G}(a, a)$ with the structure of a group.*

The proof is an exercise.

EXAMPLE B.35. If a groupoid $\mathscr{G}$ has a unique object $*$, then the data of the groupoid is really just $\mathscr{G}(*, *) = \mathrm{Mor}_{\mathscr{G}}(*, *)$. This is a group (Proposition B.34) and contains all the information of $\mathscr{G}$. We identify the two concepts: “Group” and “groupoid with one object”.

PROPOSITION B.36. *Let $\mathscr{G}$ be a groupoid, and let a, b be two objects and suppose $\mathscr{G}(a, b)$ is not empty. The group $\mathscr{G}(a, a)$ acts freely and transitively on the right of $\mathscr{G}(a, b)$ by composition:*

$$(a \xrightarrow{f} b) \cdot (a \xrightarrow{g} a) = a \xrightarrow{f \circ g} b$$

and similarly the group $\mathscr{G}(b, b)$ acts freely and transitively on the left:

$$(b \xrightarrow{h} b) \cdot (a \xrightarrow{f} b) = a \xrightarrow{h \circ f} b.$$

The two actions commute, in that $h \cdot (f \cdot g) = (h \cdot f) \cdot g$.

PROOF. The actions are well defined and commute by virtue of the rules of composition of morphisms in categories. We leave the details of this to the reader.

What we will establish is that the actions are free and transitive. It will suffice to show this for the right action, the other case being similar.

Recall that a (right) action of a group $\mathscr{G}(a, a)$ on a set $\mathscr{G}(a, b)$ is *free* if $f \cdot g = f$ implies g is the identity element. In our case, this is easy to see because f has an inverse morphism and

$$f \circ g = f \Rightarrow f^{-1} \circ (f \circ g) = f^{-1} \circ f \Rightarrow g = \mathrm{id}_a.$$

Recall that a (right) action of a group $\mathscr{G}(a, a)$ on a set $\mathscr{G}(a, b)$ is *transitive* if, for all f_1, f_2, there exists $g \in \mathscr{G}(a, a)$ such that $f_1 \cdot g = f_2$. Again, in our case, the groupoid structure implies this. For a given f_1, f_2, define $g = f_1^{-1} f_2 \in \mathscr{G}(a, a)$. Then

$$f_1 \cdot g = f_2,$$

as required. □

NOTATION B.37. A set such as $\mathscr{G}(a, b)$ equipped with a free transitive right action by a group $\mathscr{G}(a, a)$ is called a *right torsor* for that group. A *left torsor* is defined similarly. A set having commuting left- and right-torsor structures for two groups is called a *bitorsor*.

It is a general feature of bitorsors that the groups acting on the left and right must be isomorphic. Here we may prove this directly.

PROPOSITION B.38. *Suppose $\mathscr{G}$ is a groupoid and $f : a \to b$ is a morphism in $\mathscr{G}$. Then the operation*

$$\phi_f : \mathscr{G}(b,b) \to \mathscr{G}(a,a), \qquad \phi_f(h) = f^{-1} \circ h \circ f$$

is an isomorphism of groups, with inverse $\phi_{f^{-1}}$.

One proves that ϕ_f respects composition, and that $\phi_f \circ \phi_{f^{-1}}$ and $\phi_{f^{-1}} \circ \phi_f$ are identities.

REMARK B.39. Propositions B.34, B.36 and B.38 impose very strict conditions on groupoids. That is: a groupoid is a set of objects a, each carrying with it the information of a group $\mathscr{G}(a,a)$ of self maps. If a, b are two objects in $\mathscr{G}$ then there are two possibilities:

(1) There are no morphisms $a \to b$. Then $\mathscr{G}(a,a)$ and $\mathscr{G}(b,b)$ are not related.
(2) There is a morphism $f : a \to b$. Every such morphism yields an isomorphism $\mathscr{G}(b,b) \to \mathscr{G}(a,a)$. The set of morphisms $\mathscr{G}(a,b)$ is a torsor for the groups $\mathscr{G}(a,a)$ and for $\mathscr{G}(b,b)$. In particular, if f is a specified morphism, then every other such morphism can be arrived at uniquely as $f \cdot g$ and also uniquely as $h \cdot f$. Note that, in contrast to $\mathscr{G}(a,a)$ and $\mathscr{G}(b,b)$, the set $\mathscr{G}(a,b)$ does not need to have a distinguished 'identity' element, so it does not form a group.

NOTATION B.40. Say that a groupoid $\mathscr{G}$ is *connected* if for all $a, b \in \operatorname{ob}\mathscr{G}$, the set $\mathscr{G}(a,b)$ is not empty. We saw that in a connected groupoid that all the groups $\mathscr{G}(a,a)$ are isomorphic. In a disconnected groupoid, that need not be the case.

3.1. Morphisms of groupoids.

DEFINITION B.41. If $\mathscr{G}_1$ and $\mathscr{G}_2$ are two groupoids, then a *morphism of groupoids* $\phi : \mathscr{G}_1 \to \mathscr{G}_2$ is a functor. This definitions is equivalent to saying that ϕ consists of two functions

(1) a function ϕ from the objects of $\mathscr{G}_1$ to the objects of $\mathscr{G}_2$;
(2) a function (also denoted ϕ) from the morphisms of $\mathscr{G}_1$ to the morphisms of $\mathscr{G}_2$.

and these satisfy the conditions

(1) if $f : a \to b$ is an arrow in $\mathscr{G}_1$, then $\phi(f) : \phi(a) \to \phi(b)$ is an arrow in $\mathscr{G}_2$;
(2) if $g \circ f$ is a composition in $\mathscr{G}_1$, then $\phi(g \circ f) = \phi(g) \circ \phi(f)$.

REMARK B.42. You can verify that if $\mathscr{G}_1$ and $\mathscr{G}_2$ are actually groups, this recovers the definition of homomorphism of groups.

REMARK B.43. There is a category of groupoids: the objects are the groupoids and the morphisms are the morphisms of groupoids.

4. Adjoint Functors

DEFINITION B.44. Given two functors $L : \mathbf{C} \to \mathbf{D}$ and $R : \mathbf{D} \to \mathbf{C}$, we say L is *left adjoint* to R and R is *right adjoint* to L if, for any object X of $\mathbf{C}$ and Y of $\mathbf{D}$, there exists a bijection

$$\Psi_{X,Y} : \mathbf{D}(L(X), Y) \to \mathbf{C}(X, R(Y))$$

and such that the bijection Ψ is a natural isomorphism of functors $\mathbf{C}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{Set}$. More explicitly, if $f : X \to X'$ and $g : Y' \to Y$ are morphisms in the appropriate categories, then the square of sets

$$\begin{tikzcd}
\mathbf{D}(L(X'), Y') \arrow[r, "\Psi_{X',Y'}"] \arrow[d] & \mathbf{C}(X', R(Y')) \arrow[d] \\
\mathbf{D}(L(X), Y) \arrow[r, "\Psi_{X,Y}"] & \mathbf{C}(X, R(Y))
\end{tikzcd}$$

commutes.

EXAMPLE B.45. Forgetful functors often have one or both kinds of adjoint. For instance, the forgetful functor $V : \mathbf{Top} \to \mathbf{Set}$ has both a left- and a right-adjoint. The forgetful functor $V : \mathbf{Ab} \to \mathbf{Set}$ has a left adjoint, but no right adjoint.

EXAMPLE B.46. A very important family of adjunctions is modelled on the following one: fix a set X. This gives rise to two functors $\mathbf{Set} \to \mathbf{Set}$; the cartesian product functor $Y \mapsto Y \times X$, and the mapping space functor $Z \mapsto Z^X$, where Z^X is notation for the set of functions $X \to Z$. That these are indeed functors $\mathbf{Set} \to \mathbf{Set}$ is left as an exercise. We assert that they form an adjoint pair, in that there is a natural bijection

$$\mathbf{Set}(Y \times X, Z) \to \mathbf{Set}(Y, Z^X).$$

Verifying this is left to the reader.

EXAMPLE B.47. The previous example has a variant for topological spaces, provided some additional hypothesis is placed on the spaces appearing. For instance, if X is a locally compact Hausdorff space, then there is a natural bijection

$$\mathbf{Top}(Y \times X, Z) \to \mathbf{Top}(Y, \mathscr{C}(X, Z))$$

where $\mathscr{C}(X, Z)$ is the space of continuous functions $X \to Z$ given the compact–open topology.

DEFINITION B.48. Given an adjoint pair of functors $L : \mathbf{C} \to \mathbf{D}$ and $R : \mathbf{D} \to \mathbf{C}$, we can define two natural transformations.

(1) The *unit* of the adjunction $\epsilon : \mathrm{id}_{\mathbf{C}} \to R \circ L$
(2) The *counit* of the adjunction $\eta : L \circ R \to \mathrm{id}_{\mathbf{D}}$.

The unit is formed by letting $\eta_X : X \to R(L(X))$ be the element of $\mathbf{C}(X, R(L(X)))$ corresponding to $\mathrm{id}_{L(X)} \in \mathbf{D}(L(X), L(X))$ under the natural isomorphism of the adjunction. The counit is formed similarly.

REMARK B.49. We continue with the notation of the previous definition. The unit and counit have certain universal properties. In the case of the unit, suppose that there is a morphism $f : X \to R(Y)$ in $\mathbf{C}$. Since L and R are adjoint, the morphism f is equivalent to a unique morphism $g : L(X) \to Y$. This morphism can be written, tautologically, as $\mathrm{id}_{L(X)} \circ g : L(X) \to L(X) \to Y$, which, by adjunction, is equivalent to a factorization $f = R(g) \circ \epsilon_X : X \to R(L(X)) \to R(Y)$.

Dually, any morphism $h : L(X) \to Y$ factors uniquely as $\eta_Y \circ L(i) : L(X) \to L(R(Y)) \to Y$.

REMARK B.50. If $L : \mathbf{C} \to \mathbf{D}$ and $M : \mathbf{D} \to \mathbf{E}$ are two functors, each left adjoint to functors R and S respectively, then $M \circ L$ is left adjoint to $R \circ S$.

PROPOSITION B.51. *Suppose $L, L' : \mathbf{C} \to \mathbf{D}$ are two naturally isomorphic functors and R, R' are right adjoints to L and L'. Then R and R' are naturally isomorphic.*

This result applies in particular in the case where $L = L'$.

5. Diagrams, Limits and Colimits

NOTATION B.52. If $\mathbf{I}$ is a small category and $\mathbf{C}$ is a category, then a functor $D : \mathbf{I} \to \mathbf{C}$ may be called a *diagram*. If, for any morphism $f : i \to j$ in the category $\mathbf{I}$, the morphism $D(f)$ depends only on i and j, then we say the diagram is *commutative*.

EXAMPLE B.53. Not all commonly occurring diagrams are commutative. For instance, pairs of parallel morphisms $X \rightrightarrows Y$ appear often but form a commutative diagram only when the two morphisms agree.

DEFINITION B.54. Given a small category $\mathbf{I}$ and a category $\mathbf{C}$, one can define a category $\mathbf{Fun}(\mathbf{I}, \mathbf{C})$ of $\mathbf{I}$-shaped diagrams. The objects are the functors $D : \mathbf{I} \to \mathbf{C}$, and the morphisms are the natural transformations between them.

DEFINITION B.55. Give a small category $\mathbf{I}$, a category $\mathbf{C}$ and an object X of $\mathbf{C}$, we can form the *constant $\mathbf{I}$-shaped diagram with value X* by $\mathrm{const}_{\mathbf{I}}(X) : \mathbf{I} \to \mathbf{C}$ by sending all objects to X and all morphisms to id_X. In fact, $\mathrm{const}_{\mathbf{I}}$ is a functor $\mathrm{const}_{\mathbf{I}} : \mathbf{C} \to \mathscr{F}(\mathbf{I}, \mathbf{C})$.

DEFINITION B.56. Let $\mathbf{I}$ be a small category and $\mathbf{C}$ a category.

Given an $\mathbf{I}$-shaped diagram D in $\mathbf{C}$, a *limit* of D is an object $\lim D$ of $\mathbf{C}$ and a natural transformation $\Phi : \mathrm{const}_{\mathbf{I}}(\lim D) \to D$ such that for any object X of $\mathbf{C}$ equipped with a natural transformation $\Psi : \mathrm{const}_{\mathbf{I}}(X) \to D$, there is a unique map $u : X \to \lim D$ such that $\Psi = \Phi \circ \mathrm{const}(u)$.

Dually, a *colimit* of an $\mathbf{I}$-shaped diagram D is an object $\operatorname{colim} D$ of $\mathbf{C}$ and a natural transformation $\Phi : D \to \mathrm{const}_{\mathbf{I}}(\operatorname{colim} D)$ such that for any object X of $\mathbf{C}$ equipped with a natural transformation $\Psi : D \to \mathrm{const}_{\mathbf{I}}(X)$, there is a unique map $u : \operatorname{colim} D \to X$ such that $\Psi = \mathrm{const}(u) \circ \Phi$.

REMARK B.57. Strictly speaking a limit or colimit of a diagram encompasses both the object and the natural transformation of functors—which is to say, the morphisms. In practice, one often refers to the object as the limit or colimit, leaving the morphisms implicit.

REMARK B.58. It follows easily from a standard argument that if L and L' are two limits of the same diagram $D : \mathbf{I} \to \mathbf{C}$, then there is a unique isomorphism $f : L \to L'$ in $\mathbf{C}$ such that the diagram

$$\begin{tikzcd} \mathrm{const}_{\mathbf{I}} L \arrow[rr, "\mathrm{const}_{\mathbf{I}}(f)"] \arrow[dr] & & \mathrm{const}_{\mathbf{I}} L' \arrow[dl] \\ & D & \end{tikzcd}$$

commutes. A dual statement applies to colimits.

Since they are unique up to unique isomorphism, one often abuses terminology and speaks of “the limit” or “the colimit” of a diagram.

REMARK B.59. There is another view of limits and colimits that is sometimes useful. Suppose the functor $\mathrm{const}_{\mathbf{I}}$ has a right adjoint ℓ. Then a limit of D is given by the object $\ell(D)$ and the counit map $\mathrm{const}_{\mathbf{I}}\,\ell(D) \to D$.

Dually, if $\mathrm{const}_{\mathbf{I}}$ has a right adjoint colim, the *colimit* of D is the unit map $D \to \mathrm{const}_{\mathbf{I}}\,\mathrm{colim}(D)$.

EXAMPLE B.60. The language used above is technical. In practice, the idea is simple. Let us consider as a category I the standard cospan

$$\bullet \longrightarrow \bullet \longleftarrow \bullet$$

Let $\mathbf{C} = \mathbf{Top}$ be the category of topological spaces. Then the data of an I-shaped diagram D consists of three spaces and two continuous functions $X \to Y \leftarrow Z$.

The constant-diagram functor takes a space W and produces $W \to W \leftarrow W$, where the morphisms are identities. A natural transformation $\mathrm{const}(W) \to D$ is the data of continuous functions $f : W \to X$, $g : W \to Y$ and $h : W \to Z$ such that

$$\begin{tikzcd} W \arrow[r, equal] \arrow[d, "f"] & W \arrow[r, equal] \arrow[d, "g"] & W \arrow[d, "h"] \\ X \arrow[r] & Y & Z \arrow[l] \end{tikzcd}$$

commutes, or, more succinctly

(3)
$$\begin{tikzcd} W \arrow[drr, "h"] \arrow[ddr, "f"'] \arrow[ddrr, dotted, "g"] & & \\ & & X \arrow[d] \\ & Y \arrow[r] & Z. \end{tikzcd}$$

Note further that the dotted arrow is determined by either f or h, and may be omitted.

The space $\lim D$ and the natural transformation amounts to an objcet and morphisms fitting in the following diagram

(4)
$$\begin{tikzcd} \lim D \arrow[r, "h"] \arrow[d, "f"] & X \arrow[d] \\ Y \arrow[r] & Z. \end{tikzcd}$$

This diagram has the property that if W is as in Diagram (3), then there exists a unique map $W \to \lim D$ such that Diagram (5) commutes.

(5)
$$
\begin{tikzcd}
W \arrow[dr, dashed, "g"] \arrow[drr, "h"] \arrow[ddr, "f"'] & & \\
& \lim D \arrow[r] \arrow[d] & X \arrow[d] \\
& Y \arrow[r] & Z.
\end{tikzcd}
$$

This particular kind of limit is called a *fibre product* and is written $X \times_Y Z$. While our definition specifies the limit only up to unique isomorphism, we can easily construct an explicit model for $X \times_Y Z$ in the category of topological spaces. Most usually, let $X \times_Y Z$ consist of the subset of pairs $(x, z) \in X \times Z$ such that the image of x and of z in Y agree. Then endow $X \times_Y Z$ with the coarsest topology (fewest open sets) such that the evident projection maps $X \times_Y Z \to X$ and $X \times_Y Z \to Z$ are both continuous.

It is instructive to consider $X \times_Y Z$ in the following cases:

(1) When Y is a singleton space.
(2) When $X \to Y$ is the inclusion of a subspace.

REMARK B.61. By uniqueness of adjoints and of unit or counit maps, if a limit or colimit of a diagram exists, it is unique up to unique isomorphism.

NOTATION B.62. A category in which all limits can be constructed is *complete* and one in which all colimits can be constructed is *cocomplete*. The following categories are all complete and cocomplete:

(1) **Set**.
(2) **Top** and $\mathbf{Top}_\bullet$.
(3) R-**Mod**.

The full subcategory **Haus** of Hausdorff topological spaces is complete but not cocomplete.

NOTATION B.63. If D is a diagram in **C** consisting of a family of objects $\{X_i\}_{i \in I}$ and no non-identity arrows, then a limit of D is called a *product* of $\{X_i\}_{i \in I}$ and a colimit of D is called a *coproduct* of $\{X_i\}_{i \in I}$. The product of topological spaces is an example of a categorical product, and the disjoint union of topological spaces is an example of a categorical coproduct.

NOTATION B.64. If D is a diagram in **C** of the form

$$
\begin{tikzcd}
& A \arrow[d] \\
B \arrow[r] & C
\end{tikzcd}
$$

then a limit of D is called a *pullback* of D, and often denoted $A \times_C B$.

The dual concept is the *pushout*, a colimit of.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ C & \end{tikzcd}$$

PROPOSITION B.65. *Suppose $F : \mathbf{C} \to \mathbf{C}$ is a functor between complete categories such that F has a left adjoint, L. Suppose further that D is a diagram in $\mathbf{C}$. Let* $\lim D$ *be a limit of D. Then* $F(\lim D)$ *is a limit of $F(D)$.*

Dually, suppose $F : \mathbf{C} \to \mathbf{C}$ is a functor between cocomplete categories such that F has a right adjoint, R. Suppose further that D is a diagram in $\mathbf{C}$. Let $\operatorname{colim} D$ *be a limit of D. Then* $F(\operatorname{colim} D)$ *is a colimit of $F(D)$.*

REMARK B.66. Let $\mathbf{C}$ be a category. Consider the empty diagram D. If $\lim D$ exists, then it is an object $*$ such that all objects X of $\mathbf{C}$ are equipped with a unique morphism $X \to *$. Such an object $*$ is called a *terminal* object of $\mathbf{C}$. Any two terminal objects are isomorphic by a unique isomorphism.

Dually, the colimit of an empty diagram is called an *initial* object; such an object may often be denoted $\varnothing$. If an object is both initial and terminal, then it is called a *zero object*.

Exercises.

(1) The forgetful functor $V : \mathbf{Ab} \to \mathbf{Set}$ has a left adjoint, L. Describe the unit map $\epsilon : S \to V(L(S))$.
(2) Show that $V : \mathbf{Ab} \to \mathbf{Set}$ does not preserve colimits. For instance, consider the colimit of a diagram consisting of two nonzero abelian groups and no nontrivial arrows. Therefore V does not have a right adjoint.
(3) Let R be a ring and let $\mathbf{M}$ denote the category of R-modules and R-linear maps, and let $f : M \to N$ be a morphism in $\mathbf{M}$. Describe the limit of the diagram

$$\begin{tikzcd} & 0 \arrow[d] \\ M \arrow[r, "f"] & N. \end{tikzcd}$$

Express the cokernel of f as the colimit of a diagram.
(4) Consider the forgetful functor $V : \mathbf{Top}_\bullet \to \mathbf{Top}$. Describe a left adjoint to this functor. Prove that V does not have a right adjoint.
(5) Let X be a locally compact Hausdorff space, and consider the adjunction between $\times X$ and $\mathscr{C}(X, \cdot)$ in $\mathbf{Top}$. Describe the counit of this adjunction.

APPENDIX C

p-Norms

In this appendix we assume an extended real line, where ∞ is an element greater than all real numbers; the interval notation $[1,\infty]$ will be used to mean $[1,\infty) \cup \{\infty\}$.

1. The p norms on $\mathbf{R}^n$

Fix an integer $n \geq 1$. When $p \geq 1$ is a real number, we define

$$\|\mathbf{x}\|_p = \Big(\sum_{i=0}^{n} |x_i|^p\Big)^{1/p}.$$

Define

$$\|\mathbf{x}\|_\infty = \sup_i |x_i|.$$

Each of these norms has the following property: given any $\mathbf{x} \in \mathbf{R}^n$ and any $r \in \mathbf{R}$, we have

$$\|r\mathbf{x}\|_p = |r|\|\mathbf{x}\|_p. \tag{6}$$

This will be important later. It is also immediate, for $p \in [1,\infty]$, that $\|\mathbf{x}\|_p = 0$ if and only if $\mathbf{x} = 0$.

Hölder conjugates. For a given real number $p > 1$, the *Hölder conjugate* of p is the number $q > 1$ such that

$$\frac{1}{p} + \frac{1}{q} = 1;$$

this is equivalent to

$$q = \frac{p}{p-1}. \tag{7}$$

Another equivalent formulation is

$$qp - q - p = 0 \tag{8}$$

Observe that 2 is self-conjugate, but no other number is. We also declare the pair $\{1,\infty\}$ to be Hölder conjugates.

PROPOSITION C.1 (Young's Inequality). *Let p, q be a Hölder conjugate pair in $(1,\infty)$ and suppose a, b are nonnegative real numbers, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

with equality if and only if $a^p = b^q$.

EXERCISE C.2. Define

$$f(x) = \frac{x^p}{p} + \frac{b^q}{q} - bx.$$

Using calculus, prove this function has a unique global minimum on $(0,\infty)$ and find that minimum.

PROPOSITION C.3 (Hölder's Inequality). *For a given $p \in [1,\infty]$, having Hölder conjugate q, and any two vectors $\mathbf{x}, \mathbf{y}$ in $\mathbf{R}^n$, one has*

$$\sum_{i=1}^{n} |x_i y_i| \le \|\mathbf{x}\|_p \|\mathbf{y}\|_q.$$

PROOF. When $p = 1$ and $q = \infty$, or vice versa, this amounts to the triangle inequality for the absolute value on $\mathbf{R}^1$.

We therefore assume $1 < p < \infty$. By referring to (6), we see that it suffices to prove the proposition after replacing $\mathbf{x}$ and $\mathbf{y}$ by $r\mathbf{x}$ and $s\mathbf{y}$ where $0 < r$ and $0 < s$, so we may assume that $\|\mathbf{x}\|_p = \|\mathbf{y}\|_q = 1$.

By repeated use of Young's inequality, we obtain the inequality

$$\sum_{i=1}^{n} |x_i y_i| \le \sum_{i=1}^{n} \left(\frac{|x_i|^p}{p} + \frac{|y_i|^q}{q} \right),$$

which is

$$\sum_{i=1}^{n} |x_i y_i| \le \frac{\sum_{i=1}^{n} |x_i|^p}{p} + \frac{\sum_{i=1}^{n} |y_i|^q}{q} = \frac{\|\mathbf{x}\|_p^p}{p} + \frac{\|\mathbf{y}\|_q^q}{q} = \frac{1}{p} + \frac{1}{q} = 1 = \|\mathbf{x}\|_p \|\mathbf{y}\|_q.$$

□

PROPOSITION C.4 (Minkowski Inequality). *Let $p \in [1,\infty]$ and $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$, then*

$$\|\mathbf{x}+\mathbf{y}\|_p \le \|\mathbf{x}\|_p + \|\mathbf{y}\|_p.$$

PROOF. The cases of $p = 1$ and $p = \infty$ reduce immediately to the usual triangle inequality for $\mathbf{R}$.

Assume that $1 < p < \infty$.

Consider a vector $\mathbf{w}$ having i-th coordinate $w_i = |x_i + y_i|^{p-1}$. We calculate

$$\|\mathbf{w}\|_q = \left(\sum_{i=1}^{n} |x_i + y_i|^{qp-q} \right)^{1/q} = \left(\sum_{i=1}^{n} |x_i + y_i|^{p} \right)^{1/q} = \|\mathbf{x}+\mathbf{y}\|_p^{p/q}$$

Now we split up $\|\mathbf{x}+\mathbf{y}\|_p^p$ as follows:

$$\|\mathbf{x}+\mathbf{y}\|_p^p = \sum_{i=1}^{n} |x_i + y_i|^p \le \sum_{i=1}^{n} \left(|x_i||x_i + y_i|^{p-1} + |y_i||x_i + y_i|^{p-1} \right) \le \|\mathbf{x}\|_p \|\mathbf{w}\|_q + \|\mathbf{y}\|_p \|\mathbf{w}\|_q$$

□

where the last inequality is the Hölder inequality. We have a formula for $\|\mathbf{w}\|_q$, which we use to deduce

$$\|\mathbf{x}+\mathbf{y}\|_p^p \le \|\mathbf{x}\|_p \|\mathbf{w}\|_q + \|\mathbf{y}\|_p \|\mathbf{w}\|_q = (\|\mathbf{x}\|_p + \|\mathbf{y}\|_p)\|\mathbf{x}+\mathbf{y}\|_p^{p/q}.$$

Dividing through gives

$$\|\mathbf{x}+\mathbf{y}\|_p \le \|\mathbf{x}\|_p + \|\mathbf{y}\|_p,$$

which is what we wanted.

EXERCISE C.5. Show that for a given vector $\mathbf{x} \in \mathbf{R}^n$, the function $p \mapsto \|\mathbf{x}\|_p$ is decreasing on $p \in [1,\infty]$.

Show further that $\lim_{p\to\infty} \|\mathbf{x}\|_p = \|\mathbf{x}\|_\infty$.

PROPOSITION C.6. *Given $\mathbf{x} \in \mathbf{R}^n$ and any $p \in [1,\infty]$*

$$\|\mathbf{x}\|_1 \ge \|\mathbf{x}\|_p \ge \|\mathbf{x}\|_\infty \ge \frac{1}{n}\|\mathbf{x}\|_1.$$

This follows from Exercise C.5 and the observation that $\|\mathbf{x}\|_1 \le n\|\mathbf{x}\|_\infty$.

PROPOSITION C.7. *Given any $\{p,q\} \subset [1,\infty]$, there exist constants $c, C > 0$, such that for any $\mathbf{x} \in \mathbf{R}^n$, we have*

$$C\|\mathbf{x}\|_p \ge \|\mathbf{x}\|_q \ge c\|\mathbf{x}\|_p.$$

This follows immediately from C.6. It may be worthwhile to find the best possible constants c, C, but we will not do that here.

2. Norms and metrics

DEFINITION C.8. A real *normed linear space* will consist of an $\mathbf{R}$ vector space V and a norm $\|\cdot\| : V \to \mathbf{R}$ with the following properties. For all $v, w \in V$ and $r \in \mathbf{R}$:

(1) $\|v\| \ge 0$, with equality if and only if $v = 0$.
(2) $\|rv\| = |r|\|v\|$.
(3) $\|v+w\| \le \|v\| + \|w\|$.

An obvious complex analogue of the above also may be defined.

PROPOSITION C.9. *For any $n \in \mathbf{N}$ and any $p \in [0,\infty]$, the pair $(\mathbf{R}^n, \|\cdot\|_p)$ defined in the previous section is a normed linear space.*

PROOF. Easy. □

PROPOSITION C.10. *If $(V, \|\cdot\|)$ is a normed linear space, then the function $d(v,w) = \|v-w\|$ defines a metric on V.*

PROOF. This is not at all difficult.

(1) Property 1 of Definition C.8 implies immediately that $d(x,y) \ge 0$ with equality if and only if $x = y$.
(2) Property 2 of Definition C.8 with $r = -1$ shows that
$$d(x,y) = \|x-y\| = \|y-x\| = d(y,x).$$
(3) Property 3 of Definition C.8 applies to give
$$d(x,y) = \|x-y\| = \|(x-z)-(y-z)\| \le d(x,z) + d(y,z).$$

□

NOTATION C.11. The notation d_p is used for the metric associated to the normed linear space $(\mathbf{R}^n, \|\cdot\|_p)$.

3. The p-norms and product metrics

NOTATION C.12. The notation (x_n) will be used to denote a sequence (finite or infinite) of real numbers indexed by a natural number n. So (x_n) means the same thing as $(x_1, x_2, x_3, \dots)$, finite or infinite depending on context. Occasionally, we will have a need to write something complicated like the sequence $(m, m/2, m/3, \dots)$ where there is a parameter. In this case we may write the sequence as $(m/n)_n$, where the external 'n' indicates that n is the variable indexing the terms of the sequence.

DEFINITION C.13. Let $\{(X_1, d_1), \dots, (X_n, d_n)\}$ be a finite set of metric spaces. Let $X = \prod_{i=1}^n X_i$, let $p \in [1, \infty]$. Define a function $d^p : X \times X \to [0, \infty)$ by $d^p((x_i),(y_i)) = \|(d_i(x_i, y_i)\|_p$.

PROPOSITION C.14. *The functions d^p defined above are all metrics.*

PROOF. Symmetry is immediate. If $d^p((x_i),(y_i)) = \|(d_i(x_i, y_i)\|_p = 0$, then $d_i(x_i, y_i) = 0$ for all i, and since d_i is a metric, this implies $(x_i) = (y_i)$. The triangle inequality is given by combining the triangle inequalities for each d_i metric with that for $\|\cdot\|_p$, and noting that $\|\cdot\|_p$ is increasing in each variable:

$$
\begin{aligned}
d^p((x_i),(y_i)) = \|(d_i(x_i, y_i)\|_p \le \|(d_i(x_i, z_i)) + (d_i(z_i, y_i))\|_p \le \\
\le \|(d_i(x_i, z_i))\|_p + \|(d_i(z_i, y_i))\|_p = d^p((x_i),(z_i)) + d^p((z_i),(y_i))
\end{aligned}
$$

□

PROPOSITION C.15. *The metrics d^p all generate the same topologies.*

PROOF. It suffices to show that for any $p, p' \in [1, \infty]$, any ball $B_p((x_i), \epsilon)$ for the d^p metric with $\epsilon > 0$ contains a ball $B_q((x_i), \eta)$ for the $d^{p'}$ metric with $\eta > 0$ and having the same centre.

We know from Proposition C.7 that there is some constant $c > 0$ such that $c\|(d(x_i, y_i))\|_p \le \|(d(x_i, y_i))\|_{p'}$. Then $B_{p'}((x_i), c\epsilon) \subset B_p((x_i), \epsilon)$. □

EXERCISE C.16. The metric d^∞ generates the product topology; therefore all the metrics d^p generate the product topology.

REMARK C.17. It is easily seen that d_p on $\mathbf{R}^n$ from Notation C.11 is the product metric d^p for n copies of $(\mathbf{R}, |\cdot|)$. By reference to Proposition C.15, the metric spaces $(\mathbf{R}^n, d_p)$ and $(\mathbf{R}^n, d_{p'})$ all generate equivalent topologies for all $p, p' \in [1, \infty]$, and this topology is the product topology on $\mathbf{R}^n = \mathbf{R} \times \cdots \times \mathbf{R}$.

4. The p-norms on sequence spaces

DEFINITION C.18. If $p \in [1, \infty)$, we define a set $\ell^p \subset \prod_{i=1}^\infty \mathbf{R}$ to consist of those sequences (x_n) such that

$$\sum_{i=1}^\infty |x_i|^p$$

converges to a real number. For such a sequence, we define

$$\|(x_n)\|_p = \Big(\sum_{i=1}^{\infty} |x_i|^p\Big)^{1/p}.$$

PROPOSITION C.19. *The pair $(\ell^p, \|\cdot\|_p)$ is a normed linear space.*

PROOF. Conceptually, in the first place we must show that ℓ^p is a vector subspace of $\prod_{i=1}^{\infty} \mathbf{R}$. We must show it is closed under addition of vectors and under scalar multiplication. In the second, we must show that $\|\cdot\|_p$ has the properties of a norm. In practice, it is simpler to prove all these facts concerning addition together then all the facts concerning scalar multiplication.

Suppose (x_n) and (y_n) are sequences in ℓ^p, then for all $N \in \mathbf{N}$

$$\Big(\sum_{i=1}^{N} |x_i + y_i|^p\Big)^{1/p} \le \Big(\sum_{i=1}^{N} |x_i|^p\Big)^{1/p} + \Big(\sum_{i=1}^{N} |y_i|^p\Big)^{1/p}$$

by the Minkowski inequality.

Rearranging this, we deduce

$$\sum_{i=1}^{N} |x_i + y_i|^p \le \big(\|(x_n)\|_p + \|(y_n)|_p\big)^p.$$

The right hand side is the N-th partial sum of the series

$$\sum_{i=1}^{\infty} |x_i + y_i|^p \tag{9}$$

which consists of positive terms. The left hand side is independent of N, and therefore we deduce that (9) converges, and the limit is less than or equal to $(\|(x_n)\|_p + \|(y_n)\|_p)^p$. Rearranging, we deduce that

- $(x_n) + (y_n) \in \ell^p$
- $\|(x_n) + (y_n)\|_p \le \|(x_n)\|_p + \|(y_n)\|_p$.

As for scalar multiplication, it is straightforward to show that

$$\|r(x_n)\|_p = \Big(\sum_{i=1}^{\infty} |r x_i|^p\Big)^{1/p} = |r|\|(x_n)\|_p$$

which shows that

- $r(x_n) \in \ell^p$,
- $\|r(x_n)\|_p = |r|\|(x_n)\|_p$.

Finally, we observe that $\|(x_n)\| = 0$ if and only if every term of (x_n) is 0. □

DEFINITION C.20. We define ℓ^∞ to consist of those sequences (x_n) such that $\sup_{i\in\mathbf{N}} |x_i| < \infty$, i.e. the bounded sequences. We define $\|(x_n)\|_\infty$ as $\sup_{i\in\mathbf{N}} |x_i|$.

EXERCISE C.21. With the definitions above $(\ell^\infty, \|\cdot\|_\infty)$ forms a normed linear space.

DEFINITION C.22. Let c denote the set of convergent sequences of real numbers, c_0 the set of sequences of real numbers with limit 0, and $\mathbf{R}^\infty$ or c_{00} the set of sequences of real numbers having at most finitely many nonzero terms.

Unless otherwise specified, we give ℓ^p the topology induced by the (metric induced by the) norm $\|\cdot\|_p$. We give c and c_0 the topologies inherited from ℓ^∞. Which norm, metric or topology one should place on $\mathbf{R}^\infty$ is less clear, see Exercise C.30.

PROPOSITION C.23. *Suppose $p, q \in [1,\infty)$ satisfy $p < q$. Then there are strict inclusions*

$$\mathbf{R}^\infty \subset \ell^p \subset \ell^q \subset c_0 \subset c \subset \ell^\infty$$

and if $(x_n) \in \ell^p$, then $\|(x_n)\|_p \geq \|(x_n)\|_q \geq \|(x_n)\|_\infty$.

PROOF. We prove this in several steps:

(1) $\mathbf{R}^\infty \subset \ell^1$. The inclusion is immediate, and considering the sequence $(x_n) = (1/2, 1/4, 1/8, \ldots)$ for which $\|(x_n)\|_1 = 1$ but which is not in $\mathbf{R}^\infty$ shows that it is strict.

(2) Suppose $p < q \in [1,\infty)$. Suppose $(x_n) \in \ell^p$. For any initial sequence, we have

$$\sum_{i=1}^N |x_i|^q \leq \Big(\sum_{i=1}^N |x_i|^p\Big)^{q/p}$$

since $\mathbf{x} \mapsto \|\mathbf{x}\|_p$ for $\mathbf{x} \in \mathbf{R}^N$ is decreasing as a function of p. But this implies that, in the limit,

$$\sum_{i=1}^\infty |x_i|^q \leq \|(x_n)\|_p^q,$$

from which the inclusion $\ell^p \subset \ell^q$ and the inequality $\|(x_n)\|_p \geq \|(x_n)\|_q$ both follow.

We observe that if $x_n = 1/(n)^{1/p}$, then $\sum_{i=1}^\infty |x_n|^p = \sum_{i=1}^\infty 1/n$ diverges but $\sum_{i=1}^\infty |x_n|^q = \sum_{i=1}^\infty 1/n^{q/p}$ converges, both by the integral test for convergence. So the inclusion is strict.

(3) If $(x_n) \in \ell_q$, then the series $\sum_{i=1}^\infty |x_i|^q$ converges, so $\lim_{i\to\infty} x_i = 0$, so $(x_n) \in c_0$. The sequence $x_n = 1/\log(n+1)$ shows that the inclusion is strict.

(4) Any sequence coverging to 0 converges, so $c_0 \subset c$. The inclusion is clearly strict, since the constant sequence 1 converges, but not to 0.

(5) Any convergent sequence is bounded, so $c \subset \ell^\infty$. The sequence $(-1)^n$ is bounded but not convergent.

(6) Finally, we show that if $(x_n) \in \ell^p$ for $p < \infty$, then $|x_i| \to 0$ as $i \to \infty$. Assume $(x_n) \neq 0$, since the case of 0 is trivial. In particular, if $s = \sup_i |x_i| > 0$ is not attained, then there is some subsequence of (x_i) converging to $s > 0$, a contradiction. So there is some n such that $\sup_i |x_i| = |x_n|$, and for this value n, we have

$$\|(x_n)\|_\infty = \sup_{i=1}^n |x_i| \leq \Big(\sum_{i=1}^n |x_i|^p\Big)^{1/p} \leq \|(x_n)\|_p$$

by reference to Proposition C.6.

□

COROLLARY C.24. *If $p < q$ are elements of $[1,\infty]$, then the inclusions $\ell^p \subset \ell^q$ are continuous.*

PROOF. These are metric spaces, and it suffices to give an $\epsilon - \delta$ proof of continuity. Let $\epsilon > 0$ and choose $\delta = \epsilon$. For any $(x_n), (y_n) \in \ell^p$, if

$$\|(x_n) - (y_n)\|_p < \delta$$

then

$$\|(x_n)-(y_n)\|_q < \delta = \epsilon.$$

So the inclusion is indeed continuous. □

Another useful fact is the following. Each space appearing in Proposition C.23 is a subspace of the space of all sequences: $\mathbf{R}^{\mathbf{N}}$. This space may be equipped with the product topology.

PROPOSITION C.25. *Let $p \in [1,\infty]$. Then the inclusion $\ell^p \to \mathbf{R}^{\mathbf{N}}$ is continuous.*

PROOF. The product topology on $\mathbf{R}^{\mathbf{N}}$ is the weakest (or coarsest) topology making each projection continuous. In particular, if the functions $\pi_i : \ell^p \to \mathbf{R}$ that take a sequence (x_n) to the i-th term x_i are continuous, then the induced map $\ell^p \to \mathbf{R}^{\mathbf{N}}$ is continuous, and it is easy to verify that this map is indeed the inclusion.

So it suffices to show that π_i is continuous. Since the source and target are both metric spaces, an $\epsilon-\delta$ argument applies. Suppose $\|(x_n)-(y_n)\|_p < \epsilon$, then in particular $|x_i - y_i| < \epsilon$, which implies $|\pi_i((x_n)) - \pi_i((y_n))| < \epsilon$. So π_i is indeed continuous, and the result follows. □

PROPOSITION C.26. *Let $p \in [0,\infty)$. The subset $\mathbf{R}^\infty$ is dense in ℓ^p.*

PROOF. Let (x_n) be a sequence in ℓ_p, and for $m \in \mathbf{N}$ let $(x_{m,n})_n$ denote the sequence for which $x_{m,n} = x_n$ if $m \le n$ and $x_{m,n} = 0$ otherwise.

We wish to show that $\ell^p = \overline{\mathbf{R}^\infty}$. We show that every element of ℓ^p is a limit of a sequence of elements in $\mathbf{R}^\infty$—this is a sequence of sequences.

Let $\epsilon > 0$. Consider the sequence $((x_{m,n)})_m$ of elements of $\mathbf{R}^\infty$. Observe that

$$\|(x_{m,n})_n - (x_n)\|_p^p = \sum_{i=1}^{m} |x_i - x_i|^p + \sum_{i=m+1}^{\infty} |x_i|^p = \sum_{i=m+1}^{\infty} |x_i|^p.$$

Since the series $\sum_{i=1}^{\infty} |x_i|^p$ converges to $\|(x_n)\|_p^p$, we can find some N such that

$$\left|\sum_{i=1}^{m} |x_i|^p - \|(x_n)\|_p^p\right| < \epsilon^p$$

whenever $m > N$, which is equivalent to

$$\left|\sum_{i=1}^{m} |x_i|^p - \sum_{i=1}^{\infty} |x_i|^p\right| < \epsilon^p,$$

but this is equivalent to saying

$$\sum_{i=m+1}^{\infty} |x_i|^p < \epsilon^p$$

whenever $m > N$, which is to say that $\|(x_{m,n})_n - (x_n)\|_p^p < \epsilon^p$, and taking p-th roots, we see that

$$\|(x_{m,n})_n - (x_n)\|_p < \epsilon$$

whenever $m > N$. Therefore the sequence $((x_{m,n}))_m \to (x_n)$ as $m \to \infty$. □

EXERCISE C.27. Let $p \in [1,\infty)$. Let $\mathbf{Q}^\infty \subset \mathbf{R}^\infty$ denote the set of sequences having only rational-number terms and which are eventually 0.

(1) Prove $\mathbf{Q}^\infty$ is dense in ℓ^p.

(2) Give c_0 the subspace topology inherited from ℓ^∞. Prove $\mathbf{Q}^\infty$ is dense in c_0.

COROLLARY C.28. *Since $\mathbf{Q}^\infty$ is in bijection with the countable union $\bigcup_{i=1}^\infty \mathbf{Q}^i$ of countable spaces, it follows that each of the spaces ℓ^p for $p \in [1,\infty)$ or c_0 or c is separable, and since they are metric, they are second countable.*

EXERCISE C.29. Prove that ℓ^∞ is not separable (and therefore, not second countable)

The situation for infinite-dimensional spaces is therefore much more complicated than for finite-dimensional spaces. In the finite-dimensional setting, there was only one linear space for each dimension, $\mathbf{R}^n$, and each of the norms $\|\cdot\|_p$ induced the same topology. In the infinite-dimensional case, the topologies and spaces on which they are defined are all different.

It can be conceptually helpful to view the elements of $\mathbf{R}^\infty$, that is, finite sequences of some undetermined length, as the objects one is most likely to encounter in practical situations; the real world is generally finitist. Then the different spaces ℓ^p, c_0 and c are different choices of which sequences of elements in $\mathbf{R}^\infty$ one views as convergent.

EXERCISE C.30. Consider the various normed linear spaces $(\mathbf{R}^\infty, \|\cdot\|_p)$ for $p \in [1,\infty]$. Prove that these are pairwise inequivalent as metric spaces by considering which sequences of elements in $\mathbf{R}^\infty$ are convergent for the various d_p metrics.

4.1. Completeness.

EXERCISE C.31. Prove that the spaces $(\ell^p, \|\cdot\|_p)$ are complete for all $p \in [1,\infty]$. What can be said about $\mathbf{R}^\infty$, c_0 and c?

5. The p-norms for functions

This is not a course in measure theory, so we content ourselves with the following inadequate treatment.

DEFINITION C.32. Let $p \in [1,\infty)$. Suppose $f : [a,b] \to \mathbf{R}$ is a function defined on an interval $[a,b]$ for which

$$\int_a^b |f|^p \, dx$$

is defined (and finite). Then define

$$\|f\|_p = \left(\int_a^b |f|^p \, dx\right)^{1/p}.$$

REMARK C.33. The integral should really be taken in the sense of Lebesgue, but we will restrict our attention to piecewise continuous functions on closed bounded intervals, which will allow us to use only Riemann integrals, including improper Riemann integrals if necessary.

EXERCISE C.34. Let $P[a,b]$ denote the set of piecewise-continous functions on the closed bounded interval $[a,b]$. For $f \in P[a,b]$ and $p \in [1,\infty)$, show that $(P[a,b], \|\cdot\|_p)$ makes $P[a,b]$ a normed linear space.

DEFINITION C.35. We say that C is an *essential supremum* for a piecewise continuous function $f : [a,b] \to \mathbf{R}$ if the set S of values x for which $f(x) > C$ does not have any interior points—i.e., S contains no open intervals.

EXERCISE C.36. For $f \in P[a,b]$, define $\|f\|_\infty$ to be

$$\|f\|_\infty = \inf\{C : C \text{ is an essential supremum for } |f|\}.$$

Prove that $(P[a,b], \|\cdot\|_\infty)$ is a normed linear space.

EXERCISE C.37. Show that the spaces $(P[a,b], \|\cdot\|_p)$ as p varies over $[1,\infty]$ are all different.

APPENDIX D

Urysohn's Lemma and the Tietze Extension Theorem

1. Urysohn's Lemma

Suppose X is a topological space and suppose that for any two disjoint closed sets $C_0, C_1 \subset X$ we can find a continuous $f : X \to \mathbf{R}$ such that $f|_{C_0} \equiv 0$ and $f|_{C_1} \equiv 1$. Then $f^{-1}(-\infty, 1/2)$ and $f^{-1}(1/2, \infty)$ give us two open sets that separate C_0 and C_1. This implies X is normal.

THEOREM D.1 (Urysohn's Lemma). *Suppose X is a normal topological space and that C_0 and C_1 are disjoint closed sets in X. Then there exists a continuous function $f : X \to [0,1]$ such that $f(C_0) = 0$ and $f(C_1) = 1$.*

PROOF. Use normality to produce a nested sequence of open sets U_d, one for each dyadic rational $d = a/2^n$ in $[0,1] \cap \mathbf{Q}$, such that $C_0 \subset U_0$ and $C_1 \subset X \setminus \bar{U}_1$, and such that $d < d'$ implies $\bar{U}_d \subset U_{d'}$. Then define $f : X \to \mathbf{R}$ by

$$f(x) = \begin{cases} 1 \text{ if } x \notin U_1 \\ \inf_{x \in U_d} d & \text{otherwise} \end{cases}$$

To prove the function f is continuous, let $x \in X$, and fix $\epsilon > 0$. Write $t = f(x)$. We want to find an open neighbourhood V of x such that $f(V) \subset (t-\epsilon, t+\epsilon)$. The cases of $t = 0$ and $t = 1$ are exceptional. If $t = 0$, then we can find some $d < \epsilon$ such that $x \in U_d$. Then $U_d = V$ works. If $t = 1$, then we can find $d > 1-\epsilon$ such that $x \notin \bar{U}_d$. Then $X \setminus \bar{U}_d$ works. Therefore assume $t \in (0,1)$.

We can find some diadic numbers $d_1 > d_2 > t-\epsilon$ such that $x \notin U_{d_1} \supset U_{d_2}$. We can also find $d_3 < t+\epsilon$ such that $x \in U_{d_3}$. Then consider $V = U_{d_3} \setminus \bar{U}_{d_2}$. This is an open set containing x, and $f(V) \subset [d_2, d_3] \subset (t-\epsilon, t+\epsilon)$.

So f is continuous. □

EXAMPLE D.2. Let X be an uncountable space and let $p \in X$ be a point. Define the *fortissimo* topology on X as follows: a subset $C \subset X$ is closed if $p \in C$ or if C is countable.

We claim this space is normal. Two disjoint closed sets consist of two disjoint countable subsets neither containing p, or one countable subset and one subset containing p. In the first case, the sets are also open since their complements contain p. In the second, the countable set not containing p is open, and its complement is also open.

Therefore Urysohn's lemma applies to X. On the other hand, consider a continuous function $f : X \to \mathbf{R}$ such that $f(p) = 0$. Note that $\{p\}$ itself is a closed point. The sets $U_n = f^{-1}((-1/n, 1/n))$ form a countable nested family of open sets in X, each containing p. Therefore each U_n is co-countable. It follows that $f^{-1}(0) = \bigcap_{n=1}^{\infty} U_n$ is also cocountable, so that in particular, the closed set $\{p\}$ cannot be expressed $f^{-1}(0)$ for any continuous $f : X \to \mathbf{R}$.

DEFINITION D.3. If X is a topological space, a G_δ*-set* is any subset of X that can be written as an intersection of countably many open subsets.

DEFINITION D.4. A G_δ*-space* is a topological space X in which every closed subset is a G_δ-set. A space X is *perfectly normal* if it is normal and a G_δ-space.

EXERCISE D.5. A space X is perfectly normal if and only if every closed set C is the zero set of a continuous function $f : X \to \mathbf{R}$.

2. Tietze Extension

This will be filled in later

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