Math 427, 527
Homework 5
2026-03-19

1. Suppose A is an abelian group. Recall that an element $a \in A$ is said to be *torsion* if there exists a positive integer n such that $na = 0$. Let $T(A)$ denote the subgroup of torsion elements in A (you may assume these make up a subgroup). Let **Ab** denote the category of all abelian groups and homomorphisms.Prove that $T(A)$ is a naturally defined subgroup of A. Specifically: prove that $T : \mathbf{Ab} \to \mathbf{Ab}$ is a functor (you can skip verifying it preserves compositions and identities), and prove that the inclusion map $T(A) \to A$ forms part of a natural transformation from T to the identity functor. We can view this as saying that T is a *subfunctor* of the identity functor.

2. If M is a topological space and $f : M \to M$ is a homeomorphism, then the *mapping torus* T_f of f is the quotient of the space $M \times [0,1]$ obtained by making the identification $(m,0) \sim (f(m),1)$ for all $m \in M$. It can be covered by two open sets: U, V which are the homeomorphic images in T_f of $M \times (0,1)$ and $M \times ([0,1/2) \cup (1/2,1]$ respectively.

(a) Prove the following useful technical result. It holds for any category of R-modules, but this exercise is stated only in the category of abelian groups. Suppose

$$\cdots \longrightarrow A_3 \xrightarrow{d_3} A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} \cdots$$

is an exact sequence, and that $B_2 \subseteq A_2$ and $B_1 \subseteq A_1$ are subgroups with the property that d_2 maps B_2 isomorphically to B_1. Prove there is an induced exact sequence

$$\cdots \longrightarrow A_3 \xrightarrow{d'_3} A_2/B_2 \xrightarrow{d'_2} A_1/B_1 \xrightarrow{d'_1} A_0 \xrightarrow{d'_0} \cdots.$$

(b) By applying the previous technical result to a Mayer–Vietoris sequence, or otherwise, prove that for any mapping torus T_f associated to $f : M \to M$ there is a long exact sequence in homology of this form:

$$\cdots \longrightarrow \mathrm{H}_q(M) \xrightarrow{\mathrm{id}-f_*} \mathrm{H}_q(M) \longrightarrow \mathrm{H}_q(T_f) \longrightarrow \mathrm{H}_{q-1}(M) \longrightarrow \cdots.$$

(c) Let $M = S^1 \times S^1$ and let $f : S^1 \times S^1 \to S^1 \times S^1$ be the homeomorphism that is the identity of the first factor and a reflection on the second. Calculate the integral homology of the mapping torus T_f.

3. Suppose X is a path-connected space with basepoint x_0. This problem is part of a pair that will establish the following result: the Hurewicz map $\eta : \pi_1(X, x_0) \to \mathrm{H}_1(X;\mathbb{Z})$ is the abelianization of $\pi_1(X, x)$: i.e., the map η is surjective and its kernel is the normal subgroup generated by commutators $\alpha\beta\alpha^{-1}\beta^{-1}$.

You may want to use the result of Example 2.1 in the textbook in answering this question.

(a) We write $c_0^0, c_1^0 \in S_0([0,1])$ for the constant simplices with value 0 and 1. and suppose $k \in S_1([0,1])$ is a chain with the property that $d(k) = c_0^0 - c_1^0$. Prove that the class of k generates $\mathrm{H}_1([0,1],\{0,1\})$. Deduce that the image of k under the map $h : [0,1] \to S^1$ given by $h(t) = (\cos 2\pi t, \sin 2\pi t)$ generates $\mathrm{H}_1(S^1)$.

(b) For all $x \in X$, pick a 1-simplex $\tau_x : \Delta^1 \to X$ such that $\tau_x(0) = x_0$ and $\tau_x(1) = x$. Given any 1-simplex $\sigma : \Delta^1 \to X$, define a 1-cycle

$$l(\sigma) = \tau_x + \sigma - \tau_y, \quad \text{where } x = \sigma(0),\ y = \sigma(1).$$

You do not have to prove this is a 1-cycle. Prove that the homology class of $l(\sigma)$ is in the image of the Hurewicz map $\eta : \pi_1(X, x_0) \to \mathrm{H}_1(X, x_0)$.

(c) Prove that $\eta : \pi_1(X, x_0) \to \mathrm{H}_1(X)$ is surjective.

4. We say an abelian group A is *uniquely divisible* if the multiplication-by-n map $\times n : A \to A$ is an isomorphism for all positive integers n. You may assume the following easy (but boring to prove) fact: an abelian group A is uniquely divisible if and only if the abelian group structure on A extends to a $\mathbb{Q}$-vector-space structure.

(a) Let A be an abelian group and suppose that for all prime numbers p, we have $\mathbb{Z}/(p) \otimes_{\mathbb{Z}} A = 0$ and $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/(p), A) = 0$. Prove that A is uniquely divisible.

(b) You may assume that the usual multiplication map $\mu : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ is $\mathbb{Z}$-bilinear. Prove using the universal property that $\mathbb{Q}$ is a tensor product of $\mathbb{Q}$ and $\mathbb{Q}$ over $\mathbb{Z}$: in particular, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$.

(c) Suppose A is an abelian group such that $A \otimes_{\mathbb{Z}} k = 0$ and $\mathrm{Tor}_1^{\mathbb{Z}}(A, k) = 0$ for all

$$k \in \{\mathbb{Q}, \mathbb{Z}/(2), \mathbb{Z}/(3), \mathbb{Z}/(5), \ldots\}.$$

(These fields are called *prime fields*). Prove that $A = 0$.

(d) **Not to be handed in:** Give an example of a nonzero abelian group A for which $A \otimes_{\mathbb{Z}} k = 0$ for all prime fields k (and therefore, for all fields k).

5. Not to be handed in.

Let $\mathbf{C}$ and $\mathbf{D}$ be two categories. Suppose $L : \mathbf{C} \to \mathbf{D}$ and $R : \mathbf{D} \to \mathbf{C}$ are functors with the property that for all objects $c \in \mathbf{C}$ and $d \in \mathbf{D}$, there are bijections

$$\mathbf{D}(L(c), d) \leftrightarrow \mathbf{C}(c, R(d))$$

that are natural in both c and d.

(Two functors with this property are said to be *adjoint* functors, with L being the *left adjoint* and R the *right adjoint*.)

(a) If p is a pushout of $z \leftarrow x \to y$ in $\mathbf{C}$, prove that $L(p)$ is a pushout of $L(z) \leftarrow L(x) \to L(y)$ in $\mathbf{D}$.

(b) Suppose

$$A \xrightarrow{f} B \xrightarrow{q} C \to 0$$

is an exact sequence. Prove that C is a pushout of $0 \leftarrow A \xrightarrow{f} B$.

(c) Let D be another abelian group. You may assume $- \otimes_{\mathbb{Z}} D$ is left adjoint to $\mathrm{Hom}(D, -)$, both functors being $\mathrm{Ab} \to \mathrm{Ab}$. Use adjunction to prove that

$$A \otimes_{\mathbb{Z}} D \xrightarrow{f \otimes \mathrm{id}_D} B \otimes_{\mathbb{Z}} D \xrightarrow{q \otimes \mathrm{id}_D} C \otimes_{\mathbb{Z}} D \longrightarrow 0$$

is an exact sequence.