Math 426
Midterm
19 October 2022

There are 4 problems worth a total of 36 points. Answer as many as you can.
Here is a guide to the problems:

1. (12 pts) Basic definitions in topology.
2. (8 pts) Connectivity.
3. (12 pts) Compactifications and quotient spaces.
4. (4 pts) Connectivity and products.

1. We say a topological space X is *irreducible* if X cannot be written as the union of two proper closed subsets. A subspace of $A \subseteq Y$ of a topological space is said to be *irreducible* if it is irreducible in the subspace topology.

2pts (a) Let X be a topological space and let $x \in X$. Prove that $\overline{\{x\}}$ is an irreducible subspace.

4pts (b) A topological space X is said to be *sober* if the function

$$j : X \to \text{nonempty irreducible closed subsets of } X$$

given by $j(x) = \overline{\{x\}}$ is a bijection. Prove that all Hausdorff spaces are sober.

3pts (c) Let X be a set and $x \in X$ a point. The *particular-point topology* on X is defined as follows: a nonempty subset U of X is open if and only if $U \ni x$. The empty set is also open. You do not have to show this is a topology.

Prove that the particular-point topology is irreducible.

3 pts (d) Prove that the particular-point topology is sober.

(a) If $\overline{\{x\}}$ is written as a union of two closed subsets C, D, then at least one of the two must contain x itself, and therefore must contain the closure $\overline{\{x\}}$. Therefore at least one of C, D cannot be a proper subspace.

(b) First, we show j is injective. Since X is Hausdorff, singleton subsets are closed, and therefore $\overline{\{x\}} = \{x\}$. Consequently, if $j(x) = j(y)$, then $\{x\} = \{y\}$, i.e., $x = y$.

Second, we show j is surjective. Let C be an irreducible closed subset of X. We claim that C is a singleton $\{x\}$, which is in the image of j. Suppose for the sake of contradiction that $x \neq y$ are two points in C, then we can find disjoint open subsets $U \ni x$ and $V \ni y$, whereupon $C \setminus V$ and $C \setminus U$ are closed subsets of C, the first containing x and the second containing y. Furthermore, their union is $C \setminus (U \cap V) = C$. We have found a cover of C by proper closed subsets, contradicting the irreducibility of C.

Since the irreducible closed subsets of X are singletons, the function j is surjective.

(c) Write X as a union of two closed subsets $X = C \cup D$. At least one of these must contain the particular point x, but the only closed subset that contains x is X itself. Therefore X cannot be written as a union of two proper closed subsets.

(d) Every subset of X not containing x is closed. If C is an irreducible nonempty closed subset not containing x, and if y is an element of C, the decomposition $C = (C \setminus \{y\}) \cup \{y\}$ shows that $C = \{y\} = j(y)$. On the other hand, if C is an irreducible closed subset containing x, then $C \supset \overline{\{x\}} = X = j(x)$.

We see that j is indeed bijective, as required.

□

2. Let $p = (0,1) \in \mathbf{R}^2$ and $q = (0,-1) \in \mathbf{R}^2$. Let $N = \{1, 1/2, 1/3, 1/4, \ldots\}$. Let X denote the following subset of $\mathbf{R}^2$:

$$X = N \times [-1,1] \cup \{p, q\}.$$

4 pts (a) Determine, with proof, the connected components of X;

4 pts (b) Suppose $f : X \to \{0,1\}$ is a continuous function. Show that $f(p) = f(q)$, even though $\{p,q\}$ is not connected.

(a) Write A_n for $\{1/n\} \times [-1,1]$. Observe that each A_n is the image of $[-1,1]$ under a continuous map to $\mathbf{R}^2$, and each A_n is therefore connected. Consider the intersection of $(1/n-\epsilon, 1/n+\epsilon) \times [-1,1]$ with X. For sufficiently small values of ϵ, this is precisely A_n, so A_n is open in X. Similarly, considering $[1/n-\epsilon, 1/n+\epsilon] \times [-1,1] \cap X$, we see that each A_n is closed in X. Therefore no proper superset of A_n can be connected, and A_n is a connected component.

Since connected components form a partition of the space, it remains to determine the decomposition of $\{p,q\}$ into connected components, but this set is discrete and therefore disconnected. The components are $\{p\}$, $\{q\}$.

(b) Any such continuous function must be constant on connected components. Suppose without loss of generality that $f(p) = 1$. Take the sequence $(1/n, 1)_n$ in X. This sequence converges to p and so $f((1/n,1))_n \to f(p) = 1$. Therefore for some tail of this sequence $f((1/n,1))_n \equiv 1$. Since f is constant on the A_n, it follows that $f((1/n,-1)) \equiv 1$, and so $f(q) = \lim_{n\to\infty} f((1/n,-1)) = 1$ as well.

□

3. Recall that a continuous function $j : A \to B$ is a *compactification* if j is an embedding, the image of j is dense in B, and B is compact.

Suppose X, Y and $X \times Y$ are non-compact locally compact Hausdorff spaces (it is sufficient to assume X and Y have these properties).

Let $X \cup \{\infty_X\}$, $Y \cup \{\infty_Y\}$ and $(X \times Y) \cup \{\infty\}$ denote the one-point compactifications of X, Y and $X \times Y$ respectively. To keep the notation simple, we write $\hat{X}$, $\hat{Y}$ and $\widehat{X \times Y}$ for these compactifications.

4pts (a) Prove that the inclusion $i : X \times Y \to \hat{X} \times \hat{Y}$ is continuous and open.

2pts (b) Prove that i is a compactification.

2pts (c) Prove that the function $f : \hat{X} \times \hat{Y} \to \widehat{X \times Y}$ given by

$$f(x, y) = (x, y) \quad \forall (x, y \in X \times Y$$

and

$$f(\infty_X, y) = f(x, \infty_Y) = f(\infty_X, \infty_Y) = \infty \qquad \forall x \in, \forall y \in Y$$

is continuous. Possible hint: the one-point compactification has a universal property among Hausdorff compactifications of locally compact Hausdorff spaces.

4 pts (d) Prove that the spaces

$$\widehat{X \times Y} \qquad \text{and} \qquad \frac{\hat{X} \times \hat{Y}}{\left(\hat{X} \times \{\infty_Y\}\right) \cup \left(\{\infty_X\} \times \hat{Y}\right)}$$

are homeomorphic.

(a) For continuity, we use the fact that the inclusions $X \to \hat{X}$ and $Y \to \hat{Y}$ are continuous, so that the product $X \times Y \to \hat{X} \times \hat{Y}$ is continuous.

For openness: it suffices to prove that the image of a basic open set $U \times V \subset X \times Y$ is open in $\hat{X} \times \hat{Y}$, by Proposition 1.36 in the notes. The image of $U \times V$ in $\hat{X} \times \hat{Y}$ is a basic open, however, so this is trivial.

(b) Since $\hat{X}$ and $\hat{Y}$ are compact, $\hat{X} \times \hat{Y}$ is compact. We must show that $X \times Y$ is dense in $\hat{X} \times \hat{Y}$. From Homework 2, we know that $\overline{X \times Y} = \overline{X} \times \overline{Y}$, the closures being taken in $\hat{X} \times \hat{Y}$ or $\hat{X}$ and $\hat{Y}$ respectively. Since $\overline{X} = \hat{X}$ and $\overline{Y} = \hat{Y}$, this completes the proof.

(c) This is almost immediate using the universal property of the one-point compactification. We know that $\hat{X} \times \hat{Y}$ is a compactification of $X \times Y$ (proved in the previous two parts of this question) and we know that $\hat{X} \times \hat{Y}$ is Hausdorff since products of Hausdorff spaces are Hausdorff. The one-point compactification is universal among all Hausdorff compactifications of noncompact locally compact Hausdorff spaces, by Proposition 3.43 in the notes.

(d) Write $\hat{X} \wedge \hat{Y}$ for the quotient space in this question. From the previous part of this question, we know that there is a continuous function $\hat{X} \times \hat{Y} \to \widehat{X \times Y}$. This function is surjective, but not injective, because all the points in $\left(\hat{X} \times \{\infty_Y\}\right) \cup \left(\{\infty_X\} \times \hat{Y}\right)$ are mapped to ∞. By the universal property of the quotient, there is an induced continuous function $\hat{X} \wedge \hat{Y} \to \widehat{X \times Y}$. This function is bijective, and has compact source and Hausdorff target (since $X \times Y$ is locally compact). Therefore this function is a homeomorphism.

□

4. (4 pts) Suppose $\{X_i\}_{i \in I}$ is a set of connected spaces, and suppose $X_i \neq \emptyset$ for all $i \in I$. Prove that $\prod_{i \in I} X_i$ is connected.

Proof. Write X for the product, and let $f : X \to \{0, 1\}$ be a continuous function. Let $x = (x_i)_{i \in I}$ be a point in X (the assertion that such a point exists for general I relies on the axiom of choice), and suppose $f(x) = 1$.

We claim that if y differs from x only in finitely many coordinates, then $f(y) = 1$ as well. We prove this by induction on the number of coordinates in which x and y differ. The base case is $x = y$, which is trivial. For the induction step, we may assume that x and y agree in every coordinate except one, the j-th coordinate. That is x, y lie in the image of a continuous map

$$s : X_j \to X$$

given by sending z_j to the element that agrees with x and y in every coordinate except the j-th coordinate and is z_j in the j-th coordinate. Since X_j is connected, the image of $s(X_j)$ is connected, so $f(x) = f(y)$. This proves the claim.

Now suppose for the sake of contradiction that there is some $y' \in X$ such that $f(y') = 0$. Then the set $f^{-1}(0)$ of such y's is nonempty and open, and therefore contains a basic open set:

$$\bigcap_{j \in J,\, J \text{ finite}} \mathrm{proj}_j^{-1}(U_j)$$

where $U_j \subset X_j$ are open sets. In particular, we can find a point $y \in f^{-1}(0)$ that agrees with x in all coordinates except the finitely many coordinates in J. This contradicts the previous part of this proof, so no such y' exists. □