Math 427, 527
Homework 4
2026-03-02

1. This question is not to be handed in. Let X be a CW complex.

(a) Let Z be a set of points such that each cell e of X contains exactly one point from Z. Let $f_Z : Z \to \mathbb{R}$ be a function. Prove by induction on the skeleta of X that there exists a continuous function $f : X \to \mathbb{R}$ with the property that $f(z) = f_Z(z)$ for all $z \in Z$. You may assume the Tietze extension theorem.

(b) Deduce that if $K \subseteq X$ is a compact subset, then K has nonempty intersection with only finitely many cells of X.

(c) Suppose $A \subseteq X$ is a closed subset. Prove there exists a continuous function $f : X \to \mathbb{R}$ with the property that $f^{-1}(\{0\}) = A$. Deduce X is Hausdorff. You may assume points are closed.

2.

(a) **This part is not to be handed in** Establish a homotopy equivalence between $S^1 \vee S^1$ and the pair-of-pants space P illustrated in Figure 1. The space P is homeomorphic to S^2 with three disjoint open disks removed.

(b) Calculate the homology of the torus-with-two-open-ends space in Figure 2. This is homeomorphic to the torus with two open balls removed. In your presentation of the homology, what is the homology class corresponding to the indicated loop under the Hurewicz map?

(c) Calculate the homology of the genus-2 surface in Figure 3.

3. Produce a CW complex as follows. Start with a solid cube, with the "obvious" CW structure: there are eight 0-cells at the vertices, twelve 1-cells, being the edges, six 2-cells being the faces and a single interior 3-cell. Now form the quotient CW complex X by identifying opposite (closed) faces, but with a twist. For each pair of opposite closed faces A and B, identify A with B by twisting A by 90° in the positive direction from the point of view of someone looking at this face from outside the cube, and then projecting to the opposite face, B. (Observe that the identification specified by this procedure is the same if the roles of A and B are reversed.)

The resulting CW complex X has one 3-cell, three 2-cells and some number of 1- and 0-cells. Determine the number of cells of X and calculate $\mathrm{H}_*(X;\mathbb{Z})$.

4. Let n be a positive integer. Suppose A is an abelian group that appears in a short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{i} A \xrightarrow{f} \mathbb{Z}/(n) \to 0.$$

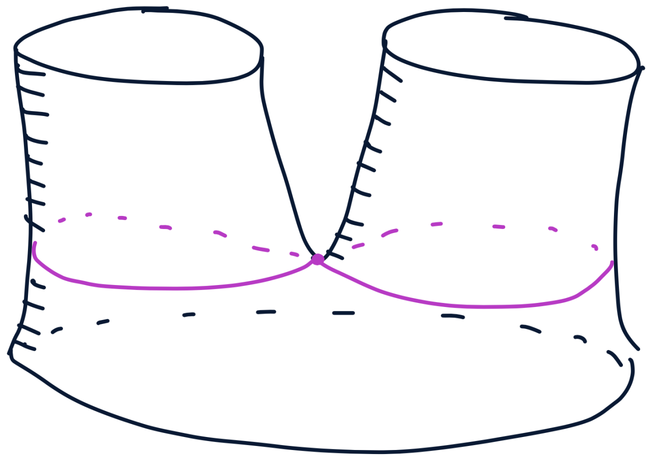

Figure 1: The “pair of pants” space P, and an embedded copy of $S^1 \vee S^1$.

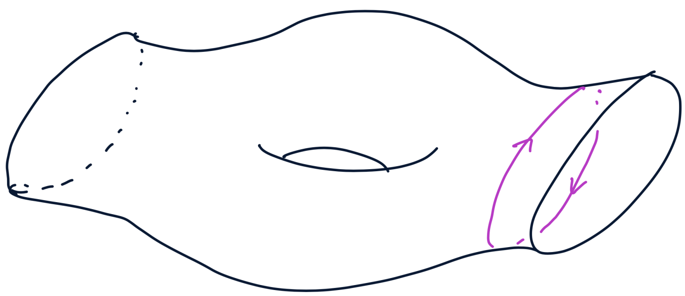

Figure 2: Torus with two open ends, and a marked embedded circle.

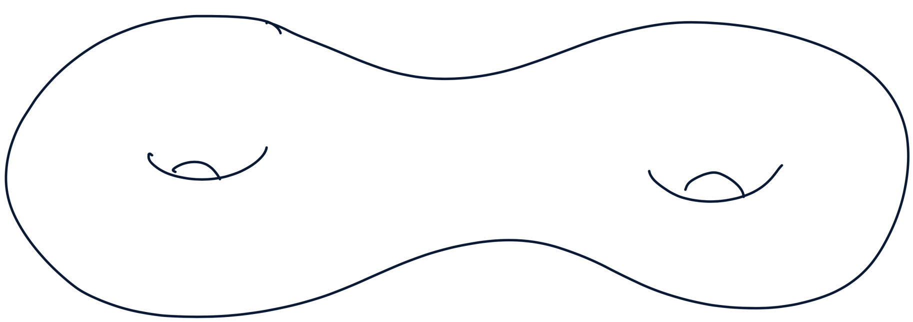

Figure 3: A genus-2 surface.

(a) Prove that A is generated by 2 elements.

(b) Prove that A is generated by 2 elements, one of which is torsion of order dividing n. Hint: if x, y generate A, so also do $m_1 x + m_2 y, n_1 x + n_2 y$ whenever m_1, m_2, n_1, n_2 are integers satisfying $m_1 n_2 - m_2 n_1 = 1$. This, combined with Bézout's lemma, may be useful.

(c) Prove that the two generators obtained in the previous step actually generate direct summands, so that $A \cong \mathbb{Z} \oplus \mathbb{Z}/(d)$ where d is a positive integer dividing n.

(d) For any positive integer d dividing n, construct a short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Z} \oplus \mathbb{Z}/(d) \xrightarrow{f} \mathbb{Z}/(n) \to 0.$$

5. Let $n \geq 2$ be an integer. Let $f : [0,1] \to S^n$ be a closed embedding (i.e., a closed continuous injective function).

(a) Let $[c] \in \bar{\mathrm{H}}_q(S^n - f([0,1]); \mathbb{Z})$ be a homology class where $q \geq 0$. Let $0 = i_0 < i_1 < i_2 < \cdots < i_{m-1} < i_m = 1$. Use the Mayer–Vietoris sequence to prove that $[c] = 0$ if and only if the images of $[c]$ in $\mathrm{H}_n(S^n - f([i_{j-1}, i_j]); \mathbb{Z})$ are 0 for all $j \in \{1, \ldots, m\}$.

(b) Observe that $f([0,1])$, being a compact set, is some positive distance away from the image of each singular n-simplex σ_k appearing in a q-chain $c = \sum_k m_k \sigma_k$. Hence or otherwise, prove that $[c] = 0 \in \bar{\mathrm{H}}_q(S^n - f([0,1]); \mathbb{Z})$.

(c) Suppose $g : S^1 \to S^n$ is a continuous injective function. Calculate $\bar{\mathrm{H}}_*(S^n - g(S^1); \mathbb{Z})$.