Math 427, 527
Homework 2
2026-02-02

1. This problem is not to be handed in.

Let X be a topological space. Recall that $\pi_0(X)$ denotes the set of path components of X: the set of equivalence classes of points $x \in X$ where $x \sim y$ if there exists a path $\gamma : I \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

Write $\mathbb{Z}\pi_0(X)$ for the free abelian group with $\pi_0(X)$ as a basis. By constructing homomorphisms each way and checking they are inverses, prove there is an isomorphism between $\mathbb{Z}\pi_0(X)$ and $\mathrm{H}_0(X;\mathbb{Z})$.

2. Suppose we are given a commutative diagram of abelian groups:

$$\begin{array}{ccccccc} A_4 & \xrightarrow{d} & A_3 & \xrightarrow{d} & A_2 & \xrightarrow{d} & A_1 \\ \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} \\ B_4 & \xrightarrow{d} & B_3 & \xrightarrow{d} & B_2 & \xrightarrow{d} & B_1. \end{array}$$

Suppose further that the rows are exact sequences, the homomorphisms f_3 and f_1 are injective, and f_4 is surjective. Prove that f_2 is injective.

3. This is part of Exercise 8.8 of Miller's notes. Suppose

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

is a short exact sequence of abelian groups. As we often do, we will identify A with its image under i. Show that the following are equivalent.

(a) There exists a homomorphism $s : C \to B$ such that $p \circ s = \mathrm{id}_C$.

(b) There exists a homomorphism $t : B \to A$ such that $t \circ i = \mathrm{id}_A$.

Prove that if s exists as above, then the homomorphism $f : A \oplus C \to B$ given by $f(a,c) = a + s(c)$ is an isomorphism.

4. Recall that $[X,Y]$ denotes the set of homotopy classes of continuous functions $X \to Y$.

Let (X, x_0) be a space X with a chosen basepoint x_0 and endow S^1 with the basepoint $s_0 = (1,0)$. Recall that $\pi_1(X, x_0)$ is the set of equivalence classes of basepoint-preserving maps $(S^1, s_0) \to (X, x_0)$ where the homotopies also satisfy $h(s_0, t) = x_0$ for all $t \in [0,1]$. There is a natural transformation $v_X : \pi_1(X, x_0) \to [S^1, X]$ that forgets the basepoint-preserving nature of maps $S^1 \to X$ and homotopies between them.

Using homotopy invariance of $\pi_1(X, x_0)$ (as presented in e.g., [1, Prop. 1.5, Lem. 1.19]), prove that $v_X(\gamma) = v_X(\delta)$ implies that γ, δ are conjugate elements in $\pi_1(X, x_0)$.

5. The "topologist's sine curve" S is a closed subset of $\mathbb{R}^2$ defined as follows. Let G denote the graph of the function $f(x) = \sin(1/x)$ on the domain $(0,1]$, indicated in blue in Figure 1. Let J denote the interval

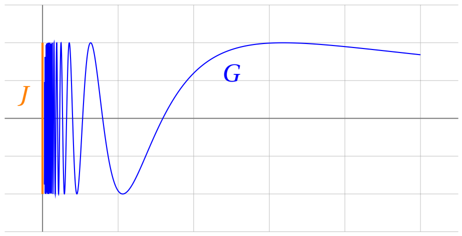

Figure 1: The topologist's sine curve, S.

$\{0\} \times [-1, 1]$, denoted in orange in Figure 1. The space S is defined to be the union $G \cup J$. The space S is well known to be connected, but to have two path components, G and J. Note that J is a closed, contractible subspace of S.

There is a continuous function $f : S \to [0, 1]$ defined by $f(x, y) = x$. Since $f(j) = 0$ for all $j \in J$, we know that f factors through the quotient map $p : S \to S/J$, i.e., we can write $f = \tilde{f} \circ p$ where $\tilde{f} : S/J \to [0, 1]$ is a continuous function.

Prove that $\tilde{f}$ is a homeomorphism. Deduce that $p : S \to S/J$ is not a homotopy equivalence.

References

[1] Hatcher (2002) *Algebraic Topology*, Cambridge University Press, Cambridge.