Math 427, 527
Homework 01
2026-01-19

Whenever you are asked to give an example, you should prove your example is correct unless otherwise instructed.

1. We define a *basis* for an abelian group F to consist of a subset $B \subset F$ with the following properties:

- (spanning) every element $f \in F$ may be written as a finite sum of $\mathbb{Z}$-multiples of elements of B, i.e.,

$$f = a_1 b_1 + \cdots + a_n b_n \quad \text{where } \{a_1, \ldots, a_n\} \subseteq \mathbb{Z} \text{ and } \{b_1, \ldots, b_n\} \subseteq B;$$

- (linear independence) The set B is linearly independent, in that a relation

$$a_1 b_1 + \ldots a_n b_n = 0 \quad \text{where } \{a_1, \ldots, a_n\} \subseteq \mathbb{Z} \text{ and } \{b_1, \ldots, b_n\} \subseteq B$$

  implies $a_1 = a_2 = \cdots = a_n = 0$.

An abelian group is *free* if it has a basis. You may assume the *invariant basis property*: any two bases of the same free abelian group have the same cardinality.

(a) Suppose F is a free abelian group with basis B. Suppose A is an abelian group. If $f : B \to A$ is a function (viewing A as its underlying set here), prove that there exists a unique homomorphism $\phi : F \to A$ with the property that $\phi(b) = f(b)$ for all $b \in B$.

(b) Give an example of a spanning set $S \subseteq \mathbb{Z}^2$ that does not contain a basis as a subset.

(c) Give an example of a linearly independent set $S \subseteq \mathbb{Z}^2$ that is not a subset of any basis.

2. The homomorphisms $d : S_{n+1}(X) \to S_n(X)$ were defined in lecture in terms of maps $d^i : \Delta^{n+1} \to \Delta^n$. Establish the identity $d^j \circ d^i = d^i \circ d^{j-1}$ when $i < j$. Deduce that $d \circ d = 0$.

3. Let **Ab** denote the category whose objects are abelian groups and whose morphisms are homomorphisms between them. Let $\mathrm{id} : \mathbf{Ab} \to \mathbf{Ab}$ denote the identity functor.

Determine with proof the set of all natural transformations $v : \mathrm{id} \to \mathrm{id}$.

4. This problem is not to be handed in.

This question takes place in some unspecified category. Prove that if the morphisms $g \circ f$ and $h \circ g$ in $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ are isomorphisms, then so are f, g, h.

Deduce that if $s : M \to N$, and $t : N \to M$ are two morphisms such that $s \circ t$ and $t \circ s$ are isomorphisms, then s and t are isomorphisms.

5. There is a category **Haus** consisting of Hausdorff topological spaces and continuous functions between them. It is a full subcategory of **Top**.

(a) Show that the inclusion $i : [0,1) \to [0,1]$ has the following property if $f, g : [0,1] \to X$ are two morphisms in **Haus** with the property that $f \circ i = g \circ i$, then $f = g$. The name for a morphism with this property is *epimorphism*.

(b) Show that i no longer has this property when we allow f, g to have target in **Top**. In particular, the inclusion functor **Haus** $\to$ **Top** does not preserve epimorphisms.