Math 427, 527
Homework 3
2026-02-17

1. This problem is not to be handed in. Prove the snake lemma to your own satisfaction.

2. Let $f : X \to Y$ be a map of spaces. Construct a space M_f as the quotient of the disjoint union of $X \times I$ and Y by the relation $(x,0) \sim f(x)$. There are continuous functions $i : X \to M_f$ given by $i(x) = (x,1)$ and $p : M_f \to Y$ given by $p(x,t) = f(x)$ for all t (including the case $t = 0$) and $p(y) = y$. You do not have to prove these are continuous, or that $p \circ i = f$.

(a) Prove that $p : M_f \to Y$ is a homotopy equivalence.

(b) Let C_f denote the quotient of M_f given by collapsing the subspace $X \times \{1\}$ to a point. Construct a long exact sequence in homology:

$$\cdots \to \mathrm{H}_n(X;\mathbb{Z}) \xrightarrow{f_*} \mathrm{H}_n(Y;\mathbb{Z}) \to \bar{\mathrm{H}}_n(C_f;\mathbb{Z}) \xrightarrow{\partial} \mathrm{H}_{n-1}(X;\mathbb{Z}) \to \cdots.$$

(c) If $Y = \mathrm{pt}$, so that f is the unique map $f : X \to \mathrm{pt}$, then C_f is also known as the *(unreduced) suspension* of X, and may be denoted SX. Give a formula relating $\bar{\mathrm{H}}_*(X)$ and $\bar{\mathrm{H}}_*(SX)$.

(d) **Not to be handed in:** Check that the constructions of M_f, C_f extend to give functors to **Top** from the category whose objects are maps $f : X \to Y$ and where a morphism $v : f \to f'$ is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \big\downarrow{\scriptstyle v_X} & & \big\downarrow{\scriptstyle v_Y} \\ X' & \xrightarrow{f'} & Y'. \end{array}$$

Deduce that suspension S gives a functor **Top** → **Top**.

3. Suppose X is a topological space and $g \in X$ is a point with the property that $\overline{\{g\}} = X$ (alternatively, g is an element of all nonempty open sets of X). Such a g is called a *generic point* of X.

(a) Define $f : X \to X$ by $f(x) = g$ for all x. Prove that f is homotopic to the identity function.

(b) Let $n \geq 1$ be an integer. Let $s \in S^n$ be a point. Prove that $\bar{\mathrm{H}}_*(S^n/(S^n - \{s\}))$ and $\mathrm{H}_*(S^n, S^n - \{s\})$ are not isomorphic.

4. When $X = Y = S^n$, the unreduced suspension functor, $S(-)$ from question 2, may be replaced by an explicit geometric construction, as we outline here.

Recall that $S^n \subset \mathbb{R}^{n+1}$, and let $\|-\|$ denote the usual norm on $\mathbb{R}^{n+1}$. Starting with a continuous function $f : S^{n-1} \to S^{n-1}$, define a continuous function

$$S'f : S^n \to S^n$$

by the formula

$$S'f(x_0,\dots,x_{n-1},x_n) = \left(\sqrt{1-x_n^2}\,f\left(\frac{1}{\sqrt{1-x_n^2}}(x_0,\dots,x_{n-1})\right),x_n\right).$$

(a) Prove that $\deg(S'f) = \deg(f)$. It may be helpful either to assert that $S'f$ is homeomorphic to Sf (no proof required) or to use the Mayer–Vietoris sequence.

(b) Suppose $f : S^n \to S^n$ is a reflection of the sphere across a hyperplane through $\mathbf{0} \in \mathbb{R}^{n+1}$. Prove that $\deg(f) = -1$.

5. The notation $\mathrm{GL}(n;\mathbb{R})$ denotes the group of invertible $n \times n$ matrices with entries in $\mathbb{R}$. Give it a topology as a subspace of $\mathbb{R}^{n\times n}$. You may assume that matrix operations such as multiplication and inversion are continuous.

By a *path* in $\mathrm{GL}(n;\mathbb{R})$ between two matrices B and C, we mean a continuous function $A : [0,1] \to \mathrm{GL}(n;\mathbb{R})$ so that $A(0) = B$ and $A(1) = C$.

(a) Give a path in $\mathrm{GL}(2;\mathbb{R})$ between $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and the identity matrix. Beyond giving the path, no further argument is required.

(b) Let n be a positive integer, let $a \in \mathbb{R}$ and $i, j \in \{1,\dots,n\}$ be indices satisfying $i \neq j$. The *elementary matrix* $E_{ij}(a)$ differs from the identity matrix only in the i,j-position, where it has the value a. Give a path in $\mathrm{GL}(n;\mathbb{R})$ from $E_{ij}(a)$ to the identity matrix. Beyond giving the path, no further argument is required.

(c) Let $A \in \mathrm{GL}(n;\mathbb{R})$ be a matrix. It is well known that A may be brought to reduced row-echelon form by elementary row operations. A variant is this: there exists a list of elementary matrices (of the kind defined above) $E_1,\dots,E_n$ such that

$$E_1E_2\cdots E_nA$$

is a diagonal matrix.

Prove that there is a path in $\mathrm{GL}(n;\mathbb{R})$ from A to a diagonal matrix whose diagonal entries are elements of $\{-1,1\}$.

(d) For the same A, sketch a proof that there is a path from A either to I_n, the $n\times n$ identity matrix, or to J, the matrix that agrees with I_n everywhere except for a -1 in the $1,1$-position.

(e) If $A \in \mathrm{GL}(n;\mathbb{R})$, then A gives us a function $f_A : S^{n-1} \to S^{n-1}$ by the formula

$$f_A(\mathbf{v}) = \frac{1}{\|A\mathbf{v}\|}A\mathbf{v}.$$

You may assume this depends continuously on $\mathbf{v}$ and on the entries of A.

Prove

$$\deg(f_A) = \frac{\det(A)}{|\det(A)|}.$$