Math 427, 527
Homework 01
2026-01-19

Whenever you are asked to give an example, you should prove your example is correct unless otherwise instructed.

1. We define a *basis* for an abelian group F to consist of a subset $B \subset F$ with the following properties:

- (spanning) every element $f \in F$ may be written as a finite sum of $\mathbb{Z}$-multiples of elements of B, i.e.,
$$f = a_1 b_1 + \cdots + a_n b_n \quad \text{where } \{a_1, \ldots, a_n\} \subseteq \mathbb{Z} \text{ and } \{b_1, \ldots, b_n\} \subseteq B;$$
- (linear independence) The set B is linearly independent, in that a relation
$$a_1 b_1 + \ldots a_n b_n = 0 \quad \text{where } \{a_1, \ldots, a_n\} \subseteq \mathbb{Z} \text{ and } \{b_1, \ldots, b_n\} \subseteq B$$
implies $a_1 = a_2 = \cdots = a_n = 0$.

An abelian group is *free* if it has a basis. You may assume the *invariant basis property*: any two bases of the same free abelian group have the same cardinality.

(a) Suppose F is a free abelian group with basis B. Suppose A is an abelian group. If $f : B \to A$ is a function (viewing A as its underlying set here), prove that there exists a unique homomorphism $\phi : F \to A$ with the property that $\phi(b) = f(b)$ for all $b \in B$.

(b) Give an example of a spanning set $S \subseteq \mathbb{Z}^2$ that does not contain a basis as a subset.

(c) Give an example of a linearly independent set $S \subseteq \mathbb{Z}^2$ that is not a subset of any basis.

(a) The letter a, possibly with subscript, always denotes an integer in this answer. We will write expressions such as
$$\sum_{b \in B} a_b b.$$
It is to be understood that the integers a_b are 0 for all except finitely many values of $b \in B$. In this way, an apparently infinite sum collapses to give a finite sum.

First we construct ϕ as a function.

Let $x \in F$. We may write $x = \sum_{b \in B} a_b b$ where the a_b are integers, all but finitely many of which are 0. The values a_b are uniquely determined by x and the basis B by the same argument as for bases for vector spaces over fields.

Define $\phi(x) = \sum_{b \in B} a_b f(b)$. Since all but finitely many of the a_b are 0, this is a finite sum, and it gives us a function $\phi : F \to A$.

We now must prove additivity of ϕ. Consider two elements of F:

$$x = \sum_{b \in B} a_b b, \quad x' = \sum_{b \in B} a'_b b.$$

Then $\phi(x + x') = \sum_{b \in B}(a_b + a'_b) f(b) = \sum_{b \in B} f(b) + \sum_{b \in B} a'_b f(b) = \phi(x) + \phi(x')$, as required.

As for uniqueness: if two homomorphisms $\phi, \phi' : F \to A$ agree on B, then for any $x = \sum_{b \in B} a_b b$, we have

$$\phi(x) = \sum_{b \in B} a_b \phi(b) = \sum_{b \in B} a_b \phi'(b) = \phi'(x).$$

Since x was arbitrary, this implies ϕ is unique.

(b) Consider the set $\{(2,0),(3,0),(0,1)\}$. Certainly this set spans $\mathbb{Z}^2$ since $(a,b) = a(3,0) - a(2,0) + b(0,1)$. However, it does not contain a basis. A basis must consist of two different elements, so there are three subsets to check: $\{(2,0),(3,0)\}$ does not have $(0,1)$ in its span. The span of the set $\{(2,0),(0,1)\}$ is the set of pairs $(2a,b)$ where $a,b \in \mathbb{Z}$, and this does not contain $(1,0)$. The argument disqualifying $\{(3,0),(0,1)\}$ is similar.

(c) Consider the set $\{(2,0),(0,1)\}$. This is linearly independent since

$$a(2,0) + b(0,1) = (2a,b) \quad \text{where } a,b \in \mathbb{Z}.$$

This is $(0,0)$ only when $a = b = 0$. Nonetheless, it is not a basis (as proved in the previous part) and it cannot form part of a basis, since it already has 2 elements.

□

2. The homomorphisms $d : S_{n+1}(X) \to S_n(X)$ were defined in lecture in terms of maps $d^i : \Delta^{n+1} \to \Delta^n$. Establish the identity $d^j \circ d^i = d^i \circ d^{j-1}$ when $i < j$. Deduce that $d \circ d = 0$.

We verify the identity directly:

$$d^j(d^i(x_0,\ldots,x_{n+1}) = d^j(x_0,\ldots,x_{i-1},0,x_i,\ldots,x_n) \quad (0 \text{ in the } i\text{-th position}),$$

and since $j > i$, the term in the j-th position on the right is x_{j-1}. Continuing, we get

$$d^j(x_0,\ldots,x_{i-1},0,x_i,\ldots,x_n) = (x_0,\ldots,x_{i-1},0,x_i,\ldots,x_{j-2},0,x_{j-1},\ldots,x_n).$$

On the other hand:

$$d^i(d^{j-1}(x_0,\ldots,x_{n+1}) = d^i(x_0,\ldots,x_{j-2},0,x_{j-1},\ldots,x_n) = (x_0,\ldots,x_{i-1},0,x_i,\ldots,x_{j-2},0,x_{j-1},\ldots,x_n).$$

This establishes the identity.

The free abelian group $S_{n+2}(X)$ has $\mathrm{Sin}_n(X)$ as a basis. It suffices to prove that $d \circ d(\sigma) = 0$ for all $\sigma \in \mathrm{Sin}_n(X)$, since the basis is a spanning set. By definition,

$$d \circ d(\sigma) = d\Big(\sum_{j=0}^{n+2} (-1)^j \sigma \circ d^j\Big)$$

and then by linearity of d this equals

$$\sum_{j=0}^{n+2}(-1)^j d(\sigma\circ d^j)=\sum_{j=0}^{n+2}(-1)^j\sum_{i=0}^{n+1}(-1)^i\sigma\circ d^j\circ d^i=\sum_{j=0}^{n+2}\sum_{i=0}^{n+1}(-1)^{i+j}\sigma\circ d^j\circ d^i.$$

Let $I=\{0,\dots,n+1\}\times\{0,\dots,n+2\}$ denote the set of pairs of indices (i,j). Divide I into two disjoint sets: the set N where $i<j$, and the set S where $i\geq j$. The function

$$f:N\to S,\quad f:(i,j)\mapsto(j-1,i)$$

has inverse

$$S\to N,\quad (i,j)\mapsto(j,i+1)$$

and therefore is a bijection between the two subsets.

The sum we wish to evaluate is therefore

$$\sum_{(i,j)\in N}[(-1)^{i+j}\sigma\circ d^j\circ d^i+(-1)^{j-1+i}\sigma\circ d^i\circ d^{j-1}]=0$$

by the identity that we proved previously. □

3. Let **Ab** denote the category whose objects are abelian groups and whose morphisms are homomorphisms between them. Let $\mathrm{id}:\mathbf{Ab}\to\mathbf{Ab}$ denote the identity functor.

Determine with proof the set of all natural transformations $\nu:\mathrm{id}\to\mathrm{id}$.

The set of natural transformations is in bijection with $\mathbb{Z}$, where $\lambda\in\mathbb{Z}$ corresponds to the transformation ν that on any given abelian group A is the homomorphism $\times\lambda:A\to A$ given by multiplying by λ.

We first verify that this does indeed define a natural transformation. That is, for all homomorphisms $\phi:A\to B$ of abelian groups, the diagram

$$\begin{array}{ccc} A & \xrightarrow{\times\lambda} & A \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\ B & \xrightarrow{\times\lambda} & B \end{array} \tag{1}$$

commutes. That is, $\phi(\lambda a)=\lambda\phi(a)$ for all $a\in A$. This holds because ϕ is a homomorphism.

Second, we verify that all natural transformations are of this form. Suppose $\nu:\mathbf{Ab}\to\mathbf{Ab}$ is a natural transformation. Consider $\nu_{\mathbb{Z}}:\mathbb{Z}\to\mathbb{Z}$. This must satisfy $\nu_{\mathbb{Z}}(1)=\lambda\in\mathbb{Z}$. Next, let A be an arbitrary abelian group, and let $a\in A$ be an element. There exists a homomorphisms $i_a:\mathbb{Z}\to A$ satisfying $i_a(1)=a$. Consider

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\nu_{\mathbb{Z}}} & \mathbb{Z} \\ \downarrow{\scriptstyle i_a} & & \downarrow{\scriptstyle i_a} \\ A & \xrightarrow{\nu_A} & A \end{array}$$

which implies that $\nu_A(i_a(1))=i_a(\nu_{\mathbb{Z}}(1))$, which simplifies to say $\nu_A(a)=i_a(\lambda)=\lambda i_a(1)=\lambda a$. Since $a\in A$ was arbitrary, we deduce that $\nu_A(a)=\lambda a$ for all elements a in all abelian groups A. That is, ν is one of the natural transformations previously described. □

4. This problem is not to be handed in.

This question takes place in some unspecified category. Prove that if the morphisms $g \circ f$ and $h \circ g$ in $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ are isomorphisms, then so are f, g, h.

Deduce that if $s : M \to N$, and $t : N \to M$ are two morphisms such that $s \circ t$ and $t \circ s$ are isomorphisms, then s and t are isomorphisms.

First, a lemma.

Lemma 0.1. *Suppose $\alpha : X \to Y$ is a morphism in a category such that α has a left inverse $\beta : Y \to X$ for which $\beta \circ \alpha = \mathrm{id}_X$ and a right inverse $\gamma : Y \to X$ for which $\alpha \circ \gamma = \mathrm{id}_Y$, then $\beta = \gamma$ and α is an isomorphism.*

Proof.

$$\beta = \beta \circ (\alpha \circ \gamma) = (\beta \circ \alpha) \circ \gamma = \gamma$$

which establishes the first claim. Then $\beta \circ \alpha = \mathrm{id}_X$ and $\alpha \circ \beta = \mathrm{id}_Y$, establishing the second. □

Now we prove g is an isomorphism. It has a left-inverse: $(h \circ g)^{-1} \circ h \circ g = \mathrm{id}_X$ and a right inverse: $g \circ f \circ (g \circ f)^{-1} = \mathrm{id}_Y$. The lemma proves that g is an isomorphism.

Next we show that f is an inverse for the isomorphism $j := (g \circ f)^{-1} \circ g$. it is immediate that f is a right inverse for j. Since j is an isomorphism, it has a left inverse j^{-1} as well, and the lemma tells us that $f = j^{-1}$, so that f is an isomorphism.

Similarly, h is a left inverse for the isomorphism $k := g \circ (h \circ g)^{-1}$, and therefore $h = k^{-1}$ by the lemma.

Finally, if we apply the preceding results in the case where $f = h = s$ and $g = t$, we see that if $s \circ t$ and $t \circ s$ are isomorphisms, then s and t are isomorphisms. □

5. There is a category **Haus** consisting of Hausdorff topological spaces and continuous functions between them. It is a full subcategory of **Top**.

(a) Show that the inclusion $i : [0,1) \to [0,1]$ has the following property if $f, g : [0,1] \to X$ are two morphisms in **Haus** with the property that $f \circ i = g \circ i$, then $f = g$. The name for a morphism with this property is *epimorphism*.

(b) Show that i no longer has this property when we allow f, g to have target in **Top**. In particular, the inclusion functor **Haus** → **Top** does not preserve epimorphisms.

(a) If $f \circ i = g \circ i$, then $f(x) = g(x)$ for all $x \in [0,1)$. To prove that $f = g$, it suffices to show that $f(1) = g(1)$. Therefore consider the sequence $(1 - 1/n)$ which converges to 1. Since f, g are both continuous, it must be the case that $f(1-1/n) \to f(1)$ and $g(1-1/n) \to g(1)$. Since $f(1-1/n) = g(1-1/n)$, we see that $f(1)$ and $g(1)$ are both limits of the sequence $f(1-1/n)$, and since X is Hausdorff, the limit of the sequence is unique and we conclude $f(1) = g(1)$.

(b) Let $T = [0,1]/[0,1)$ be the quotient space. As a set: $T = \{b, 1\}$, and the topology on T is that $\{b\}$ is open, along with $\emptyset$ and T itself.

The quotient map $q : [0,1] \to T$ is continuous. It satisfies $q(x) = b$ if $x < 1$ and $q(1) = 1$. But the constant map $f : [0,1] \to T$ given by $f(x) = b$ for all x is also continuous. The maps q, f are different, but they agree on $[0,1)$, so that $q \circ i = f \circ i$, as required.

□