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Chapter 1

CW Complexes

1.1 CW Structures

A "map" means a continuous function.

Notation 1.1. The following notation will be used throughout:

$$D^n = \{x \in \mathbb{R}^n \mid \|x\|_2 \leq 1\}$$

and

$$S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid \|x\|_2 = 1\}.$$

If basepoints are called for, both can be given the basepoints $(1, 0, \ldots, 0)$.

Definition 1.2. A *CW Structure* on a Hausdorff topological space X is a set of maps, called *characteristic maps*

$$\Phi_\alpha : D^n \to X$$

satisfying the following properties.

1. Each Φ_α restricts to a homeomorphism from $\mathrm{Int}D^n \to X$. The image of this homeomorphism is denoted e_α and is called a *cell* of the structure (or of X). The closure $\bar{e}_\alpha$ is a *closed cell.*

2. The cells e_α are all disjoint and their union is X.

3. For each characteristic map Φ_α, the image $\Phi_\alpha(\partial D^n)$ is contained in a finite number of cells of dimension less than n.

4. A subset of X is closed if and only if it meets the closed cels of X in closed sets.

Remark 1.3. Suppose X is equipped with a CW structure. Fix $n \geq 1$. Let X_n (Hatcher has X^n) denote the unions of all cells of dimension n or less in X. This is called the *n-skeleton* of (the CW structure on) X. Let A be an indexing set, so that $\{\Phi_\alpha\}_{\alpha \in A}$ is the set of all characteristic maps $\Phi : D^n \to X$. The following diagram is a push out diagram

$$\begin{tikzcd}
\coprod_{\alpha \in A} \partial D^n \arrow[r, "\phi_\alpha"] \arrow[d] & X_{n-1} \arrow[d, hook] \\
\coprod_{\alpha \in A} D^n \arrow[r, "\Phi_\alpha"] & X
\end{tikzcd}$$

Here ϕ_α is the restriction of Φ_α to the boundary of the disk. The maps ϕ_α are called *attaching maps* for obvious reasons.

This means that the CW structure gives us a sequence of spaces, each one a closed subspace of the subsequent spaces, called *skeleta* of X

$$X_0 \subseteq X_1 \subseteq \cdots \subseteq X$$

The topology on X is the *weak* topology, i.e., the colimit topology. That is, to specify a continuous function $X \to Y$ is the same as specifying a compatible sequence of continuous functions $X_n \to Y$ for all n.

Remark 1.4. If a space admits a CW structure, unless it is is 0-dimensional (i.e., discrete), it will admit infinitely many CW structures. A space admitting a CW structure will be called a *CW complex*, and generally we will assume a particular fixed CW structure, but we should not lose sight of the fact that there are many possibilities.

Remark 1.5. CW complexes are Hausdorff by definition. They are also normal—disjoint closed subspaces can be separated by open neighbourhoods—and locally contractible—every point has a local basis consisting of subspaces homotopy equivalent to a point.

Remark 1.6. CW complexes have the property that to test if a subset $C \subset X$ is closed, it is sufficient to test if $C \cap \operatorname{im}(\Phi_\alpha)$ is closed for each characteristic function. Since the images of the characteristic functions are compact, it suffices to test if $C \cap K$ is closed in K as K ranges over the compact subspaces of X. Therefore X is *compactly generated*.

Definition 1.7. If X is a CW complex, then a *subcomplex* $Y \subseteq X$ is a subspace of X that is a CW complex by means of a subset of the characteristic functions of a structure on X. Equivalently, $Y \subseteq X$, and Y has a CW structure such that $Y_n \subseteq X_n$ for all n.

If X and Y are two CW complexes, then a *cellular map* $f : Y \to X$ is a map that restricts to maps $f : Y_n \to X_n$ for all n.

1.2 Products

Remark 1.8. It may be the case that X and Y are compactly generated topological spaces, but $X \times Y$ is not compactly generated. In fact, one can find an example of this behaviour where X

and Y are CW complexes. This looks like bad news, because the product topology on $X \times Y$ cannot be the topology of a CW complex.

Definition 1.9. If X is a topological space, let kX denote the topological space that has the same elements (points) as X, but where a subset $C \subset X$ is closed in kX if and only if $C \cap K$ is closed in K for all compact Hausdorff subsets of X. It is an exercise to verify that this defines a topology on X. There is a map $kX \to X$, which is the identity if X is already compactly generated and Hausdorff.

Notation 1.10. Suppose X and Y are Hausdorff spaces. Let $X \times_c Y$ denote the space $k(X \times Y)$.

Remark 1.11. If both X and Y are compactly generated and at least one is locally compact , then $X \times Y$ is compactly generated, so $k(X \times Y) = X \times Y$.

Proposition 1.12. *If X, Y are CW complexes with characteristic maps Φ_α and Ψ_β, then $X \times_c Y$ is a CW complex with characteristic maps $\Phi_\alpha \times \Psi_\beta$.*

Exercise 1.13. Find two CW complexes X and Y so that $X \times Y$ and $X \times_c Y$ are not homeomorphic.

1.3 Categorical constructions

Notation 1.14. A *based* or *pointed space* is a pair (X, x_0), where X is a topological space and $x_0 \in X$. A map of based spaces $f : (X, x_0) \to (Y, y_0)$ must satisfy $f(x_0) = y_0$). The basepoints will not always be written.

If $A \subset X$ is a subspace of X, then X/A is a based space with the image of A serving as the basepoint. The notation X_+ will be used to denote the space X with a disjoint basepoint.

Notation 1.15. We will write **Top** for the category of topological spaces and $\mathbf{Top}_*$ for the category of pointed spaces.

Notation 1.16. If (X, x_0) and (Y, y_0), then we can form

1. $(X \times_c Y, (x_0, y_0))$
2. $X \vee Y, *$
3. $X \wedge_c Y, *$

Note that under most circumstances the subscript c is unnecessary.

We write $\mathscr{C}(X, Y)$ for the space of continuous functions from X to Y, endowed with the compact open topology. This behaves best if X and Y are compactly generated (weak) Hausdorff spaces—for instance, if X, Y are CW complexes. For three such spaces, there is a natural bijection (an adjunction of functors)

$$\mathrm{Map}(X \times_c Y, Z) \approx \mathrm{Map}(X, \mathscr{C}(Y, Z)).$$

The subspace of $\mathscr{C}(X,Y)$ consisting of functions satisfying $f(x_0) = y_0$ is closed (exercise). We write $\mathscr{C}_*(X,Y)$ for this subspace—the basepoints having been suppressed.

The previous adjunction yields a pointed adjunction

$$\mathrm{Map}_*(X \wedge_c Y, Z) \approx \mathrm{Map}_*(X, \mathscr{C}_*(Y,Z))$$

Remark 1.17. If X and Y are CW complexes, then $X \vee_c Y$ and $X \wedge Y$ are both equipped with obvious CW structures.

Remark 1.18. Some facts about smash products. Let (X,x_0), (Y,y_0), (Z,z_0) be based spaces

1. $X \wedge (Y \wedge Z) \approx (X \wedge Y) \wedge Z$
2. $X \wedge Y \approx Y \wedge X$. Warning: this is not an equality.
3. If $X \to Y$ is an inclusion of based spaces then $(Y \wedge Z)/(X \wedge Z) \approx (Y/X) \wedge X$. Both are the universal space for maps from $Y \times Z$ sending $X \times Z$ and $Y \times \{z_0\}$ both to the basepoint.

Notation 1.19. If (X,x_0) is a based space then the notation ΣX is used for $X \wedge S^1$. It is an exercise to prove that $S^n \wedge S^1 \approx S^{n+1}$

Definition 1.20. An *unpointed homotopy* between maps $f,g : X \to Y$ is a map $H : X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$. Homotopy between maps is an equivalence relation, and is compatible with composition.

Therefore we can form a *Homotopy category* **H**, the objects are topological spaces and the set of morphisms $X \to Y$ are the homotopy classes of continuous functions.

Definition 1.21. There is also a notion of *pointed homotopy*. A homotopy between two maps $f,g : (X,x_0) \to (Y,y_0)$ is a map $H : X \wedge I_+ \to Y$ (of based spaces). Unwinding, this means that H is the same data as a map $H : X \times I \to Y$ with the property that $H(x_0,t) = y_0$ for all t. There is a *pointed homotopy category*, $\mathbf{H}_*$, where the objects are based spaces, and the morphisms $X \to Y$ are pointed homotopy classes of pointed maps. The set of maps $X \to Y$ in thi category is often denoted $[X,Y]$.

Notation 1.22. An isomorphism in **H** or $\mathbf{H}_*$ is called a *homotopy equivalence*. A space that is homotopy equivalent to a point is said to be *contractible*.

Notation 1.23. Suppose $A \subseteq X$ are two topological spaces and there is a map $X \to A$ so that the composite $A \to X \to A$ is the identity, then A is said to be a *retract* of X. If, moreover, $X \to A \to X$ is homotopic to the identity, then A is said to be a *deformation retract* of X.

Exercise 1.24. Let **C** be a category and let

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

be a diagram in which $g \circ f$ and $h \circ g$ are isomorphisms. Show that f, g, h are isomorphisms.

Exercise 1.25. Show that a deformation retract is a homotopy equivalence. Give an example of a retract that is not a homotopy equivalence.

Example 1.26. $\pi_1(X, x_0) = [S^1, X]$. In fact, we define $\pi_n(X, x_0) = [S^n, X]$ for all $n \geq 0$.

Exercise 1.27. Suppose X is a topological space with a continuous action of $I = [0,1]$ on X. Namely, there exists a map $\alpha : I \times X \to X$—such a map arises whenever X is a topological vector space over either $\mathbb{R}$ or $\mathbb{C}$. Suppose further that the action is multiplicative, in that $(st) \cdot x = s(tx)$ for all $s, t \in I$. Let X_0 denote the subset of X consisting of points of the form $0 \cdot x$ for $x \in X$. Show that $X_0 \subseteq X$ is a closed deformation retract of X.

In particular, all topological vector spaces are contractible.

Use this exercise to show that $S^n \subset \mathbb{R}^{n+1} \setminus \{0\}$ is a deformation retract.

1.4 CW complexes and Homotopy

We remark that a deformation retraction of X onto a subspace A can be encapsulated by a single map $H : X \times I \to X$ such that $H(x, 0) = x$ and $H(x, 1) \in A$ with $H(a, 1) = a$. It is said to be a *strong* deformation retraction of $H(a, t) = a$ for all t. We will almost always be using strong deformation retractions, just incidentally. When we say we have a "deformation retraction $X \to A$" we really mean a map $X \times I \to X$ of the form just discussed.

Proposition 1.28. *Suppose K is a compact subset of a CW complex X. Then K meets only finitely many open cells of X.*

Proof. Suppose not, then find a sequence of points (x_n), one in the intersection of each of countably infinitely many cells with K. Verify that this has the discrete topology, a contradiction. □

Definition 1.29. A CW pair (X, A) means a CW complex X and a subcomplex A.

The following is [Hat10, Proposition 0.16], even down to the proof.

Proposition 1.30. *If (X, A) is a CW pair, then $X \times \{0\} \cup A \times I$ is a (strong) deformation retract of $X \times I$.*

Proof. First we do the special case where $X = D^n$ and $A = \partial D^n = S^{n-1}$. Here it is obvious (radial projection for instance).

This gives a deformation retraction of $X_n \times I$ on $X_n \times \{0\} \cup (X_{n-1} \cup A_n) \times I$, just do the other thing for each cell in X_n but not A_n. In fact, we can produce a deformation retraction

$$X \times \{0\} \cup ((X_n \cup A) \times I) \to X \times \{0\} \cup ((X_{n-1} \cup A) \times I)$$

Perform that last deformation retraction above in the time interval $[2^{-n-1}, 2^{-n}]$. Observe that this actually gives a deformation retraction

$$X \times \{0\} \cup (X \cup A) \times I \to X \times \{0\} \cup ((X_{-1} \cup A) \times I)$$

and $X_{-1} = \emptyset$. □

Proposition 1.31 (The Homotopy Extension Property)**.** *Suppose* (X, A) *is a CW pair,* Y *is a space, and suppose there is a map* $H_0 : X \times \{0\} \cup A \times I \to Y$*. Then* H_0 *extends to a map* $H : X \times I \to Y$*.*

Proof. This is a corollary of the previous statement. □

Theorem 1.32 (The Cellular Approximation Theorem)**.** *Suppose* (X, A) *is a CW pair and* $f : X \to Y$ *is a map between CW complexes. Suppose* $f|_A : A \to Y$ *is cellular. Then* f *is homotopic to a map* g *so that* $f|_A = g|_A$ *and* $g : X \to Y$ *is cellular.*

Proof from Hatcher. We will do most of the proof. One highly technical, point-set part will be left for the homework.

We work by induction. Suppose we've already done this on X_{n-1} and let e^n be an n-cell of X. Write $\bar{e}^n$ for the closure of e^n, which is compact, and so $f(e^n)$ can intersect only finitely many cells of Y. Let e^k be a cell of maximal dimension meeting $f(e^n)$. If $k \leq n$, there's nothing to do.

We want to show that there is some homotopy of $f|_{X_{n-1} \cup e^n}$ to a different map, let's call it $f' : X_{n-1} \cup e^n \to Y$ for now, such that $f|_{X_{n-1}} = f'|_{X_{n-1}}$ but so that $f'(e^n) \subset \partial \bar{e}^k$—that is, $f'(e^n)$ has been deformed to miss e^k altogether. The main idea is the following: if there is some point $p \in e^k$ such that $p \not\in f(e^n)$, then we can deformation-retract $Y^k - p$ onto $Y^k - e^k$, and composing this with f, we get the desired map.

The problem is a surprising one: we don't know that there is a point $p \in e^k$ that is not in the image of $f(e^n)$. For instance, f might restrict to a space-filling curve $\mathbb{R} \to \mathbb{R}^2$. To get around this, we have to deform f to be somewhat well-behaved on the interior of e^n. Hatcher produces a piecewise-linearization in Lemma 4.10. We omit this.

Repeat the above procedure as often as necessary for e^n to avoid all the cells e^k of dimension $k > n$. In fact, we can do this for all n-cells of X simultaneously. This gives us a homotopy between $f|_{X_n \cup A} : X_n \cup A \to Y$ and a cellular map $f'|_{X_n \cup A} : X_n \cup A \to Y$. Use the homotopy extension property to give a homotopy between two maps f, f' on $X \cup A$, where the first is cellular on $X_{n-1} \cup A$ and the second is cellular on $X_n \cup A$, and notice that on the cells where the thing was already cellular (e.g., on A) nothing will change.

Then compose the countably many homotopies required together using the usual trick. This gives cellular approximation. □

Corollary 1.33. *Suppose* X *is a (based) CW complex having no cells of dimension greater than* d*. Suppose* Y *is a based CW complex. Then the natural map*

$$[X, Y^{d+1}] \to [X, Y]$$

is a bijection.

Corollary 1.34. $\pi_n(S^{n+k}, s_0) = 0$ *when* $k > 0$*.*

1.4.1 Sundries

Each S^n includes in S^{n+1} as an equator. The colimit is S^∞, given the colimit topology. That is, a subset of S^∞ is closed if and only if it meets each S^n in a closed subset. Let $i_n : S^n \to S^\infty$ denote the inclusion.

S^∞ consists of infinite strings, $(a_1, \dots)$ where almost all terms are 0. The cells are sets of the form $(a_1, \dots, a_n, 0, \dots)$ where $a_n > 0$ or $a_n < 0$. There are therefore 2 cells of each dimension. A subset is closed if and only if it meets each S^n in a closed subset.

S^∞ is given the basepoint $(1, 0, \dots)$, in keeping with the basepoints of all the S^n.

Proposition 1.35. *The sphere S^∞ is contractible.*

Proof. Let $h_n : S^\infty \to S^\infty$ be a map that is cellular and such that $h_n(S^n) = s_0$. Then h_n is homotopic to a map h_{n+1} that is cellular and such that $h_{n+1}(S^{n+1}) = s_0$.

This is proved as follows. The restriction of h_n to S^{n+1} lies in S^{n+1}, by cellularity. The inclusion of $S^{n+1} \subset S^\infty$ is nullhomotopic. Compose $h_n|_{S^{n+1}} \times \mathrm{id} : S^{n+1} \times I \to S^{n+1}$ with the nullhomotopy to give a homotopy from $h_n|_{S^{n+1}} : S^{n+1} \to S^\infty$ to the null map $h' : S^{n+1} \to S^\infty$. Then use HEP to extend to a homotopy from h_n to a map $h' : S^\infty \to S^\infty$ that is null on the n-skeleton. Use cellular approximation (if we were precise, we wouldn't need this) to make a homotopic cellular map $h_{n+1} : S^\infty \to S^\infty$ contracting the $n+1$-skeleton.

Set $h_{-1} = \mathrm{id}_{S^\infty}$.

Once all this is set up, do the homotopy from h_n to h_{n+1} speeded up, in the interval $[1 - 2^{-n-1}, 1 - 2^{-n-2}]$. This gives us a continuous function $H : S^\infty \times [0,1) \to S^\infty$. For each point p of S^∞, there exists some time t_p such that H sends p to the basepoint after time t_p, so we extend H to a homotopy on all of $[0,1]$ by defining it to be the constant map to a basepoint at time 1. The map H is continuous when restricted to any finite skeleton, so is continuous. □

Remark 1.36. In the homework, a criterion is given for when a topological vector space V satisfies the condition that $V \setminus 0$ is contractible. Specifically, if there exists a non-surjective linear self map $T : V \to V$ such that T has no eigenvector in V, then $V \setminus 0$ is contractible. If V is a normed vector space, then

$$v \mapsto \frac{v}{1 + t\|v\| - 1}$$

gives a deformation retract of $V \setminus 0$ onto the unit sphere in V, denoted $S(V)$.

It follows that if V is a normed vector space with a non-surjective self map T without any eigenvectors, then $S(V)$ is contractible. This yields many examples of "infinite spheres", all of which are contractible.

1.5 Cones, Cylinders, Quotients

Definition 1.37. Let $f : X \to Y$ be a map of spaces. We define the *mapping cylinder* $M(f)$ as the quotient of $X \times I \coprod Y$ by the relation $(x, 1) \sim f(x)$. The map f can be factored $X \to M(f) \to Y$, by including $X = X \times \{0\}$ in $M(f)$. The space Y is a deformation retract of $M(f)$.

Remark 1.38. It can be proved that f is a homotopy equivalence if and only if X is a deformation retract of $M(f)$. In particular, if f is a homotopy equivalence, then $M(f)$ is a space such that both X and Y are deformation retracts.

Definition 1.39. Let $f : X \to Y$ be a map of CW complexes. Define the *mapping cone* of f, denoted $C(f)$, to be the quotient of $M(f)$ obtained by collapsing $X \times \{0\}$ to a point.

Proposition 1.40. *If f is a cellular map between CW complexes, then $C(f)$ is a CW complex.*

Remark 1.41. There is a category where the objects are maps $f : X \to Y$ and the morphisms are commuting squares

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

The formation of $M(f)$ and $C(f)$ is functorial in this category.

Definition 1.42. A square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"] & Y \arrow[d, "j"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

is said to be *homotopy commutative* if there exists a homotopy Φ from $j \circ f$ to $f' \circ i$.

Remark 1.43. Suppose given a homotopy-commutative square, as above. Then we may use the data of i, j and Φ to produce a map $C(f) \to C(f')$.

Specifically, for $t \leq 1/2$, define the map by $(x,t) \mapsto (i(x), 2t))$ and for $t \geq 1$, define it by $\Phi(x, 2t-1)$. (this makes more sense when drawn out).

Proposition 1.44. *Let $f, g : X \to Y$ be two maps, homotopic via $H : X \times I \to Y$. Then $C(f) \simeq C(g)$.*

Hatcher proves this indirectly. The proof here is adapted from [Ark11].

Proof. Use the fact that we have a homotopy-commutative square

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"] & X \arrow[d, "g"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

and use the construction in the previous remark to get the map $\phi : C(f) \to C(g)$.

The same construction with the reverse of H gives a map $\psi : C(g) \to C(f)$.

Showing that ϕ and ψ are homotopy inverses is a tricky business. We sketch the argument that $\psi \circ \phi \simeq \mathrm{id}_{C_f}$.

For this, we should determine what $\psi \circ \phi$ actually does. For $t \in [0, 1/4]$, we have

$$\psi \circ \phi(x, t) = (x, 4t)$$

For $t \in [1/4, 1/2]$ we have

$$H(x, 4t - 1)$$

and for $t \in [1/2, 1]$ we have $H(x, 2 - 2t)$.

Then since we are composing H with $\bar{H}$, we can remove both, and deform this map to the identity. This can be written down precisely, but doing so is a waste of time in class.

The argument for $\phi \circ \psi$ is identical. □

Lemma 1.45. *Let $A \subset X$ be a CW pair and suppose A is contractible (deformation retracts onto a point). Then the map $X \to X/A$ is a homotopy equivalence.*

Proof. There's a homotopy of $\mathrm{id} : A \to A$ to a map $A \to *$. Extend this (HEP) to a homotopy $H : X \times I \to X$ from id to a map $s : X \to X$, which takes A to a point. Since $H(A, t) \subset A$ for all time t, there is a reduction of H to a homotopy $h : X/A \times I \to X/A$, starting at $\mathrm{id}_{X/A}$

Note that $s : X \to X$ actually factors through a map $f : X/A \to X$. We claim that this is a homotopy inverse to the reduction $X \to X/A$.

The composite $f : X/A \to X \to X/A$ is actually the end of the homotopy h, so is homotopic to the identity. The composite $X \to X/A \to X$ is similarly homotopic to the identity, via H. □

Proposition 1.46. *Let $i : A \hookrightarrow X$ be a CW pair. Then $C(i) \simeq X/A$.*

Proof. Consider the subcomplex of $C(i)$ consisting of $C(\mathrm{id}_A)$. This is contractible, and $C(i)/C(\mathrm{id}_A) \approx X/A$. □

Chapter 2

Cellular Homology

2.1 Homological Algebra

Let R denote a commutative, unital, associative ring. In practice, R will be a quotient ring of a subring of the rational numbers, of which the most common examples are $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}/(p) = \mathbb{F}_p$, or R may occasionally be $\mathbb{R}$ or $\mathbb{C}$.

We assume you know what a left R-module is. Since R is commutative, we can be sloppy about left- versus right-R-modules. There is a category R-**Mod** of left R-modules. This category is *additive*, in that

- One can add two maps of R-modules: $f, g : M \to N$ to get $f + g$. In fact, $\mathrm{Hom}_R(M, N)$ carries an R-module structure.

- There is a direct sum operation $M \oplus N$ that is both a product and a coproduct (it's a biproduct).

It is furthermore *abelian*, which is to say that a map $f : M \to N$ has a kernel and a cokernel: $\operatorname{coker} f = N/(\operatorname{im} N)$, and the kernel and cokernel satisfy the isomorphism theorems.

As a special case, when $R = \mathbb{Z}$, the category of R-modules is the same as the category of abelian groups.

Definition 2.1. A *chain complex* of R-modules is a sequence of R-modules and maps, either finitely many or infinitely many,

$$M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}}$$

such that $d_i \circ d_{i+1} = 0$ for all applicable i. The maps d_i are called the *differentials* and are often written simply as d.

In the *homological convention* the indices are written as subscripts and the differentials reduce degree. There is also a *cohomological convention* where the indices are written as superscripts and the differentials increase degree.

Definition 2.2. If $(M_\bullet, d)$ is a chain complex, then the elements $\ker d_i$ (for any i) are called *cycles* and are written $Z_i \subset M_i$. The elements $\operatorname{im} d_{i+1} \subset Z_i$ are called *boundaries* and are written B_i. The *homology* of the complex at i is

$$H_i(M_\bullet) = Z_i / B_i$$

The intuition here is that d_i takes elements in M_i to their 'boundaries' in M_{i-1}. A thing without a boundary is a cycle. The image of d_i consists entirely of boundaries. This might become clearer once we start doing the topology.

Definition 2.3. A chain complex is *exact* if the homology vanishes, which is equivalent to saying that the kernel of each differential is the image of the preceding one. One can say a complex is exact at i if $\mathrm{H}_i(M_\bullet)$ in particular vanishes. An exact chain complex is called an *exact sequence*. If the chain complex is infinite (in at least one direction) it is called a *long exact sequence*—and sometimes if it is simply very long, it can be called this.

Definition 2.4. A *short exact sequence* of R-modules

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is an exact chain complex of 5 terms, the two extreme terms of which are 0.

Remark 2.5. The exactness here implies that f is an injection, g is a surjection and $B/\operatorname{im}(f) \to C$ is an isomorphism. Loosely, this says $A \subset B$ and $C = B/A$ (but strictly, all this is true only up to isomorphism).

Remark 2.6. Given an exact sequence

$$M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{0} M_{i-1} \xrightarrow{d_{i-1}} M_{i-2}$$

where the indicated map is 0, then d_{i+1} is surjective and d_{i-1} is injective. As a special case, if

$$0 \to M \to N \to 0$$

appears in an exact sequence, then $M \to N$ is an isomorphism.

Definition 2.7. A *free* R-module F is an R-module that is a (possibly infinite) direct sum of modules isomorphic to R.

Remark 2.8. We assume the axiom of choice. Our definition of a free R-module is therefore equivalent to the existence of a subset $B \subseteq F$ with the property that every element of F can be written uniquely as a finite R-linear combination of elements of B. We will call such a B a *basis* for F. If R is a field, then every R-module is free. Over a commutative ring that is not a field, there is a nontrivial ideal $\mathfrak{p} \subset R$, and $R/(\mathfrak{p})$ is not free as an R-module since a nonzero element $p \in \mathfrak{p}$ annihilates $R/\mathfrak{p}$ but does not annihilate R or any free R-module. In less fancy terms, the abelian groups $\mathbb{Z}/(n)$ are not free.

Example 2.9. If R is a PID—and all quotient rings of subrings of $\mathbb{Q}$ are PIDs, since they are all quotients of localizations of $\mathbb{Z}$—, then it is a theorem that every submodule of a free module is free. In particular, if R is a PID and M is an R-module, we can find a free module F_0 and surjection $s_0 : F_0 \to M$ and $\ker s_0 = F_1$ is also free. There is a short exact sequence

$$0 \to F_1 \to F_0 \to M \to 0.$$

Remark 2.10. While $M \times N$ and $M \oplus N$ mean the same thing for R-modules, the infinite analogues differ. The notation $\prod M_i$ means the module of sequences (m_i) where $m_i \in M_i$, and $\bigoplus M_i$ means the submodule where almost all elements are nonzero. The module $\prod_{\mathbb{N}} \mathbb{Z}$ is not free.

2.2 Brief resumé on functors

Definition 2.11. Given two categories $\mathbf{C}$ and $\mathbf{D}$, a *(covariant) functor* $F : \mathbf{C} \to \mathbf{D}$ is an assignment of an object $F(c)$ of $\mathbf{D}$ to each object c of $\mathbf{C}$, and, for all $\phi : c \to c'$ in $\mathbf{C}$, a morphism $F(\phi) : F(c) \to F(c')$ so that F preserves

1. Identity morphisms,
2. Function composition.

Remark 2.12. It follows from the above that functors preserve

- Isomorphisms
- Split monomorphisms—morphisms with a right inverse—in the case of topological spaces, these are the retracts
- Split epimorphisms.

2.3 Homology

Fix a ring R. Most often, the ring is $\mathbb{Z}$.

Definition 2.13. A *reduced ordinary homology theory for CW complexes* with coefficients in R is a sequence of functors indexed by $i \in \{0, 1, \dots\}$

$$\tilde{\mathrm{H}}_i(\cdot; R) : \mathbf{H}_\bullet \to R\text{-}\mathbf{Mod}$$

satisfying the following further properties

1. Long exact sequence: if $j : A \hookrightarrow X$ is a CW pair, then there is a natural long exact sequence

$$\longrightarrow \tilde{\mathrm{H}}_i(A; R) \xrightarrow{j} \tilde{\mathrm{H}}_i(X; R) \xrightarrow{q} \tilde{\mathrm{H}}_i(X/A; R) \xrightarrow{\partial} \tilde{\mathrm{H}}_{i-1}(A; R) \longrightarrow$$

2. Wedge sum axiom: Given an infinite wedge sum $\bigvee_i X_i$, the projection maps $\bigvee_i X_i \to X_i$ assemble to give an isomorphism

$$\tilde{\mathrm{H}}_n(\bigvee X_i; R) \xrightarrow{\cong} \bigoplus_i \tilde{\mathrm{H}}_n(X_i; R)$$

3. Normalization: $\tilde{H}_0(S^0; R) = R$ and $\tilde{H}_n(S^0; R) = 0$ otherwise.

Remark 2.14. Strictly speaking, the above definition is incorrect, because the maps ∂ associated to the pair (X, A) and the integer i are part of the data, not simply a property.

Remark 2.15. In this course, we will not discuss extraordinary homology theories, but they do exist. In anticipation of these, we might extend the indexing set of $\tilde{\mathrm{H}}_i(X; R)$ to $i \in \mathbb{Z}$, declaring the negatively-graded groups to be 0.

Remark 2.16. *Naturality* means that if (X, A) is a CW pair and (Y, B) is another, and there is a compatible map $X \to Y$, which in the case at hand means that the square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

homotopy commutes, then there is a commutative diagram of long exact sequences (a ladder).

Theorem 2.17. *For any ring R, a reduced ordinary homology theory exists.*

We will postpone proving this theorem for a long time. Instead, we will concentrate on the implications of the axioms.

Theorem 2.18. *If X is a pointed space that is homotopy equivalent to a pointed CW complex, then $\tilde{\mathrm{H}}_i(X; R)$ is determined by the axioms.*

We will prove this result somewhat sooner, but first, we will prove some simpler results and make some definitions.

Proposition 2.19. $\tilde{\mathrm{H}}_i(*; R) = 0$

Proof. Since $S^0 \vee * = S^0$, this follows from the wedge sum axiom. (For finite wedge sums, the wedge sum axiom can be deduced from the l.e.s. axiom.) □

Proposition 2.20 (Homology is stable)**.** *Let (X, x_0) be a pointed CW complex. Then $\tilde{\mathrm{H}}_i(X; R) \cong \tilde{\mathrm{H}}_{i+1}(\Sigma X; R)$ for all i.*

Proof. This follows from the long exact sequence associated to $X \to C_{\mathrm{id}X} \to SX \simeq \Sigma X$, and the fact that $C_{\mathrm{id}X} \simeq *$, so has 0 homology. □

Corollary 2.21. $\tilde{\mathrm{H}}_i(S^n;R) = 0$ *if* $n \neq i$ *and* $\tilde{\mathrm{H}}_n(S^n;R) = R$.

Corollary 2.22. *Let* n *and* m *be nonnegative integers. If* $S^n \simeq S^m$*, then* $n = m$*. Similarly, if* $\mathbb{R}^n \approx \mathbb{R}^m$*, then* $n = m$.

Proof. The first part is immediate. As for the second, if there is a homeomorphism $\mathbb{R}^n \to \mathbb{R}^m$ then there's an induced homeomorphism on the one-point compactifications, $S^n \to S^m$. □

Proposition 2.23. *The boundary* S^n *is not a retract (deformation or otherwise) of* D^{n+1}.

2.4 Infinite CW complexes and the Mapping Telescope

We will want to use induction arguments on CW skeleta in order to prove things "for all CW complexes", but this runs into a difficulty when it encounters a CW complex without a bound on the dimension of the cells appearing.

Proposition 2.24. *Suppose* X *is a CW complex,* X_n *is the* n*-skeleton and* $k < n$*. Then* $\mathrm{H}_k(X_n;R) \to \mathrm{H}_k(X;R)$ *is an isomorphism. Moreover,* $\mathrm{H}_n(X_n;R) \to \mathrm{H}_n(X;R)$ *is an epimorphism.*

Proof. First suppose X is finite-dimensional. We make use of the following fact, which will reappear over the course of the term: $X_{n+1}/X_n \approx \bigvee S^{n+1}$, one sphere for each $n+1$-cell of X. By means of the long exact sequence, we deduce that $\tilde{\mathrm{H}}_k(X_n;R) \cong \tilde{\mathrm{H}}_k(X_{n+1};R)$, and similarly for the epimorphism statement. If X is finite-dimensional, then $X = X_N$ for some N, and the result is proved.

Now suppose X is not necessarily finite dimensional. By applying the l.e.s. to $X_n \to X \to X/X_n$, we see that it is sufficient to prove that X/X_n has vanishing k-homology for $k \le n$. It also has trivial n-skeleton.

Take the inclusions $X_0 \to X_1 \to X_2 \to \dots$, and form the *mapping telescope* T of this. That is, start with $\coprod_{i=0}^{\infty} X_i \times I$ and then identify $X_i \times \{1\}$ with $X_{i+1} \times \{0\}$. It may be helpful to imagine all this as a subspace of $X \times [0,\infty)$ by putting $X_i \times I$ in as $X_i \times [i, i+1]$.

We claim that $T \to X \times [0,\infty)$, this inclusion, is a homotopy equivalence. We establish this by proving that there is a deformation retraction. We will actually show that $T \cup X \times [n,\infty) \to T \cup X \times [n+1,\infty)$ is a deformation retract—just use Proposition 1.30. Then string all these homotopies together in unit time using the usual trick. This establishes the homotopy equivalence.

Now take X_0, assumed to be a point, and treat it as the basepoint of X. Let S denote the ray $X_0 \times [0,\infty)$ in T and let F denote the fish-skeleton $S \cup \bigcup_{i=1}^{\infty} X_1 \times \{i\}$.

The space T/F is an infinite wedge sum of spaces, each being a suspension of X_i for some i, and each X_i has trivial n-skeleton. By the wedge-sum axiom, therefore,

$$\tilde{H}_{k+1}(T/F;R) = \bigoplus_i \tilde{H}_{k+1}(\Sigma X_i;R) = \bigoplus_i \tilde{\mathrm{H}}_k(X_i;R) = 0$$

when $k \le n$, since the homology of the n-skeleton surjects onto each homology group appearing, and $X_n = *$. It follows that $\tilde{\mathrm{H}}_k(F;R) \to \tilde{\mathrm{H}}_k(T;R)$ is an isomorphism. So it suffices to prove the vanishing result for $\tilde{\mathrm{H}}_k(F;R)$, and since S is contractible, for F/S, but F/S is a wedge sum $\bigvee_{i=1}^{\infty} X_i$, and from this we get the result. □

The preceding proof indicates another result.

Proposition 2.25. *Suppose $A_0 \to A_1 \to A_2 \to \ldots$ is a sequence of maps of CW complexes, each one of which is a cellular inclusion, but not necessarily an inclusion of a skeleton. Then the union A, with the CW topology, is homotopy equivalent to the mapping telescope T.*

2.5 The Eilenberg–Steenrod Axioms

Definition 2.26. We define the *unreduced homology* or just *homology* of a space X, denoted $\mathrm{H}_i(X;R)$, to be the reduced homology $\tilde{\mathrm{H}}_i(X_+;R)$.

Many people might think this definition is backwards, and they'd be right! But here we are.

Definition 2.27. Let $A \subseteq X$ be a CW pair. We define the *relative homology* $\mathrm{H}_i(X,A;R)$ to be $\tilde{\mathrm{H}}_i(X/A;R)$.

Remark 2.28 (Eilenberg–Steenrod axioms). The following axioms for a homology theory lead to a slightly different definition from what we have given before, but they agree for all spaces having the homotopy type of a CW complex. They are called the Eilenberg–Steenrod axioms. The data are the functors $\mathrm{H}_i(\cdot;R) : \mathbf{Pairs} \to R\text{-}\mathbf{Mod}$ from the category of pairs $A \subseteq X$ of topological spaces to R-modules—or more classically, to abelian groups—and the boundary maps $\partial : \mathrm{H}_i(X,A;R) \to \mathrm{H}_{i-1}(A;R)$. The notation $\mathrm{H}_i(X;R)$ means $\mathrm{H}_i(X,\emptyset;R)$. These satisfy the following 5 axioms:

1. Homotopy: homotopic maps of pairs induce the same map in homology. We get around this by defining the functor on the homotopy category in the first place.

2. Excision: if $K \subset A \subset X$ is a sequence of containments so that the closure of K is contained in the interior of A, then the functorial map $\mathrm{H}_i(X \setminus K, A \setminus K; R) \to \mathrm{H}_i(X,A;R)$ is an isomorphism. This is not an axiom that is possible to state in the homotopy category, but we instead exploit the fact that $(X \setminus K)/(A \setminus K) \to X/A$ is a homeomorphism, and therefore a homotopy equivalence. We defined $\mathrm{H}_i(X,A;R)$ to be $\tilde{\mathrm{H}}_i(X/A;R)$. We will return to this point.

3. Long exact sequence:

$$\longrightarrow \mathrm{H}_{i+1}(A;R) \longrightarrow \mathrm{H}_{i+1}(X;R) \longrightarrow \mathrm{H}_{i+1}(X,A;R) \longrightarrow \mathrm{H}_i(A;R) \longrightarrow$$

   This is directly equivalent to ours.

4. Additivity: $\mathrm{H}\left(\coprod_\alpha X_\alpha;R\right) \to \bigoplus_\alpha \mathrm{H}_i(X_\alpha;R)$. This is equivalent to our wedge-sum axiom.

5. Normalization: $\mathrm{H}_0(*;R) \cong R$ if $i = 0$ and $\mathrm{H}_i(S^0;R) = 0$ otherwise.

Remark 2.29. In this framework, if (X, x_0) is a pointed space, then $\tilde{\mathrm{H}}_i(X;R)$ is defined $\mathrm{H}_i(X, \{x_0\}; R)$.

Proposition 2.30. *Assume* H *is a homology theory satisfying the Eilenberg–Steenrod axioms—possibly excluding the normalization axiom. Let $A \subset X$ be an inclusion of a closed subset of X, such that there exists an open neighbourhood $V \supseteq A$ with the property that $A \subseteq V$ is a deformation retract*[1]*. Then the map* $\mathrm{H}_i(X, A;R) \to \mathrm{H}_i(X/A, A/A;R) = \tilde{\mathrm{H}}_i(X/A;R)$ *is an isomorphism.*

Any CW pair satisfies the hypotheses. The case of $A = \emptyset$ is a little different, at least psychologically, so we might just check it works and assume A is not empty.

Proof. We first claim that $\mathrm{H}_i(X, A;R) \to \mathrm{H}_i(X, V;R)$ is an isomorphism. This follows from the 5-lemma and the long exact sequence. Similarly, $\mathrm{H}_i(X/A, A/A;R) \to \mathrm{H}_i(X/A, V/A;R)$ is an isomorphism. Moreover, by functoriality, the diagram

$$\begin{tikzcd}
\mathrm{H}_i(X,A) \arrow[r] \arrow[d] & \mathrm{H}_i(X,V) \arrow[d] \\
\mathrm{H}_i(X/A, A/A;R) \arrow[r] & \mathrm{H}_i(X/A, V/A;R)
\end{tikzcd}$$

commutes.

Therefore it is sufficient to prove the reduction map $(X, V) \to (X/A, V/A)$ induces an isomorphism. Now we use excision. There is a subset, $A \subset V$, so that the closure of A is contained in the interior of V. Moreover, the point A/A is contained in the interior of V/A in X/A. We may replace the map under investigation by $(X \setminus A, V \setminus A) \to ((X/A) \setminus (A/A), (V/A) \setminus (A/A)$. But this map is a homeomorphism of pairs, whence the result. $\square$

Definition 2.31. If X is a space and $x \in X$ is a point, then we define the *local homology of X at x* to be

$$\mathrm{H}_n(X|x;R) = \mathrm{H}_n(X, X \setminus \{x\};R).$$

Remark 2.32. Typically, $X \setminus \{x\}$ is not a CW complex, even if X is. Therefore one might replace $X \setminus \{x\}$ by $X \setminus U$ where U is a contractible neighbourhood of x, and such that one can place a CW structure on X such that $X \setminus U$ is a CW subcomplex. [Hat10, Proposition A.4] says that CW complexes are locally contractible.

Proposition 2.33. *Suppose a homology theory satisfying the Eilenberg–Steenrod Axioms exists. Let $U \subseteq \mathbb{R}^n$ be a nonempty open set and $V \subseteq \mathbb{R}^m$ be an open set. If $U \approx V$, then $n = m$.*

Proof. We calculate the local homology $\mathrm{H}_i(U|x;\mathbb{Z}) = \mathrm{H}_i(B|x;\mathbb{Z})$ where B is a small closed Euclidean ball around x, by excision. Then $\mathrm{H}_i(B|x;\ ZZ) \cong \mathrm{H}_i(D^n, D^n \setminus \mathrm{Int} D^n;\mathbb{Z})$, by homotopy and the 5-lemma. Bu $D^n/S^{n-1} \approx S^n$, so $\mathrm{H}_i(U|x;\mathbb{Z}) = \mathbb{Z}$ if $i = n$ and 0 otherwise. $\square$

[1] Hatcher calls (X, A) a *good pair*

Proposition 2.34. *Suppose (X, x_0) is a connected CW complex (note that CW complexes are locally path connected, so that this implies X is path connected). Then* $\tilde{\mathrm{H}}_0(X; R) = 0$.

Proof. There is a running assumption that $\{x_0\}$ is in the 0-skeleton of X. Let y be any other point of the 0-skeleton of X, and consider a path $\gamma : I \to X$ starting at x and ending at y. The map γ is a map between cell complexes, and it restricts to a cellular map on the subcomplex $\partial I = \{0, 1\}$. By cellular approximation, γ is homotopic to a path $\gamma' : I \to X_1$. That is, between any two points in the 0-skeleton of X, there is a path in the 1-skeleton. The 1-skeleton, therefore, forms a connected graph. We can find a spanning tree T of this graph, and we may then form X/T, a CW complex for which $X \to X/T$ is a homotopy equivalence (since T is contractible) and where X/T has a CW structure with only one 0-cell. So without loss of generality, we may assume X has only one 0-cell.

By Proposition 2.24, $\tilde{\mathrm{H}}_0(X; R) = \tilde{\mathrm{H}}_0(X_1; R)$, but X_1 is a wedge of S^1s, so the result follows. □

Corollary 2.35. *Let X be a CW complex, then* $\mathrm{H}_0(X; R) = \bigoplus_{\pi_0(X)} R$

Proof. It follows from the wedge sum axiom that $\mathrm{H}_0(\pi_0(X); R) = \bigoplus_{\pi_0(X)} R$.

For each connected component of X, choose a point in the 0-skeleton of that component. This amounts to a cellular map $\pi_0(X) \to X$ splitting the evident surjection $X \to \pi_0(X)$, so we obtain a (split) inclusion $\mathrm{H}_0(\pi_0(X); R) \to \mathrm{H}_0(X; R)$. Observe that $X/\pi_0(X)$ is a connected CW complex, so that $\mathrm{H}_0(\pi_0(X); R) \to \mathrm{H}_0(X; R)$ is also surjective, by virtue of the long exact sequence. The result follows. □

Remark 2.36. The following problem will recur in this course. We know that $\tilde{\mathrm{H}}^n(S^n; \mathbb{Z})$ is an infinite cyclic group, and therefore is generated by a single element. There are, however, two different elements in this group one could take as a generator. Let us fix, once and for all, a generator in $\tilde{\mathrm{H}}^0(S^0; \mathbb{Z})$ so we can write $\tilde{\mathrm{H}}^0(S^0; \mathbb{Z}) = \mathbb{Z}$, calling the generator 1. The other choice of generator is then -1. Use the long exact sequence associated to $S^n \to D^{n+1} \to S^{n+1}$ to fix a distinguished generator $1 \in \tilde{\mathrm{H}}^n(S^n; \mathbb{Z})$ for all n. This will turn out later to be equivalent to fixing an orientation on S^n.

Definition 2.37. Let $n \geq 0$ and let (X, x_0) be a pointed topological space. There is a map $\eta : \pi_n(X, x_0) \to \mathrm{H}_n(X; \mathbb{Z})$, the *Hurewicz map*, defined as follows. Every element of $\pi_n(X, x_0)$ corresponds to a homotopy class of maps (basepoint preserving) $f : S^n \to X$. We therefore obtain a homomorphism $f_* : \tilde{\mathrm{H}}_n(S^n; \mathbb{Z}) \to \tilde{\mathrm{H}}_n(X; \mathbb{Z})$. Define $\eta(f) = f_*(1)$. This homomorphism forms part of a natural transformation of functors.

Proposition 2.38. *When $n \geq 1$, the Hurewicz map is a homomorphism.*

Proof. We will do the case $n = 1$. The other cases are similar, but not required for this course.

Suppose given two maps $f : S^1 \to X$ and $g : S^1 \to X$. Consider the element represented by fg in $\pi_1(X, x_0)$. [Draw the pinch map]. This is a represented by a composite $S^1 \to S^1 \vee S^1 \to X$, and in homology we obtain

$$\tilde{\mathrm{H}}_1(S^1; \mathbb{Z}) \to \tilde{\mathrm{H}}_1(S^1; \mathbb{Z}) \oplus \tilde{\mathrm{H}}_1(S^1; \mathbb{Z}) \xrightarrow{f_*, g_*} \tilde{\mathrm{H}}_1(X; \mathbb{Z})$$

Each of the two evident composites $S^1 \to S^1 \vee S^1 \to S^1$ is homotopic to the identity, so we deduce that $\eta(fg) = \eta(f) + \eta(g)$ as required. □

Proposition 2.39. *Let* $(X, x_0$ *be a space. The Hurewicz map* $\eta : \pi_1(X, x_0) \to \tilde{\mathrm{H}}_1(X; \mathbb{Z})$ *factors*

$$\pi_1(X, x_0) \to \pi_1(X, x_0)^{ab} \to \tilde{\mathrm{H}}_1(X; \mathbb{Z}).$$

If X *is a connected CW complex, then* $\pi_1(X, x_0)^{ab} \to \tilde{\mathrm{H}}_1(X; \mathbb{Z})$ *is an isomorphism.*

We postpone the proof until after we have developed more cellular machinery. It would be possibly to give it now, but we would end up repeating ourselves.

Remark 2.40. We can see immediately, however, that the Hurewicz map is an isomorphism when $X = S^1$.

2.6 Degree

Definition 2.41. Suppose given a map $f : S^n \to S^n$, where $n \geq 1$. Under the induced map $f_* : \mathrm{H}_n(S^n; \mathbb{Z}) \to \mathrm{H}_n(S^n; \mathbb{Z})$, the image $f_*(1) \in \mathbb{Z}$ is an integer. This integer is the *degree* of f.

Remark 2.42. It is implicit above that S^n is the same space in both source and target. The following properties of the degree are easily proved:

1. $\deg(\mathrm{id}_{S^n}) = 1$.
2. $\deg(f)$ depends only on the homotopy class of f.
3. If f is not surjective, then $f : S^n \to S^n$ factors through $D^n \subset S^n$, and so $\deg(f) = 0$.
4. If $f : S^n \to S^n$ is a map, then $\deg(f) = \deg(Sf) = \deg(\Sigma f)$.
5. The degree of a map $f : S^1 \to S^1$ takes on the usual meaning of winding number.
6. If $f : S^n \to S^n$ is a map and $g : S^n \to S^n$ another, then $\deg(g \circ f) = \deg(g)\deg(f)$

Proposition 2.43. *The inclusion* $SO(n) \subset \mathrm{SL}_n(\mathbb{R})$ *is a deformation retract, and* $\mathrm{SL}_n(\mathbb{R})$ *is path connected (when given the topology of a subspace of* $\mathbb{R}^{n^2}$.

Proof. The deformation retraction statement is proved using the Gram–Schmidt method, applied to the columns of an $n \times n$ matrix. This certainly produces an orthogonal matrix, and leaves the sign of the determinant unchanged.

To see $\mathrm{SL}_n(\mathbb{R})$ is path connected, observe that every matrix $A \in \mathrm{SL}_n(\mathbb{R})$ is a product of elementary matrices $E_{ij}(a)$, and each of these may be connected to I_n by a path. □

Corollary 2.44. *Let* $A \in O(n)$ *act on* $S^{n-1} \subset \mathbb{R}^n$*, giving a map* $A : S^{n-1} \to S^{n-1}$*. Then* $\deg(A) = \det(A)$.

Proof. If $A \in SO(n)$, then $A \simeq \mathrm{id}$. If $A \in O(n)_-$, then $\deg(A)$ agrees with $\deg(f)$ where $f : S^{n-1} \to S^{n-1}$, where f is the map flippling the sign of the first coordinate. But this is $\Sigma^{n-2}(f')$ where $f' : S^1 \to S^1$ is the direction-reversal map, which has degree -1. □

Definition 2.45. Let $f : S^n \to S^n$ be a map, let $x \in S^n$ be a point and suppose there exists a neighbourhood $U \ni x$ such that $f(x) \notin f(U \setminus x)$. This is the case if f restricts to a homeomorphism onto its image in a neighbourhood of x, for instance. Then define the *local degree* of f at x, denoted $\deg_x f$, by the degree of

$$\mathrm{H}_n(U, U \setminus x; \mathbb{Z}) \to \mathrm{H}_n(S^n, S^n \setminus f(x); \mathbb{Z})$$

Remark 2.46. Why is the above well defined? First observe we can shrink U by use of excision, so we may assume U is homeomorphic to a ball in $\mathbb{R}^n$. The space $\mathrm{H}_n(U, U \setminus x; \mathbb{Z})$ is equipped with an isomorphism to $\mathrm{H}_n(U, U \setminus V; \mathbb{Z})$, where V is a small contractible neighbourhood of x—this is the 5-lemma and homotopy. Moreover, we can compare $\mathrm{H}_n(U, U \setminus V)$ to $\mathrm{H}_n(S^n, S^n \setminus V)$, and they are isomorphic by virtue of excision, and the latter is isomorphic to $\tilde{\mathrm{H}}_n(S^n / S^n \setminus V)$ which is, in turn, isomorphic to $\tilde{\mathrm{H}}_n(S^n)$. A similar story applies to the target.

Proposition 2.47. *Let $f : S^n \to S^n$ be a map, and let $y \in S^n$ be such that $f^{-1}(y)$ consists of finitely many points $x_1, x_2, \ldots, x_r$. Then*

$$\deg(f) = \sum_{i=1}^{r} \deg_{x_i} f$$

Remark 2.48. Do example with winding number.

Proof of proposition. For each x_i choose an open U_i around x_i in such a way that the U_is are pairwise disjoint and choose an open V around y so that $f(U_i \setminus \{x_i\}) \subset V \setminus \{y\}$. Now take any of the values i and form the following commutative diagram ($\mathbb{Z}$ coefficients are understood, but not written).

$$\begin{tikzcd}
 & \mathrm{H}_n(U_i, U_i \setminus x_i) \arrow[r, "f_*"] \arrow[dl, "\cong"'] \arrow[d] & \mathrm{H}_n(V, V \setminus y) \arrow[d, "\cong"] \\
\mathrm{H}_n(S^n, S^n \setminus x_i) & \mathrm{H}_n(S^n, S^n \setminus f^{-1}(y)) \arrow[l] \arrow[r, "f_*"] & \mathrm{H}_n(S^n, S^n \setminus y) \\
 & \mathrm{H}_n(S^n) \arrow[r, "f_*"] \arrow[u] \arrow[ul, "\cong"] & \mathrm{H}_n(S^n) \arrow[u, "\cong"']
\end{tikzcd}$$

The lower map here is f_* by definition, and is measuring $\deg f$. The middle map, also labelled f_*, is induced by f as well. The upper f_* is measuring the local degree of f at x_i. The different isomorphisms are either given by excision or by means of homotopy invariance and the 5-lemma.

Now let us write the same diagram but showing the isomorphism classes of the groups in question. Each of the infinite cyclic groups here has a distinguished generator, and may be identified with $\mathbb{Z}$ by means of that.

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[r, "f_*"] \arrow[d] \arrow[dl, "\cong"'] & \mathbb{Z} \arrow[d, "\cong"] \\
\mathbb{Z} & \mathbb{Z}^r \arrow[l] \arrow[r, "f_*"] & \mathbb{Z} \\
 & \mathbb{Z} \arrow[r, "f_*"] \arrow[u] \arrow[ul, "\cong"] & \mathbb{Z} \arrow[u, "\cong"']
\end{tikzcd}$$

In the right hand column, the upper map is $\mathbb{Z} \to \mathbb{Z}^r$ given by inclusion on the i-th factor. On the other hand, the lower map is the diagonal inclusion $a \mapsto (a, a, \dots, a)$. The upper horizontal map is the local degree, and commutativity of this square implies that $(0, \dots, 0, 1, 0 \dots, 0)$ is sent to $\deg_{x_i}(f)$ in the middle right $\mathbb{Z}$. Commutativity of the lower square says that $\deg f = f_*(1, 1, \dots, 1)$, and putting these two observations together, we see that $\deg f = \sum_{i=1}^r \deg_{x_i} f$.

□

Remark 2.49. This whole section was done for $\mathbb{Z}$, since that is what the *degree* of a map of spheres means, but there is no impediment to defining an R-degree for any ring, as the degree of $\mathrm{H}_n(S^n; R) \to \mathrm{H}_n(S^n; R)$.

2.7 The Cellular Chain Complex

Throughout this section, we fix a ring R. This ring may as well be taken to be $R = \mathbb{Z}$ on first reading.

Definition 2.50. Let X be a CW complex. For each n, define the *cellular chains of dimension n* of X (really of the CW structure) to be

$$C_n^{\mathrm{cell}}(X; R) := \tilde{\mathrm{H}}_n(X_n / X_{n-1}; R).$$

For these purposes, the -1-skeleton is $\emptyset$.

Remark 2.51. The space X_n / X_{n-1}, the quotient of one skeleton by another, is homeomorphic to a bouquet of spheres, one for each n-cell of X. Therefore

$$C_n^{\mathrm{cell}}(X; R) = \bigoplus_{n\text{-cells}} R.$$

Definition 2.52. We define *cellular differential* maps $d_n : C_n^{\mathrm{cell}}(X; R) \to C_{n-1}^{\mathrm{cell}}(X; R)$ by the formula

$$\begin{tikzcd}
\tilde{\mathrm{H}}_n(X_n / X_{n-1}; R) \arrow[rr] \arrow[dr, "\partial"] & & \mathrm{H}_{n-1}(X_{n-1} / X_{n-2}; R) \\
 & \mathrm{H}_{n-1}(X_{n-1}; R) \arrow[ur, "q_*"] &
\end{tikzcd}$$

using two different long exact sequences woven together. One for $X_{n-1} \to X_n \to X_n/X_{n-1}$ and one for $X_{n-2} \to X_{n-1} \to X_{n-1}/X_{n-2}$.

Proposition 2.53. *The resulting structure $C_*(X;R), d_*$ is a chain complex of R-modules, and the homology in degree n is $\mathrm{H}_n(X;R)$.*

Remark 2.54. In fact, the chain complex is functorial in R and in (the CW structure on) X.

Proof. To prove this, we identify $\ker d_n$ and $\operatorname{im} d_{n+1}$. To keep things brief, we omit R.

First the kernel. We can factor d_n as $\tilde{\mathrm{H}}_n(X_n/X_{n-1}) \to \mathrm{H}_{n-1}(X_{n-1}) \to \tilde{\mathrm{H}}_{n-1}(X_{n-1}/X_{n-2})$. Observe that the second map belongs in a long exact sequence

$$0 = \mathrm{H}_{n-1}(X_{n-2}) \to \mathrm{H}_{n-1}(X_{n-1}) \to \tilde{\mathrm{H}}_{n-1}(X_{n-1}/X_{n-2})$$

and therefore is injective. Therefore the kernel of d_n agrees with the kernel of the first map, which also belongs in a long exact sequence

$$0 = \mathrm{H}_n(X_{n-1}) \to \mathrm{H}_n(X_n) \to \tilde{\mathrm{H}}_n(X_n/X_{n-1}) \to \mathrm{H}_{n-1}(X_{n-1})$$

and so the kernel of d_n agrees with the image of the injective map $\mathrm{H}_n(X_n) \to \tilde{\mathrm{H}}_n(X_n/X_{n-1})$. Identify $\mathrm{H}_n(X_n)$ with its image under this injective map, for convenience of notation—otherwise you have to keep many injective maps around and it gets messy.

Now let us consider the image of d_{n+1}. Again there is a factorization

$$\tilde{\mathrm{H}}_{n+1}(X_{n+1}/X_n) \to \mathrm{H}_n(X_n) \to \tilde{\mathrm{H}}_n(X_n/X_{n-1})$$

and we saw already that the second map is injective. The image of d_{n+1} therefore agrees with the image of the map $\tilde{\mathrm{H}}_{n+1}(X_{n+1}/X_n) \to \mathrm{H}_n(X_n) \subseteq \tilde{\mathrm{H}}_n(X_n/X_{n-1})$.

Observe that we have incidentally shown that $\operatorname{im} d_{n+1} \subset \ker d_n$.

Now to calculate $\ker d_n / \operatorname{im} d_{n+1} = \mathrm{H}_n(X_n)/\operatorname{im}(\mathrm{H}_{n+1}(X_{n+1}/X_n) \to \mathrm{H}_n(X_n))$. This calculation is already available to us in the form of the long exact sequence for $X_n \to X_{n+1} \to X_{n+1}/X_n$.

$$\tilde{\mathrm{H}}_{n+1}(X_{n+1}/X_n) \to \mathrm{H}_n(X_n) \to \mathrm{H}_n(X_{n+1}) \to \tilde{\mathrm{H}}_n(X_{n+1}/X_n) = 0$$

from which we see that the R-module in question is $\mathrm{H}_n(X_{n+1})$, but we established in Proposition 2.24 that this is naturally isomorphic to $\mathrm{H}_n(X)$. □

Remark 2.55. Observe that the calculations we carried out in the proposition above depend only on the (reduced) set of axioms for $\tilde{\mathrm{H}}_*(X;R)$ that we first gave. Therefore this result establishes Theorem 2.18.

Remark 2.56. How do we understand the differentials in $C_*^{\text{cell}}(X;R)$? We have a decomposition of $C_n^{\text{cell}}(X;R)$ into summands, one for each n-cell. Each such n-cell has an attaching map, $S^{n-1} \to X_{n-1}$. We compose this with the quotient $X_{n-1} \to X_{n-1}/X_{n-2} \approx \bigvee S^{n-1}$, where in this bouquet of spheres, we have one sphere for each $n-1$-cell. For each pair, therefore, of n-cell and $n-1$-cell, we loosely-speaking get a map $S^{n-1} \to S^{n-1}$. Strictly, the target is not actually the $n-1$-sphere, merely a space homeomorphic to it. A specific homeomorphism also imposes an orientation. One therefore has to choose orientations on all the cells in order to make the calculations. The answer does not depend (up to isomorphism) on the specific orientations chosen, but you have to be consistent in your choice of orientation on any given cell, you can't orient it one way for incoming differentials and another for outgoing.

Anyway, when the orientations have been chosen, for each n-cell and each $n-1$-cell, we get a map $S^{n-1} \to S^{n-1}$ (up to homeomorphism) and this map has a degree in R.

The behaviour on 0-cells and attaching 1-cells is a little different (do example).

Example 2.57. Klein bottle with integer homology. Klein bottle with $\mathbb{Z}/(2)$ homology.

Example 2.58. Let $\mathbb{C}P^n$ denote complex projective space. A point in this space amounts to an equivalence class of $n+1$-tuples of complex numbers $[z_0 : z_1 : \cdots : z_n]$. This space has a cellular structure where the locus where $z_n \neq 0$ (or, equivalently, z_n is scaled to 1, is the open top cell. This cell is a $2n$-dimensional cell, since the open cell is homeomorphic to $\mathbb{C}^n$. The $2n-1$ skeleton consists of those points $[z_0 : z_1 : \cdots : z_{n-1} : 0]$, which is homeomorphic to $\mathbb{C}P^{n-1}$. We deduce that $\mathbb{C}P^n$ has a cell structure with a single cell in each even dimension $0, \ldots, 2n$, and no other cells. The homology can be calculated immediately from this.

Some further remarks: the surjective map $\mathbb{C}^n \setminus \{0\} \to \mathbb{C}P^n$ (quotient being by action of $\mathbb{C}^\times$) restricts to a quotient map defined on the unit sphere in $\mathbb{C}^n \setminus \{0\}$, i.e., $S^{2n-1} \to \mathbb{C}P^n$ (the quotient being by S^1). The complex projective spaces are compact complex manifolds.

The space $\mathbb{C}P^1$ is homeomorphic to S^2, by consideration of the cells for instance. It is also a 1-point compactification of $\mathbb{C}^1$.

We have therefore constructed an interesting map $S^3 \to S^2$, essentially the map $\mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1$. This is the *Hopf map*. The cell structure we placed on $\mathbb{C}P^2$ has three cells. A 0-cell, a 2-cell (attached in the only possible way to the 0-cell) and a 4-cell, attached by some map $\partial D^4 = S^3 \to \mathbb{C}P^1 = S^2$. Unproved assertion: this attaching map is the Hopf map (at least up to homotopy). Second unproved assertion: $\mathbb{C}P^2$ is not homotopy equivalent to $S^4 \vee S^2$, although it has the same $\mathbb{Z}$-homology. It follows from these two assertions that the cone on the Hopf map (which is $\mathbb{C}P^2$) is not equivalent to the cone on the trivial map $S^3 \to S^2$, which is $S^4 \vee S^2$. But we know homotopic maps have equivalent cones, so we deduce that the Hopf map is a map $S^3 \to S^2$ that is not nullhomotopic.

We see here the limits of what homology can detect. The cellular chain complex remembers the information of degrees of maps of spheres arising from the attaching of one cell to a lower-dimensional skeleton, but if some cell is attached by a map $S^{n+k} \to S^n$ for positive k, then this is not going to appear in the homology.

Example 2.59. A more prosaic example is given by $\mathbb{R}P^n$. Here the induction arguments shows there is a cell in each dimension $0, \ldots, n$. The attaching maps are therefore slightly harder to calculate. One method of doing this is to present $\mathbb{R}P^n$ as a quotient of S^n, rather than $\mathbb{R}^n \setminus \{0\}$, by the antipodal action of C_2, the cyclic group of order 2.

In order that this action be cellular, we put a CW structure on S^n with two cells in each dimension 0 to n. It is a fun exercise to write down what the differentials on S^n should be. These differentials depend on the orientations chosen on the various cells, so there are several possibilities. One consistent choice causes the differentials $C_i^{\text{cell}}(S^n;R) \to C_{i-1}^{\text{cell}}(S^n;R)$ to work out as

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad i \text{ even}, \quad \text{and} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \quad i \text{ odd}$$

in the applicable range (Do the example of S^2). With this structure, one can then take the C_2-quotient, noting that the orientations on now-identified cells are compatible, and so we can read off the maps in the cell structure for the quotient, giving the following cellular chain complex for $\mathbb{R}P^n$:

$$0 \to \mathbb{Z} \to \cdots \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

Thus we have

$$\mathrm{H}_i(\mathbb{R}P^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \text{ or } i = n \text{ is odd} \\ \mathbb{Z}/(2) & \text{if } i \text{ is odd and } i < n \\ 0 & \text{otherwise} \end{cases}$$

Chapter 3

Singular Homology and Homological Algebra

3.1 Definition

Definition 3.1. We fix the *standard topological n-simplex* Δ^n as the closed subset of $\mathbb{R}^{n+1}$ determined by the equation $\sum_{i=1}^{n+1} x_i = 1$ and the condition $0 \leq x_i \leq 1$ for all i. From $\Delta^{n-1} \to \Delta^n$ there are $n+1$ *coface maps*, d^i, given by inclusion by setting each of the $n+1$ coordinates equal to 0.

Definition 3.2. Let X be a topological space and let A be an abelian group (in particular, A can be a ring). We define the *singular n-simplices* of X to be the maps $\Delta^n \to X$, and we define the *singular n-chains of X with coefficients in A* to be the "A-linear" formal sums of singular n-simplices. That is, a singular n-chain is an element of $\bigoplus_{\sigma:\Delta^n \to X} A\sigma$. The set of such singular n-chains is denoted $C_*(X;A)$.

Definition 3.3. Given a singular n-simplex σ, we define its *boundary (with coefficients in A)*, denoted $d_n(\sigma)$ to be

$$\sum_{i=1}^{n+1} (-1)^i \sigma \circ d^i$$

the sum being taken in $C_{n-1}(X;A)$. If $\alpha = \sum_j a_j \sigma_j$ is an A-linear formal sum of such singular n-simplicies, i.e., α is a singular n-chain, then we define $d_n(\alpha)$ by A-linearity.

Proposition 3.4. *The sequence of groups and maps $C_*(X;A), d_*$ forms a chain complex. If A is a commutative ring R, then it forms a chain complex of R-modules.*

Proof. We first show that $d_{n-1} \circ d_n = 0$. This is very simple, however, since $d_{n-1} \circ d_n$ is a sum over faces-of-faces, and each term appears twice and with alternating signs. The statement about R-modules follows from the fact that $C_*(X;R)$ is defined as a free R-module and we specify R-module maps by declaring what happens on a basis. □

Definition 3.5. We define the *singular homology with coefficients in A* to be the homology of this chain complex: $\mathrm{H}_i^{\mathrm{sing}}(X;A)$.

Remark 3.6. We would like to show that the theory we have produced satisfies the Eilenberg–Steenrod axioms, and therefore can be used to produce a theory that computes cellular homology when applied to CW complexes. Most of the axioms are largely "algebraic", which is to say that there is minimal geometric input required in checking them, but the excision axiom is hard.

3.2 Basic Homological Algebra

Fix a ring R.

All our chain complexes of R-modules will be *bounded below*, which is to say they look like $\to C_n \to C_{n-1} \to \dots$ where $C_n = 0$ for all $n < N$ for some N (often, $N = 0$). We will also occasionally make the assumption that R is a PID. A feature of PIDs is that submodules of free modules are free. This does not hold for domains that are not PI: for instance $(2, 1+\sqrt{-5}) \subset \mathbb{Z}[\sqrt{-5}]$ is a submodule of a free module that is not itself free.

Remark 3.7. We can define the following notions for chain complexes of R-modules:

1. The shift: $C_*[a]$ is the chain complex having C_{i-a} in the i-th position.
2. A morphism: $f : C_* \to D_*$ is the data of a sequence of R-module maps $C_i \to D_i$, commuting with the differentials.
3. Kernels and cokernels (DO A DIAGRAM).
4. Direct sums of chain complexes.
5. Exact sequences of chain complexes

Remark 3.8. The category of chain complexes of R-modules is a *abelian category*, which means (loosely speaking) that the same isomorphism theorems hold for chain complexes of R-modules as hold for abelian groups. For instance, the image of a map $f : C_* \to A_*$ is isomorphic to the quotient of C_* by the kernel of f.

Proposition 3.9. *If $f : C_* \to D_*$ is a map of chain complexes, then there is an induced map of homology $f_* : \mathrm{H}_*(C_*) \to \mathrm{H}_*(D_*)$. In fact, H_* is a functor.*

Definition 3.10. A map of chain complexes $f : C_* \to D_*$ is a *quasi-isomorphism* if f induces an isomorphism on homology in all degrees.

Example 3.11. Over $R = \mathbb{Z}$:

1. $$0 \longrightarrow \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \longrightarrow 0$$

   is quasi-isomorphic to 0. In fact, any exact sequence is quasi-isomorphic to the 0-sequence.

2.
$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "1\mapsto p"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/(p) \arrow[r] & 0
\end{tikzcd}$$

is a quasi-isomorphism.

Remark 3.12. The big idea, which may seem harebrained at this stage, is that there is actually a "homotopy theory" for chain complexes, just as there is for topological spaces. If you set up the right theoretical framework, e.g. model categories or ∞-categories, then both can be seen as special cases of a general idea. We will not devote time to setting up a general machine, but you can still see the deep ideas if you look for them.

For instance, a chain complex of free R-modules can be thought of as rather like a CW complex. The things in degree i are like the i-cells. Our next task is to define a homotopy between maps of chain complexes $C_* \to A_*$. To do this, we should define it to be a map from "$C_* \otimes I$", which should be a chain complex.

This complex should have the following form: $C_* \otimes I$ in level n is $C_n \oplus C_{n-1} \oplus C_n$ (DRAW PICTURE)

$$\begin{tikzcd}
C_n \arrow[d, "d"] & C_{n-1} \arrow[dl, "-\mathrm{id}"'] \arrow[d, "-d"] \arrow[dr, "\mathrm{id}"] & C_n \arrow[d, "d"] \\
C_{n-1} & C_{n-2} & C_{n-1}
\end{tikzcd}$$

One can include C_* into $C_* \otimes I$ as either of the two endpoints. A chain homotopy from $f : C_* \to A_*$ to $g : C_* \to A_*$ should be a sequence of morphisms $C_n \oplus C_{n-1} \oplus C_n \to A_n$ restricting to f and g on the two outer summands and compatible with the differential in $C_* \otimes I$ and A_*. There is some mild algebraic simplification one can do to these data. We leave the simplification as an exercise.

Definition 3.13. A *chain homotopy* Ψ from $f : C_* \to A_*$ to $g : C_* \to A_*$ is a sequence of maps $\Psi_{n-1} : C_{n-1} \to A_n$ such that for all $c \in C_n$ we have $\Psi(dc) + d\Psi(c) = g(c) - f(c)$.

Proposition 3.14. *If Ψ is a chain homotopy between f and g, then f and g induce the same map on homology.*

Proof. Exercise. □

Remark 3.15. With this notion of homotopy, it becomes possible to define a "homotopy equivalence" between chain complexes. $f : C_* \to A_*$ is a homotopy equivalence if there is a "homotopy inverse" $g : A_* \to C_*$ with the property that $f \circ g \simeq \mathrm{id}$ and $g \circ f \simeq \mathrm{id}$. It follows from the functoriality of homology and the previous proposition that a homotopy equivalence is a quasi-isomorphism.

A quasi-isomorphism between bounded-below complexes of free modules is a homotopy equivalence (this requires proof). On the other hand, a general quasi-isomorphism need not be a homotopy equivalence at all.

The situation is analogous to the topological situation, where cellular maps between CW complexes have "good" homotopical behaviour, but general spaces may behave poorly.

Proposition 3.16 (The snake lemma)**.** *Suppose given a diagram of R-modules*

$$\begin{tikzcd} & A_1 \arrow[r, "f_1"] \arrow[d, "\alpha"] & B_1 \arrow[r, "g_1"] \arrow[d, "\beta"] & C_1 \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A_2 \arrow[r] & B_2 \arrow[r] & C_2 & \end{tikzcd}$$

in which the rows are exact, then there is a natural long exact sequence

$$\ker(\alpha) \to \ker(\beta) \to \ker(\gamma) \xrightarrow{\partial} \operatorname{coker}(\alpha) \to \operatorname{coker}(\beta) \to \operatorname{coker}(\gamma)$$

Proof. Exercise. □

Proposition 3.17. *Suppose*

$$0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$$

is a short exact sequence of chain complexes of R-modules. There is a natural long exact sequence of homology groups

$$\ldots \xrightarrow{f_*} \mathrm{H}_n(B) \xrightarrow{g_*} \mathrm{H}_n(C) \xrightarrow{\partial} \mathrm{H}_{n-1}(A) \xrightarrow{f_*} \mathrm{H}_{n-1}(B) \xrightarrow{g_*} \ldots$$

Proof. Write d^A for the differential in A, etc. The first step is to verify that the diagram

$$\begin{tikzcd} & A_n/(\operatorname{im} d^A_{n+1}) \arrow[r, "f"] \arrow[d] & B_n/(\operatorname{im} d^B_{n+1}) \arrow[r, "g"] \arrow[d] & C_n/(\operatorname{im} d^C_{n+1}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \ker(d^A_{n-1}) \arrow[r, "f"] & \ker(d^B_{n-1}) \arrow[r, "g"] & \ker(d^C_{n-1}) & \end{tikzcd}$$

satisfies the conditions of the snake lemma.

The second step is applying the snake lemma—the result just falls out. □

3.3 Singular homology satisfies the Eilenberg–Steenrod axioms

Definition 3.18. Let $A \subset X$. Use the inclusion of A in X to define an inclusion $C_*(A;R) \subset C_*(X;R)$. Define $C_*(X,A;R)$ as the quotient $C_*(X;R)/C_*(A;R)$. Define $\mathrm{H}_n^{\mathrm{sing}}(X,A;R)$ as the homology of this chain complex. Note that $\mathrm{H}^{\mathrm{sing}}(X,\emptyset;R) = \mathrm{H}^{\mathrm{sing}}(X;R)$ as previously defined.

Throughout the rest of this section we fix a ring R, and $\mathrm{H}_*(X,A) = \mathrm{H}_*^{\mathrm{sing}}(X,A;R)$ will denote singular homology with R coefficients.

We start with the easiest axioms:

Proposition 3.19 (Dimension Axiom). *$\mathrm{H}_0(*) = R$ and $\mathrm{H}_n(*) = 0$ if $n \neq 0$.*

Proof. For each $n \geq 0$, there is a unique map $\Delta^n \to *$, and so the singular chain complex of $*$ takes the form

$$\cdots \to R \to R \to R \to 0 \to 0 \to \dots$$

with the last nonzero R-module being in dimension 0. The differentials $C_n(*;R) \to C_{n-1}(*;R)$ are given as an alternating sum of $n+1$ copies of the identity map: so $\sum_{i=0}^{n}(-1)^i : R \to R$. When n is odd, we get $0 : R \to R$ and when n is even, we get $1 : R \to R$, the identity map. The calculation of homology is now easy. □

Proposition 3.20 (Additivity). *If $X = \coprod_\alpha X_\alpha$, then $C_*(X;R) = \bigoplus_\alpha C_*(X_\alpha;R)$, so $\mathrm{H}_*(X;R) = \bigoplus_\alpha \mathrm{H}_*(X_\alpha;R)$.*

Proof. Easy. □

Proposition 3.21 (Long Exact Sequence of a Pair). *Suppose $A \subset X$ is a subspace, then there is a natural short exact sequence of chain complexes*

$$0 \to C_*(A;R) \to C_*(X;R) \to C_*(X,A;R) \to 0$$

and therefore an attendant long exact sequence of homology.

Proof. The R-module $C_*(X,A;R)$ was defined to make this work. □

Now the moderately hard one: Homotopy invariance.

Proposition 3.22. *Let $f \simeq g : X \to Y$. Then the maps $f_*, g_* : C_*(X;R) \to C_*(Y;R)$ are chain homotopic.*

Proof. Let H be a homotopy $H : X \times I \to Y$ from f to g. We use H to produce a chain homotopy $P : C_*(X;R) \to C_{*+1}(Y;R)$ between f_* and g_*.

In order to do this, we really have to subdivide the spaces $\Delta^n \times I$ into $n+1$-simplices. A good way of doing this as follows: Let $[v_0,\dots,v_n]$ denote the vertices of $\Delta^n \times \{0\}$ and $[w_0,\dots,w_n]$ the vertices of $\Delta^n \times \{1\}$. An $n+1$- or n-simplex in $\Delta^n \times I$ may be specified by naming the vertices, in order. Then the simplex is taken to be the image of a homeomorphism $\Delta^n \to \Delta^n \times I$ or $\Delta^{n+1} \to \Delta^n \times I$, mapping the vertices to the vertices in order, and behaving linearly elsewhere.

The following all describe $n+1$-simplices embedded in $\Delta^n \times I$:

$$A_j = [v_0, \dots, v_j, w_j, \dots, w_n]$$

as j varies from 0 to n. (DRAW PICTURE IF POSSIBLE)

We wish to produce a chain homotopy P. To do this, suppose $\tau : \Delta^n \to X$ is a singular n-simplex. We will produce a singular $n+1$-chain $P\tau \in C_{n+1}(Y;R)$, and then extend to all of $C_n(X;R)$ by linearity. To get going, we consider the map

$$H' := H \circ (\tau \times \mathrm{id}) : \Delta^n \times I \to X \times I \to Y$$

The map H' is not a map from an $n+1$-simplex, but rather, a map from the $n+1$ different $n+1$-simplices $A_j \to Y$. Let us define $P(\tau)$ to be

$$P(\tau) = \sum_{j=0}^{n} (-1)^j H'|[v_0, \dots, v_j, w_j, \dots, w_n].$$

This extends by linearity to a perfectly good homomorphism $C_n(X;R) \to C_{n+1}(Y;R)$.

Now we show this is actually a chain homotopy, by calculating $dP\tau$ and $Pd\tau$

$$dP(\tau) = \sum_{j=0}^{n} \left(\sum_{i=0}^{j} (-1)^{i+j} H'|[v_0, \dots, \hat{v}_i, \dots, v_j, w_j, \dots, w_n] + \sum_{i=j}^{n} (-1)^{i+j+1} H'|[v_0, \dots, v_j, w_j, \dots, \hat{w}_i, \dots, w_n] \right)$$

This is a large sum, and there are cancellations. With the exception of the two terms $H'|[v_0, \dots, v_n]$ and $H'|[w_0, \dots, w_n]$, all the terms $H'|[v_0, \dots, v_j, w_{j+1}, \dots, w_n]$ with consecutively-numbered vertices appear twice, and with opposite signs. Note further that $H'|[w_0, \dots, w_n] = g_*(\tau)$ and $H'|[v_0, \dots, v_n] = f_*(\tau)$. With this simplification, we can write

$$\begin{aligned} dP(\tau) = H'|[w_0, \dots, w_n] + (-1)^{2n+1} H'|[v_0, \dots, v_n] + \sum_{i<j} (-1)^{i+j} H'|[v_0, \dots, \hat{v}_i, \dots, v_j, w_j, \dots, w_n] + \\ + \sum_{i>j} (-1)^{i+j+1} H'|[v_0, \dots, v_j, w_j, \dots, \hat{w}_i, \dots, w_n] = \\ = g_*(\tau) - f_*(\tau) + \text{other terms} \end{aligned}$$

On the other hand, we calculate $Pd\tau$

$$\begin{aligned} P \sum_{i=0}^{n} (-1)^i H'|[v_0, \dots, \hat{v}_i, \dots, v_n] = \sum_{i<j} (-1)^{i+j} H'[v_0, \dots, v_j, w_j, \dots, \hat{w}_i, \dots, w_n] + \\ + \sum_{i>j} (-1)^{i+j-1} H'|[v_0, \dots, \hat{v}_i, \dots, v_j, w_j, \dots, w_n] \end{aligned}$$

which happens to be precisely -1 times the "other terms" referenced above. So we see that

$$dP(\tau) + Pd(\tau) = g_*(\tau) - f_*(\tau)$$

Since the elements τ form a basis for $C_n(X;R)$, this identity holds on the whole free module, and so $dP + Pd = g_* - f_*$, giving the required chain homotopy. $\square$

And now for the hard one: Excision.

Definition 3.23. Let X be a topological space and let $\mathscr{U}$ be an open cover of X. Say that a singular simplex $\sigma : \Delta^n \to X$ is *subordinate* to $\mathscr{U}$ if there is an open set $U_i \in \mathscr{U}$ such that $\operatorname{im}(\sigma) \subset U_i$. Let $C_*^{\mathscr{U}}(X;R)$ denote the chain complex obtained by restricting $C_*(X;R)$ to singular simplices subordinate to $\mathscr{U}$.

Hatcher weakens this to simply requiring that the interiors of the sets of $\mathscr{U}$ should form a cover.

Lemma 3.24. *The inclusion $C_*^{\mathscr{U}}(X;R) \to C_*(X;R)$ is a quasi-isomorphism (in fact, a homotopy equivalence).*

Proof. The proof of this hard technical fact will not be done in class. Notes will be made available. □

Proposition 3.25. *Let $Z \subseteq Y \subseteq X$ be a sequence of spaces, where Z is closed in X and Y is open. Then $C_*(X \setminus Z, Y \setminus Z) \to C_*(X, Y; R)$ is a quasi-isomorphism. This establishes the excision theorem.*

Proof. Recall that $C_*(X,Y)$ is defined as a quotient

$$C_*(Y) \to C_*(X) \to C_*(X,Y).$$

Define an open cover of X by Y and $W = X - Z$. Then $C_*^{\{Y,W\}}(X)$ is a subcomplex of $C_*(X)$ and has the same homology. Note that $C_*(Y)$ is also a subcomplex of this subcomplex, and so we get a diagram of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & C_*(Y) \arrow[r] \arrow[d, equal] & C_*^{\{Y,W\}}(X) \arrow[r] \arrow[d] & C_*^{\{Y,W\}}(X,Y) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_*(Y) \arrow[r] & C_*(X) \arrow[r] & C_*(X,Y) \arrow[r] & 0
\end{tikzcd}$$

Here we have used this diagram to define $C_*^{\{Y,W\}}(X,Y)$. There is an induced map of long exact sequences of homology groups, and the 5-lemma says that $C_*^{\{W,Y\}}(X,Y) \to C_*(X,Y)$ is a quasi-isomorphism.

Now consider $C_n(X-Z) \to C_n^{\{Y,W\}}(X)$. The former has a basis consisting of all n-simplices with image in $X - Z = W$, the latter has a basis consisting of all n-simplices with image in either Y or in W. In particular, we can say $C_n^{\{Y,W\}}(X)$ is equal to the sum of two submodules $C_n(W) + C_n(Y)$. The second isomorphism theorem then produces a (natural) isomorphism $C_n(W)/(C_n(W) \cap C_n(Y)) \to C_n^{\{Y,W\}}(X)/C_n(Y)$. But this is an isomorphism $C_n(X-Z, Y-Z) \to C_n^{\{Y,W\}}(X,Y)$. This isomorphism is compatible with differentials (check, or use naturality), and so establishes the result. □

3.4 Functoriality of homology in the coefficients

In these notes, we defined $\mathrm{H}_*^{\mathrm{sing}}(X;A)$ when A is merely an abelian group, but we've proved most of our results under the further assumption that A is a ring. This is really only for convenience, the results hold also for general A. The most important abelian groups are the finitely generated ones, which are all direct sums of abelian groups $\mathbb{Z}/(n)$ carrying an evident ring structure.

Lemma 3.26. *Given two abelian groups A and B, there is a natural isomorphism $\mathrm{H}_*^{\mathrm{sing}}(X;A\oplus B) \cong \mathrm{H}_*^{\mathrm{sing}}(X;A) \oplus \mathrm{H}_*^{\mathrm{sing}}(X;B)$.*

Proof. This follows from a similar level at the singular-chain level. For all n, there is an isomorphism $C_n^{\mathrm{sing}}(X;A\oplus B) \cong C_n^{\mathrm{sing}}(X;A) \oplus C_n^{\mathrm{sing}}(X;B)$, and the behaviour of the differential is compatible with this decomposition. That is, if d_n^A is a differential for $C_*(X;A)$ and d_n^B is the corresponding differential for $C_n(X;B)$, then $d_n^A \oplus d_n^B$ is the differential for $C_*(X;A\oplus B)$. Then direct verification shows that $\mathrm{H}_*(X;A\oplus B) \cong \mathrm{H}_*(X;A) \oplus \mathrm{H}_*(X;B)$. □

Lemma 3.27 (The 3×3 lemma)**.** *Suppose given a diagram*

$$\begin{tikzcd}
A_{11} \arrow[r] \arrow[d] & A_{21} \arrow[r] \arrow[d] & A_{31} \arrow[d] \\
A_{12} \arrow[r] \arrow[d] & A_{22} \arrow[r] \arrow[d] & A_{32} \arrow[d] \\
A_{13} \arrow[r] & A_{23} \arrow[r] & A_{33}
\end{tikzcd}$$

in which all columns are short exact sequences, and in which $A_{12} \to A_{32}$ is the 0 map. Then if any two rows are short exact sequences, so is the third.

Note that the condition on $A_{12} \to A_{32}$ is required only when the middle row is not the one assumed exact.

Proof. Exercise. □

Proposition 3.28. *Let $A \to B$ be a map of abelian groups. Then for all n and all pairs of spaces (X,Y), there is a natural map $\mathrm{H}_n^{\mathrm{sing}}(X,Y;A) \to \mathrm{H}_n^{\mathrm{sing}}(X,Y;B)$. These natural maps are compatible with the boundary map $\mathrm{H}_n^{\mathrm{sing}}(X,Y;A) \to \tilde{\mathrm{H}}_{n-1}(Y,\emptyset;A)$, and if*

$$0 \to A \to B \to Q \to 0$$

is a short exact sequence of abelian groups then there is a (natural) long exact sequence

$$\cdots \to \mathrm{H}_n^{\mathrm{sing}}(X;A) \to \mathrm{H}_n^{\mathrm{sing}}(X;B) \to \mathrm{H}_n^{\mathrm{sing}}(X;Q) \to \mathrm{H}_{n-1}^{\mathrm{sing}}(X;A) \to \dots$$

Proof. The map of groups $A \to B$ yields a map of chain complexes $C_*^{\text{sing}}(X, Y; A) \to C_*^{\text{sing}}(X, Y; B)$. Homology being a functor for chain complexes, we get the first assertion.

The compatibility with the boundary map is a diagram chase, which is long but not especially interesting.

Given the short exact sequence of coefficients, we get a (natural) short exact sequence of chain complexes

$$0 \to C_*^{\text{sing}}(X; A) \to C_*^{\text{sing}}(X; B) \to C_*^{\text{sing}}(X; Q) \to 0$$

for any X.

From there, we get a short exact sequence

$$0 \to C_*^{\text{sing}}(X, Y; A) \to C_*^{\text{sing}}(X, Y; B) \to C_*^{\text{sing}}(X, Y; Q) \to 0$$

by using the 3×3 lemma (DO OUT).

Associated to a short exact sequence of chain complexes, there is a long exact sequence of homology groups, establishing the result. □

Example 3.29. We'll see more about the above when we talk about "Universal coefficients". For now, we'll content ourselves with seeing an example relating $\mathbb{Z}$-homology and $\mathbb{Z}/(2)$-homology for an interesting space: say $\mathbb{R}P^3$.

The homology $\mathrm{H}_*(\mathbb{R}P^3; \mathbb{Z})$ was calculated earlier. It is $\mathbb{Z}$ in dimension 3 and 0, and $\mathbb{Z}/(2)$ in dimension 1. It vanishes in other dimensions. Write H_i instead of $\mathrm{H}_i(\mathbb{R}P^3; \mathbb{Z}/(2))$ for brevity. The long exact sequence we get from the short exact coefficient sequence

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/(2) \to 0$$

takes the form

$$0 \to \mathrm{H}_4 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathrm{H}_3 \to 0 \to 0 \to \mathrm{H}_2 \to \mathbb{Z}/(2) \xrightarrow{\times 2} \mathbb{Z}/(2) \to \mathrm{H}_1 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathrm{H}_0 \to 0$$

This gives the homology: it's $\mathbb{Z}/(2)$ in all dimensions 0 to 3, which can also be calculated directly from the cell structure.

Chapter 4

Further properties and uses of homology

4.1 The Hurewicz map

We first pay off a debt, namely, the proof of the Hurewicz map.

First, a recollection on notation. Let G, H be groups and $W \to G$ and $W \to H$ be group homomorphisms. Then $G *_W H$ denote the pushout group so that for any group K the following two sets are in bijection

1. the set of a pairs of group homomorphisms $G \to K$ and $H \to K$ such that the composites $W \to G \to K$ and $W \to H \to K$ agree
2. The set of homomorphisms $G *_W H \to K$.

We recall the van Kampen theorem.

Theorem 4.1. *Let $X = U \cup V$ be a union of two path-connected open subspaces for which $U \cap V$ is also path connected. Fix a basepoint $x_0 \in U \cap V$, which is suppressed from the notation henceforth. Then the evident map $\pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V) \to \pi_1(X)$ is an isomorphism.*

Recall that G^{ab} denotes the abalianization of G, i.e., the universal abelian group equipped with a map $G \to G^{\mathrm{ab}}$.

Lemma 4.2. *Let G be a group and $g \in G$ an element. Let $\bar{g}$ denote the image of g in G^{ab}. There is a natural isomorphism $(G *_{(g)} \{e\})^{ab} \to G^{\mathrm{ab}}/(\bar{g})$.*

In fact, it is not difficult to see that $G^{\mathrm{ab}} *_{\bar{g}} \{e\} = G^{\mathrm{ab}}/(\bar{g})$, so that the lemma is really saying that the abelianization functor commutes with the construction $G *_{(g)} \{e\}$. One can read in the appendix about left-adjoint functors, and see that ab is left adjoint to the functor forgetting the "abelian" structure of abelian groups, so this is a special case of a result saying left-adjoint functors preserve colimits. This explains the formal quality of the following argument.

Proof. Both groups $G^{\mathrm{ab}}/(\bar{g})$ and $(G *_{(g)} \{e\})^{\mathrm{ab}}$ are universal groups for maps $G \to K$ with the property that K is abelian and the image of g is 0. (DO SLOWLY IN CLASS) □

We now establish the Hurewicz isomorphism in the most geometrically easy case. We will later reduce all cases to this.

Proposition 4.3. *Hurewicz isomorphism in the geometric case] Let T be a finite 2-dimensional CW complex for which $T_0 = \{t_0\}$ is a singleton. Then the Hurewicz map $\eta : \pi_1(T, t_0)^{\mathrm{ab}} \to \mathrm{H}_1(T;\mathbb{Z})$ is an isomorphism.*

Proof. We prove this by induction on the number of 2-cells. If there are none, then we have the case of a wedge of circles

$$T = \bigvee_{\alpha \in A} S^1$$

The van Kampen theorem says $\pi_1(T, t_0) = F$, the free group on the classes $[\alpha]$ (or $[\mathrm{id}_\alpha]$ if you prefer). The homology, by virtue of the axioms is the free abelian group on the same classes.

Now we consider what happens when a 2-cell is added to T by means of a gluing map $\phi : \partial D^2 \to T_1$. The space constructed T' is the mapping cone of ϕ, and (up to homotopy equivalence) depends only on the homotopy class of $[\phi] = g \in \pi_1(T_1, t_0)$. The van Kampen theorem says that $\pi_1(T', t_0) = \pi_1(T, t_0) *_{(g)} \{e\}$, whereas the long exact sequence for homology says that $\mathrm{H}_1(T';\mathbb{Z}) = \mathrm{H}_1(T;\mathbb{Z})/(\bar{g})$. Then, by the lemma, the result holds. □

Proposition 4.4. *Let X be a simply connected CW complex with basepoint x_0, and consider the map $\eta : \pi_1(X, x_0) \to \tilde{\mathrm{H}}_1(X;\mathbb{Z})$. Then η induces an isomorphism (also denoted η))*

$$\eta : \pi_1(X, x_0)^{\mathrm{ab}} \to \mathrm{H}_1(X;\mathbb{Z}).$$

Proof. The map η was previously produced and is natural. We can make some reductions. First of all, we can assume X is 2-dimensional (has no cells of dimension 3 or higher), since the inclusion of the 2 skeleton induces isomorphisms on the invariants being computed. Second, we can contract a spanning tree in the 1-skeleton, this does not change the homotopy type of X and therefore we may assume X_0 is a point, and X_1 is a wedge sum of circles.

Suppose therefore that X is a CW complex with $X = X_2$, and where $X_1 = \bigvee_{\alpha \in A} S^1_\alpha$. Moreover, X_2 is given as a pushout square:

$$\begin{tikzcd} \coprod_{\beta \in B} \partial D^2_\beta \arrow[r, "\phi"] \arrow[d] & X_1 \arrow[d] \\ \coprod_{\beta \in B} D^2 \arrow[r, "\Psi"] & X \end{tikzcd}$$

The (cellular) homology H_1 of this space is given by the homology of the sequence

$$0 \longrightarrow \bigoplus_{\beta \in B} \mathbb{Z}_\beta \longrightarrow \bigoplus_{\alpha \in A} \mathbb{Z}_\alpha \xrightarrow{0} \mathbb{Z}$$

where $\mathbb{Z}_\beta \to \mathbb{Z}_\alpha$ is the degree of a particular map $\partial D^2_\beta \to S^1_\alpha$. Moreover, $\mathrm{H}_1(X_1;\mathbb{Z}) = \bigoplus \mathbb{Z}_\alpha$, so $\mathrm{H}_1(X_1;\mathbb{Z}) \to \mathrm{H}_1(X;\mathbb{Z})$ is surjective.

The homotopy $\pi_1(X, x_0)$ can also be presented, but this is a little more delicate. We know from the van Kampen theorem (Math 426) that $\pi_1(X_1, x_0)$ is the free group (not free abelian) F generated by the circles S^1_α. By virtue of the cellular approximation theorem, any map $S^1 \to X$ can be deformed to have image in X_1, so that $\pi_1(X_1, x_0) \to \pi_1(X, x)$ is surjective.

We have a diagram

$$\begin{tikzcd} \pi_1(X_1) \arrow[r, two heads] \arrow[d] & \pi_1(X) \arrow[d] \\ \mathrm{H}_1(X_1) \arrow[r, two heads] & \mathrm{H}_1(X) \end{tikzcd}$$

The left vertical map is the map from F, a free group, to the abelianization, a free abelian group.

Suppose we have a class in $a \in \mathrm{H}_1(X)$, then we can find a preimage in $\mathrm{H}_1(X_1)$, which we can lift to $\pi_1(X_1)$, and then map to $\pi_1(X)$. This shows that η is surjective.

Finally we can show that η is injective. This amounts to a reduction to the previous proposition. Suppose $z \in \pi_1(X, x_0)$ is such that $\eta(z) = 0$. We can assume as before that z is represented by a cellular map, and of course this map meets finitely many 1-cells. We can also find finitely many 2-cells $e^2_1, \dots, e^2_j$ of X such that the image of z in $\mathrm{H}_1(X_1;\mathbb{Z})$ is in the image of the boundary maps of the e^2_is. Therefore we can produce a finite subcomplex of X, denoted T, so that $z \in \mathrm{im}(\pi_1(T, x_0) \to \pi_1(X, x_0)$ and $\eta(z) = 0 \in \mathrm{H}_1(T;\mathbb{Z})$—just add in enough cells to include all the e^2_is, their boundaries, and whatever else was used to define z. But we know that $\eta : \pi_1(T, x_0)^{\mathrm{ab}} \to \mathrm{H}_1(T;\mathbb{Z})$ is an isomorphism, so $z = 0$ in $\pi_1(T, x_0)^{\mathrm{ab}}$, and since abelianization is a functor, this implies $z = 0$ in $\pi_1(X, x_0)^{\mathrm{ab}}$, which is what we wanted to show. □

4.2 Euler Characteristic

This is the first-discovered topological invariant.

Definition 4.5. Let C_* be a chain complex of vector spaces over a field k, such that $\bigoplus_n C_n$ is finite dimensional. Then the *Euler characteristic*, $\chi(C_*)$ is the alternating sum

$$\sum_{n=-\infty}^{\infty} (-1)^n \dim_k C_n.$$

Remark 4.6. By treating the homology of a chain complex as a chain complex again, but this time with differentials all 0, we may define the Euler characteristic

$$\chi(\mathrm{H}_*(C_*)) = \sum_{n=-\infty}^{\infty} (-1) \dim_k \mathrm{H}_k(C_*).$$

While the definition of χ can be made over a ring as well as over a field, simply by requiring the modules that appear to be projective and therefore to have a well defined rank, one may not be able to make sense of the Euler characteristic of the homology in that generality.

Proposition 4.7.

$$\chi(C_*) = \chi(\mathrm{H}_*(C_*))$$

Proof. Exercise. □

Definition 4.8. Fix a field k. If X is a space such that $\bigoplus \mathrm{H}_*(X;k)$ is a finite dimensional k vector space, then define the *k-Euler characteristic*, $\chi(X)_k$, of X to be $\chi(\mathrm{H}_*(X;k))$. If no field is specified, $k = \mathbb{Q}$ is understood.

Proposition 4.9. *Let Y be a space with a CW structure having a_n cells in dimension n, where almost all a_ns are 0. Then*

$$\chi(Y)_k = \sum_{n=0}^{\infty} (-1)^n a_n.$$

In particular, $\chi(Y)_k$ is independent of k. In fact, $\chi(X)_k$ is independent of k even if X is merely homotopy equivalent to a finite CW complex.

Proof.

$$\chi(Y)_k = \sum_{n=0}^{\infty} (-1)^n C_n(Y;k) = \sum_{n=0}^{\infty} (-1)^n a_n$$

□

Example 4.10. Euler's formula $f - e + v = 2$ for connected planar graphs is a special case of the Euler characteristic.

4.3 The Künneth Formula

Construction 4.11. Fix a ring R. Suppose C_* and C'_* are chain complexes of R-modules. We define $C_* \otimes_R C'_*$ as the chain complex having

$$(C \otimes_R C')_n = \bigoplus_{i=-\infty}^{\infty} C_i \otimes_R C'_{n-i}$$

and where $d : C_i \otimes C'_{n-1} \to (C \otimes_R)_{n-1}$ is $d(c \otimes c') = d(c) \otimes c' + (-1)^i c \otimes d(c')$.

Remark 4.12. One notes that $d^2 = 0$. Moreover, defining $B_i \subseteq Z_i \subseteq C_i$ to be the boundaries and cycles, and $B'_i \subseteq Z'_i \subseteq C'_i$ similarly, we see that the evident map $C_i \otimes C_{n-i} \to C_n$ restricts to $Z_i \otimes Z'_{n-1} \to Z''_n$, and also $B_i \otimes Z'_{n-i}$ and $Z_i \otimes B'_{n-i}$ both map to B''_n.

In this way we can get a map

$$\frac{Z_i \otimes_R Z'_{n-i}}{Z_i \otimes_R B'_{n-i} + B_i \otimes_R Z'_{n-i}} \to \frac{Z''_n}{B''_n}$$

or (after a small amount of algebra) gives rise to a map

$$\mathrm{H}_i(C_*) \otimes_R \mathrm{H}_{n-i}(C'_*) \to \mathrm{H}_n(C_* \otimes_R C'_*)$$

Summing over all i, we have

$$\bigoplus_{i=-\infty}^{\infty} \mathrm{H}_i(C_*) \otimes_R \mathrm{H}_{n-i}(C'_*) \to \mathrm{H}_n(C_* \otimes_R C'_*)$$

Proposition 4.13 (Algebraic Künneth). *If C_* and C'_* are chain complexes of free R-modules where R is a PID, then there is a short exact sequence*

$$0 \to \bigoplus_i \mathrm{H}_i(C) \otimes_R \mathrm{H}_{n-i}(C') \to \mathrm{H}_n(C \otimes C') \to \bigoplus_i \mathrm{Tor}^R(\mathrm{H}_i(C), \mathrm{H}_{n-i-1}(C')) \to 0$$

Proof shamelessly taken from Hatcher. First, let us do the case where C is concentrated in a single degree. This case is easy, since free modules are flat. In this case, there is no Tor term.

Second, the case where C is a direct sum of complexes, each concentrated in a single degree—that is, C has only 0-differentials. This follows easily from the previous case. Again there is no Tor term.

For each i, we can write short exact sequences $0 \to Z_i \to C_i \to B_{i-1} \to 0$ and since $B_{i-1} \subseteq C_{i-1}$, this sequence is moroever split. In fact, we can make

$$\begin{tikzcd}
0 \arrow[r] & Z_i \arrow[r] \arrow[d, "0"] & C_i \arrow[r] \arrow[d, "d_i"] & B_{i-1} \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_{i-1} \arrow[r] & C_{i-1} \arrow[r] & B_{i-1} \arrow[r] & 0
\end{tikzcd}$$

and generalizing, we can make $0 \to Z_* \to C_* \to B_* \to 0$, a short exact sequence of chain complexes. The outer complexes have 0 differentials.

Now apply $\otimes_R C'_*$. This results in a short exact sequence of chain complexes:

$$0 \to Z_* \otimes_R C'_* \to C_* \otimes_R C'_* \to B_* \otimes_R C'_* \to 0$$

(again, levelwise split).

Take homology to get a long exact sequence:

$$\cdots \to Z_* \otimes_R \mathrm{H}_*(C') \to \mathrm{H}_*(C \otimes_R C') \to B_* \otimes_R \mathrm{H}_*(C') \to \dots$$

Here there are bundary maps $B_* \otimes_R \mathrm{H}_*(C') \to Z_* \otimes_R \mathrm{H}_*(C')$. On the level of actual named groups, this consists of maps $B_{i-1} \otimes_R \mathrm{H}_*(C') \to Z_{i-1} \otimes_R \mathrm{H}_*(C')$—at first blush the groups are sums of these and the maps might interact between the B_i and the Z_i, but in fact they do not. Indeed, if you trace everything through using the snake lemma, the boundary map is just induced from all the inclusions

$$B_{i-1} \otimes_R \mathrm{H}_{n-i}(C'_*) \to Z_{i-1} \otimes_R \mathrm{H}_{n-i}(C'_*).$$

Let i_n denote the direct sum of all these maps. Exactness above gives us short exact sequences

$$0 \to \operatorname{coker} i_n \to \mathrm{H}_n(C \otimes_R C') \to \ker i_{n-1} \to 0$$

The calculation of coker $i_n = \bigoplus_i \mathrm{H}_i(C_*) \otimes_R \mathrm{H}_{n-i}(C'_*)$ is direct.

The determination of the kernel is not much harder. (DO IN CLASS). This establishes the short exact sequence. □

Remark 4.14. The short exact sequence of the Künneth formula is *natural*. That is, given maps $C \to D$ and $C' \to D'$ of chain complexes, you get induced maps of short exact sequences. This can be seen by tracing through everything in the proof.

Remark 4.15. The short exact sequence in the Künneth formula is split, but not necessarily naturally split. To see this, we produce a splitting map

$$\mathrm{H}_n(C \otimes_R C') \to \bigoplus_i \mathrm{H}_i(C_*) \otimes \mathrm{H}_{n-i}(C'_*).$$

To do this, we produce maps $C_i \to \mathrm{H}_i(C_*)$ and $C'_{n-i} \to \mathrm{H}_{n-i}(C'_*)$—using the fact we're over a PID—and so, induced maps on the tensored homology.

Lemma 4.16. *Let $f : S^n \to S^n$ and $g : S^m \to S^m$ be basepoint preserving maps. Then $f \wedge g : S^n \wedge S^m \to S^n \wedge S^m$ is of degree* $\deg(f)\deg(g)$.

Proof. We can write $f \wedge g$ as $f \wedge \mathrm{id} \circ \mathrm{id} \wedge g$. Then we would like to know that $\deg(\mathrm{id} \wedge g) = \deg(g)$, but this is an iterated application of the fact that the suspension preserves degree. □

Theorem 4.17 (Künneth formula)**.** *Let X and Y be CW complexes, and let R be a PID. Then there are short exact sequences*

$$0 \to \bigoplus_{i=0}^{n} \mathrm{H}_i(X;R) \otimes_R \mathrm{H}_{n-i}(X;R) \to \mathrm{H}_n(X \times Y;R) \to \bigoplus_{i=0}^{n-1} \mathrm{Tor}^R(\mathrm{H}_i(X;R), \mathrm{H}_{n-i-1}(Y;R)) \to 0$$

This short exact sequence is natural, and split, but not naturally split.

Proof. To prove this theorem, we prove that $C_*(X;R) \otimes_R C_*(Y;R) \cong C_*(X \times Y;R)$ for the cellular chain complexes of X and Y. The identification of the cells is easy: given an i-cell e^i_X of a X and an $n-i$-cell e^{n-i}_Y of Y we get an n-cell of $X \times Y$, denoted $e^i_X \times e^{n-i}_Y$.

Now we should show that the differentials do what we want. Following [Hat10], we establish this first in the case where X and Y are both products of intervals.

We give I the obvious structure with two 0-cells and a 1-cell. Then we give I^n the induced product structure. Let e_i denote the i-th 1-cell of I^n, and write 0_i and 1_i for the vertices at the boundary of these 1 cells. The top cell of I^n is $e_1 \times \cdots \times e_n$. We try to calculate $d(e_1 \times \cdots \times e_n)$. This has to be of the form

$$d(e_1 \times \cdots \times e_n) = \sum_{i=1}^{n} (-1)^{a(i)} e_1 \times \cdots \times de_i \times \cdots \times e_n$$

Either swapping the ends of an edge, or swapping two adjacent edges, induces a reflection on the boundary, and therefore a degree-−1 map, so that up to an overall sign we must have

$$d(e_1 \times \cdots \times e_n) = \sum_{i=1}^{n} (-1)^i e_1 \times \cdots \times de_i \times \cdots \times e_n.$$

The overall sign is a matter of convention, ultimately (Hatcher has already fixed a convention previously at this point). We choose it so that the formula works out.

Now if we consider $I^a \times I^{n-a}$, we give the first I^a a cell structure with top cell $E = e_1 \times \cdots \times e_a$ and the second I^{n-a} the top cell $F = e_{a+1} \times \cdots \times e_n$, then

$$d(E \times F) = d(E) \times F + (-1)^a E \times d(F).$$

This is what we wanted to establish in general, in a special case.

Now suppose we have a general situation, with two CW complexes X and Y having cells e^i_α and e^{n-i}_β respectively. We then have a cell $e^i_\alpha \times e^{n-i}_\beta$ of $X \times Y$. We want to write down a boundary map $d(e^i_\alpha \times e^{n-i}_\beta)$.[1]

Cells usually are represented by disks D^i and D^{n-i}, but we can use I^i and I^{n-i} as models for these disks. Let $\phi : \partial I^i \to X_{i-1}$ and $\psi : \partial I^{n-i} \to Y_{n-i-1}$ be attaching maps for the cells above, and Φ and Ψ the characteristic maps. Write e^i for the top cell of I^i and e^{n-i} for the top cell of I^{n-i}. We may assume the attaching maps, and therefore Φ and Ψ, are cellular, using cellular approximation for instance.

Now we can calculate

$$d(e_\alpha \otimes e_\beta) = d(\Phi \times \Psi(e^i \times e^j)) = (\Phi \times \Psi)_*(de^i \times e^j + (-1)^i e^i \times d(e^j))$$

This is great as far as it goes, but we should understand $(\Phi \times \Psi)_*$ as it applies to a cell.

Lemma 4.18. *If $\Phi : W \to X$ and $\Psi : Z \to Y$ are cellular maps, then $(\Phi \times \Psi)_* = \Phi_* \otimes \Psi_*$*

Proof of Lemma. Suppose $\Psi_*(e^i_\alpha) = \sum_\gamma m_{\alpha\gamma} e^i_\gamma$ and $\Psi_*(e^j_\beta) = \sum_\delta n_{\beta\delta} e^j_\delta$. Then we determine $(\Phi \times \Psi)_*(e^i_\alpha \times e^j_\beta)$. Specifically, we determine the coefficient of $e^i_\gamma \times e^j_\delta$ in this sum, hoping to show it is $m_{\alpha,\gamma} n_{\beta,\delta}$. Since this is the coefficient of $e^i_\gamma \times e^j_\delta$ in $\Phi_* \otimes \Psi_*$, establishing this identity would establish the lemma.

How does this coefficient come about? Define $\Phi_{\alpha\gamma} : S^i \to W_i / W_{i-1} \to X_i / X_{i-1} \to S^i$ given by characteristic maps for α, γ and the map Ψ. Define $\Psi_{\beta,\delta}$ similarly. Then $\deg \Phi_{\alpha,\gamma} = m_{\alpha,\gamma}$ and $\deg \Psi_{\beta,\delta} = n_{\beta,\delta}$. The coefficient calculation we want is the degree the top cell in $\Phi \wedge \Psi : S^i \wedge S^j \to W_i \times Z_i / \sim \to X_i \times Y_j / \sim \to S^i \wedge S^j$. We already established this fact in a previous lemma. This proves the result. $\square$

[1] In this argument, the cells $e_\alpha \times e_\beta$ appear both as cells per se as well as elements of $C_n(X \times Y)$, where they may be identified with $e_\alpha \otimes e_\beta$. We are therefore not careful in distinguishing $e_\alpha \times e_\beta$ and $e_\alpha \otimes e_\beta$.

Now we finish the proposition. We have established that

$$d(e_\alpha \otimes e_\beta) = (\Phi \times \Psi)_*(de^i \times e^i + (-1)^i e^i \times d(e^j)) = \Phi_*(de^i) \times \Psi(e^j) + (-1)^i \Phi_*(e^i) \times \Psi(de^j)$$

and finally, use the fact that Φ and Ψ are cellular maps to conclude that this is:

$$de^i_\alpha \times e^j_\beta + (-1)^i e^i_\alpha \times de^j_\beta$$

which is what we wanted to prove. □

Remark 4.19. The result also holds for singular homology, using a result called the Eilenberg–Zilber Theorem.

Theorem 4.20 (Universal Coefficients for Homology)**.** *Let A be an abelian group, then for any space there is a short exact sequence of singular homology groups*

$$0 \to A \otimes_{\mathbb{Z}} \mathrm{H}_n(X;\mathbb{Z}) \to \mathrm{H}_n(X;A) \to \mathrm{Tor}^{\mathbb{Z}}(A, \mathrm{H}_{n-1}(X;\mathbb{Z})) \to 0$$

again, this sequence is natural, split, but not naturally split.

Chapter 5

Cohomology

5.1 Cohomology

For simplicity, we work with $\mathbb{Z}$-modules.

Construction 5.1. Suppose C_* is a chain complex of R-modules. Let A be another R-module. We can form the complex $\mathrm{Hom}_{\mathbb{Z}}(C_*, A)$, and take the homology. We will call this the *cohomology* of C_* with coefficients in A, and we will denote it $\mathrm{H}^*(C_*; A)$.

Proposition 5.2. *Let R be a PID, and let C_* be a chain complex of free R-modules. Then there is a natural short exact sequence*

$$0 \to \mathrm{Ext}^1_R(\mathrm{H}_{i-1}(C_*), A) \to \mathrm{H}^i(C_*; A) \to \mathrm{Hom}_R(\mathrm{H}_i(C_*), A) \to 0.$$

This sequence is split, but not naturally split.

Definition 5.3. If X is a topological space and R a ring. We define the *singular cohomology* of X with coefficients in R as the cohomology $\mathrm{H}^*(C(X;\mathbb{Z}); R)$.

Remark 5.4. If A is an abelian group, then a group homomorphism $f : A \to R$ corresponds to an R-linear map $\mathrm{id}\otimes f : R\otimes_{\mathbb{Z}} A \to R$. Therefore $\mathrm{Hom}_{\mathbb{Z}}(C^{sing}_*(X;\mathbb{Z}), R)$ is isomorphic to $\mathrm{Hom}_R(C^{sing}_*(X;R), R)$, and so the definition you might want to give of $\mathrm{H}^*(X;R)$ as the cohomology of $C^{sing}_*(X : R)$ with coefficients in R also works.

Proposition 5.5 (Universal Coefficients for Cohomology)**.** *For any space X and any ring R there is a short exact sequence*

$$0 \to \mathrm{Ext}^1_R(\mathrm{H}_{i-1}(X;\mathbb{Z}), R) \to \mathrm{H}^i(X; R) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_i(X), R) \to 0.$$

This sequence splits, but not naturally.

Proposition 5.6. *Let k be a field. Then there is an isomorphism*

$$\mathrm{H}^i(X;k) \cong \mathrm{Hom}(\mathrm{H}_i(X;k), k)$$

Remark 5.7. The above constructions for spaces can equally well be made for pairs of spaces, based on the chain complex $C_*(X,A;\mathbb{Z})$. Of course, $\mathrm{H}^i(X,\emptyset;R) = \mathrm{H}^i(X;R)$

5.1.1 Eilenberg–Steenrod Axioms for Cohomology

There are Eilenberg–Steenrod Axioms for Cohomology. Fix a ring R (the axioms can be stated for abelian groups, but we don't do that here).

Remark 5.8. Cohomology with coefficients in R gives a contravariant functor from **Top** to the category of graded R-modules. That is, if $f : X \to Y$ is a map of spaces, then there is an induced map $f^* : \mathrm{H}^*(Y;R) \to \mathrm{H}^*(X;R)$. Similarly, given a continuous map of pairs of spaces $f : (X,A) \to (Y,B)$, there is a contravariant map in cohomology. These satisfy the following properties.

Homotopy Invariance If $f \simeq g$, then $f^* = g^*$.

Product axiom $\mathrm{H}^i(\coprod_{\alpha\in A} X_\alpha;R) \cong \prod_{\alpha\in A} \mathrm{H}^i(X_\alpha;R)$.

Long Exact Sequence of a Pair Associated to a pair of spaces (X,A), there is a natural long exact sequence

$$\cdots \to \mathrm{H}^{i-1}(A;R) \to \mathrm{H}^i(X,A;R) \to \mathrm{H}^i(X;R) \to \mathrm{H}^i(A;R) \to \ldots$$

(as a consequence, there is a suspension isomorphism for cohomology: $\tilde{\mathrm{H}}^i(X;R) \cong \tilde{\mathrm{H}}^{i+1}(\Sigma X;R)$.

Excision Let $Z \subseteq A \subseteq X$ be a sequence of spaces such that the closure of Z is contained in the interior of A. Then $\mathrm{H}^i(X,A;R) \to \mathrm{H}^i(X-Z,A-Z;R)$ is an isomorphism.

Dimension $\tilde{\mathrm{H}}^0(S^0;R) = R$.

Remark 5.9. As before, the excision and long exact sequence axioms imply that for a CW pair, $\mathrm{H}^*(X,A;R) = \tilde{\mathrm{H}}^*(X/A;R)$. Moreover, there is a *cellular cohomology theory*. One can form the *cellular cochains* of a CW complex. With coefficients in R, one has $C^*_{cell}(X;R) = \mathrm{Hom}_{\mathbb{Z}}(C^{cell}_*(X;\mathbb{Z}),R)$. The axioms therefore determine the cohomology of a CW complex.

The situation is most pleasant when X is a CW complex having only finitely many cells in each dimension. In this case, $C^{cell}_*(X;R) = \mathrm{Hom}_R(C^{cell}_*(X;R),R)$ is a free R-module in each degree. When X has infinitely many cells in degree i, then $C^i_{\mathrm{cell}}(X;R)$ will be the dual of a free R-module, and may not be free, depending on R.

Remark 5.10. Let X be a space. From the universal coefficients theorem, we see that $\mathrm{H}^0(X;R) = \mathrm{Hom}_R(\mathrm{H}_0(X;\mathbb{Z}),R)$. In fact, since $\mathrm{Hom}_0(X;\mathbb{Z})$ is a free abelian group with coefficients in $\mathbb{Z}$, we see that $\mathrm{H}^0(X;R) = R^{\pi_0(X)}$. In this sense it clearly has an R-module structure—and in fact, it has an R-algebra structure given by componentwise multiplication.

5.2 Cup Product

There are two approaches to cup product. At this stage of the term, I'm willing to skip some details, farming them out to references. The approach in [Hat10, Section 3.2] is a good one, requiring little technology, but it's not the intuitive way of seeing the cup product. We sketch another method here.

Remark 5.11. Let X and Y be spaces and R a ring. Then there are natural maps

$$\phi : C_*(X \times Y; R) \to C_*(X; R) \otimes_R C_*(Y; R), \quad \psi : C_*(X; R) \otimes_R C_*(Y; R) \to C_*(X \times Y; R)$$

of singular chain complexes satisfying the following properties:

- ϕ and ψ are normalized so that they induce the 'obvious' isomorphism on C_0.
- ϕ and ψ are inverse chain homotopy equivalences.
- ϕ and ψ are determined up to chain homotopy by the two properties above.

A wiser presentation of the course material might have used these to produce the Künneth formula in singular homology.

Proposition 5.12. *Let X and Y be spaces. Let R be a coefficient ring. Then there is a natural map*

$$\chi : \bigoplus_{i=0}^{n} \mathrm{H}^i(X; R) \otimes_R \mathrm{H}^{n-i}(Y; R) \to \mathrm{H}^n(X \times Y; R)$$

Proof. There are two ideas in the proof. The map defined above is the degree-n part of a map that is actually easier to define in general.

First there is a map

$$C^*(X; R) \otimes_R C^*(Y; R) = Hom_R(C_*(X), R) \otimes_R \mathrm{Hom}_R(C_*(Y); R) \to \mathrm{Hom}_R(C_*(X) \otimes_R C_*(Y), R)$$

given by sending $f \otimes g$ to the function sending $\alpha \otimes \beta \in C_*(X) \otimes_R C_*(Y)$ to $f(\alpha)g(\beta)$, and extending by R-linearity.

Then there is the map $\phi^* : \mathrm{Hom}_R(C_*(X) \otimes_R C_*(Y), R) \to \mathrm{Hom}_R(C_*(X \times Y; R), R) = C^*(X \times Y; R)$.

Then taking the homology gives us the map we wanted. □

Definition 5.13. The natural map $\bigoplus_i \mathrm{H}^i(X; R) \otimes_R \mathrm{H}^{n-i}(Y; R) \to \mathrm{H}^n(X \times Y; R)$ is called the *exterior product* in cohomology. It can be constructed even more easily in cellular cohomology (where the analogue of the map ϕ is an isomorphism), and in the case where $R = k$ is a field (or where the homology is free over $\mathbb{Z}$ etc) and finitely generated in each degree, the exterior product is an isomorphism.

Proposition 5.14. *The exterior product is associative, in that the two obvious maps*

$$\mathrm{H}^*(X;R)\otimes_R \mathrm{H}^*(Y;R)\otimes_R \mathrm{H}^*(Z;R)\to \mathrm{H}^*(X\times Y\times Z;R)$$

agree. This can be proved precisely (if you believe Remark 5.11 at any rate) but is just going to be more algebra.

Construction 5.15. Let $\Delta : X \to X\times X$ denote the diagonal. Let R be a ring. We can construct a map

$$\smile : \mathrm{H}^*(X;R)\otimes_R \mathrm{H}^*(X;R)\to \mathrm{H}^*(X\times X;R)\xrightarrow{\Delta^*}\mathrm{H}^*(X;R)$$

called the *cup product.*

Restricted to specific integers, we get $\smile : \mathrm{H}^i(X;R)\otimes_R \mathrm{H}^{n-i}(X;R)\to \mathrm{H}^n(X;R)$.

Remark 5.16. The cup product has the following properties:

1. $\smile$ is natural in X and in R.

2. $\smile$ is associative.

3. If X is path connected, then we can identify $\mathrm{H}^0(X;R) = R$, and the action $\mathrm{H}^0(X;R)\times \mathrm{H}^n(X;R)\to \mathrm{H}^n(X;R)$ is the usual R-action. If X has multiple components, then $\mathrm{H}^*(X;R) = \prod_{X_i\in\pi_0(X)}\mathrm{H}^*(X_i;R)$ and $R^{\pi_0(X)} = \mathrm{H}^0(X;R)$ works component by component.

4. $\smile$ is graded-commutative. This is a property of graded rings that bears further explanation. Let $\xi\in \mathrm{H}^i(X;R)$ and $\eta\in \mathrm{H}^j(X;R)$, then $\xi\smile\eta = (-1)^{ij}\eta\smile\xi$. The proof of this appears in Hatcher as [Hat10, Theorem 3.14], and the proof there is long and involved, but this is the cost of doing things in the elementary way.

   In our case, one can argue as in the following sketch.

   Recall that if C_* and C'_* are chain complexes of R-modules, then $C_*\otimes_R C'_*$ has differential satisfying $d(c\otimes c') = d(c)\otimes c' + (-1)^{\deg c} c\otimes d(c')$. We can define an isomorphism $C\otimes_R C'\to C'\otimes_R C$ by the formula $(c\otimes c')\mapsto (-1)^{\deg(c)\deg(c')}c'\otimes c$ when c and c' are homogeneous elements. The sign is necessary to make the differentials work out.

   There are two maps $C_*(X\times X;R)\to C_*(X;R)\otimes_R C_*(X;R)$, one is ϕ and the other is obtained by composing ϕ with the the twist isomorphism that swaps the order of the two factors, and potentially introduces a sign. That is, if α is in $C_n(X\times X;R)$, then $\phi(\alpha) = \sum \beta_i\otimes\gamma_{n-i}$ for various homogeneous classes of the indicated degree. Then the twist map replaces $\beta_i\otimes\gamma_{n-i}$ with $(-1)^{i(n-i)}\gamma_{n-i}\otimes\beta_i$.

   Then when we build the cup product in this way, this sign propagates all the way through and we obtain the graded-commutativity as advertised.

Remark 5.17. A major consequence of the naturality of the cup product is the following: if $f : X \to Y$ is a homotopy equivalence of spaces, then there is an isomorphism $f^* : \mathrm{H}^n(Y;R) \to \mathrm{H}^n(X;R)$ for each n. This gives an isomorphism of R-algebras

$$\mathrm{H}^*(Y;R) \to \mathrm{H}^*(X;R)$$

Actually opinion differs on whether $\mathrm{H}^*(X;R)$ should mean $\bigoplus_{i=0}^{\infty} \mathrm{H}^i(X;R)$ (a graded R-algebra) or $\prod_{i=0}^{\infty} \mathrm{H}^*(X;R)$. In this course, the former is meant.

Example 5.18. Let R be a ring. Then, as an R-algebra

$$\mathrm{H}^*(S^n;R) = R[e]/(e^2), \quad |e| = n$$

This is forced for reasons of dimension.

Example 5.19. Let $X = Y \coprod Z$ be a disjoint union. We know that $\mathrm{H}^n(X;R) = \mathrm{H}^n(Y;R) \times \mathrm{H}^n(Z;R)$. By chasing diagrams around, we can establish that if $(y,z) \in \mathrm{H}^n(Y;R) \times \mathrm{H}^n(Z;R)$ and $(y',z') \in \mathrm{H}^{n'}(Y;R) \times \mathrm{H}^{n'}(Z;R)$, then $(y,z) \smile (y',z') = (y \smile y', z \smile z')$.

By comparison using the map $Y \coprod Z \to Y \vee Z$, we deduce the same result for $Y \vee Z$ away from degree 0.

Example 5.20. Two examples that we'll actually compute later:

$$\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;R) = R[x]/(x^{n+1}), \quad |x| = 2$$

and similarly $\mathrm{H}^*(\mathbb{C}\mathrm{P}^\infty;R) = R[x]$.

Accepting this for the time being, recall that the mapping cone of the Hopf map $\eta : S^3 \to S^2 = \mathbb{C}P^1$ is homeomorphic to $\mathbb{C}P^2$. It was observed long ago that this space has homology isomorphic to $\mathrm{H}_*(S^4 \vee S^2;R)$, where $S^4 \vee S^2$ is the mapping cone on the trivial map $S^3 \to S^2$. We know enough now to see that the cup-product structures on the spaces are different, and so we see that η is not homotopic to the constant map.

Example 5.21. Similarly

$$\mathrm{H}^*(\mathbb{R}P^n;\mathbb{F}_2) = \mathbb{F}_2[x]/(x^{n+1}), \quad |x| = 1.$$

Remark 5.22. Recall above that we produced a cross product: $\mathrm{H}^*(X;R) \times \mathrm{H}^*(Y;R) \to \mathrm{H}^*(X \times Y;R)$. This was produced as a map of R-modules. In fact, it is a map of rings. I proved this one year in 527, and it is very boring.

We should remark on the definition. If A and B are graded R-algebras, then $A \otimes_R B$ can be given a ring structure: if $a, c \in A$ are homogeneous elements and $b, d \in B$ are homogeneous elements, then define

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd$$

and extending linearly to the entire tensor product.

In good circumstances, for instance, if X and Y are CW complexes and at least one of them has f.g. free (co)homology in each dimension, then $\mathrm{H}^*(X;R) \otimes_R \mathrm{H}^*(Y;R) \to \mathrm{H}^*(X \times Y;R)$ is an isomorphism. (DO IN CLASS).

Example 5.23. Let R be a ring. The notation $\Lambda_R(x_1,\dots,x_n)$ means an R-algebra generated by variables $x_1,\dots,x_n$ satisfying the relations $x_i^2=0$ and $x_ix_j=-x_jx_i$. Let

$$X = S^{n_1}\times\cdots\times S^{n_k}$$

be a product of spheres where $n_1,\dots,n_k$ are all odd integers. Then by induction

$$\mathrm{H}^*(X;R)\cong\Lambda_R(x_1,\dots x_k)$$

where $|x_i|=n_i$.

Similarly if

$$Y = S^{m_1}\times\cdots\times S^{m_j}$$

is a product of even spheres, then

$$\mathrm{H}^*(Y;R)\cong R[y_1,\dots,y_l]/(\{y_iy_l\}_{1\le i\le l,1\le j\le l}$$

that is, the relation $x_ix_j=-x_jx_i$ is replaced by $y_iy_j=y_jy_i$.

If we have $Z=X\times Y$, then

$$\mathrm{H}^*(Z;R)=\Lambda_R(x_1,\dots x_k)\otimes_R R[y_1,\dots,y_l]/(\{y_iy_l\}_{1\le i\le l,1\le j\le l}$$

5.3 The Cap Product

Throughout we use singular homology and cohomology.

Remark 5.24. Let X be a space and R a ring. Just from the definition $C^n(X;R)$ is the R-module of R-linear functions $\xi: C_n(X;R)\to R$. This gives an evaluation map $e: C^n(X;R)\otimes_R C_n(X;R)\to R$ One verifies directly that if $\xi\otimes\alpha$ is an elementary tensor in $C^n(X;R)\otimes_R C_n(X;R)$ such that both ξ and α are (co)cycles and at least one is a (co)boundary, then $e(\xi\otimes\alpha)=\xi(\alpha)=0$. Therefore the evaluation map descends to an evaluation map $\mathrm{H}^n(X;R)\otimes_R\mathrm{H}_n(X;R)\to R$.

We have already seen this map in the special case $R=\mathbb{Z}$ in the universal coefficients formula, part of which is a map

$$\mathrm{H}^n(X;\mathbb{Z})\to\mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_n(X;\mathbb{Z}),\mathbb{Z})$$

Now we combine this *evaluation pairing* with the diagonal in order to produce an action of cohomology on homology.

Construction 5.25. Let X be a space and let R be a ring. Consider the following composite map (R coefficients are taken throughout, and are omitted)

$$\begin{aligned} C^n(X)\otimes_R C_m(X) &\xrightarrow{\Delta_*} C^n(X)\otimes_R C_m(X\times X)\\ &\xrightarrow{\mathrm{id}\otimes\psi} C^n(X)\otimes\Big(\bigoplus_{i=-\infty}^{\infty} C_i(X)\otimes_R C_{m-i}(X)\Big)\\ &\to C^n(X)\otimes_R C_n(X)\otimes_R C_{m-n}(X)\\ &\to R\otimes_R C_{m-n}(X)\cong C_{m-n}(X)\end{aligned}$$

If we start with an elementary tensor $\xi \otimes \alpha$ where α is a cycle, then the output of this procedure is a cycle—this is because ψ is a map of complexes. If both ξ and α are (co)cycles and either ξ of α is a (co)boundary, then the output is a boundary. This requires detailed verification, not written up here, but it's true. We have an induced map

$$\frown : \mathrm{H}^n(X;R) \otimes_R \mathrm{H}_m(X;R) \to \mathrm{H}_{m-n}(X;R)$$

called the *cap product*.

Remark 5.26. In general this gives us a map $\mathrm{H}^n(X;R) \otimes \mathrm{H}_n(X;R) \to \mathrm{H}_0(X;R)$. Composing with the map $\mathrm{H}_0(X;R) \to \mathrm{H}_0(*;R)$ induced by $X \to R$ recovers the evaluation map.

The proofs of the following two propositions are a matter of homological algebra.

Proposition 5.27. *If $f : X \to Y$ is a map of spaces, if $\alpha \in \mathrm{H}_m(X;R)$ and $\xi \in \mathrm{H}^n(Y;R)$ are homology and cohomology classes, then $f_*(f^*(\xi) \frown \alpha) = \xi \frown f_*(\alpha)$.*

Proposition 5.28. *If ξ, η are cohomology classes and α a homology class, the $(\xi \smile \eta) \frown \alpha = \xi \frown (\eta \frown \alpha)$.*

Remark 5.29. There are relative cap products:

$$\mathrm{H}^n(X;R) \otimes_R \mathrm{H}_m(X,A;R) \to \mathrm{H}_{m-n}(X,A;R)$$

(the construction of this is not at all difficult) and

$$\mathrm{H}^n(X,A;R) \otimes_R \mathrm{H}_m(X,A;R) \to \mathrm{H}_{m-n}(X;R).$$

This is a little more surprising, but exists because if you have $\xi \in C^n(X,A;R)$—cochains on X vanishing when restricted to A—and $\alpha \in C_m(A;R)$ then the pairing map gives 0, so it follows that there is an induced map $C^n(X,A;R) \otimes_R C_m(X;R)/C_m(A;R) \to C_{m-n}(X;R)$. This is the asserted map.

5.4 Compactly supported cohomology

We use singular cohomology throughout and let R be a ring.

Definition 5.30. Let X be a space, and let $\xi \in C^n(X;R)$ be a cochain. We say ξ is *compactly supported* if there exists a compact set $K \subseteq X$ such that for all chains $\alpha \in C_n(X-K) \subseteq C_n(X)$, we have $\xi(\alpha) = 0$. Observe that if ξ is compactly supported—with support in K—and if $\alpha \in C_{n+1}(X-K)$ then $d(\xi)(\alpha) = \xi(d\alpha) = 0$, so that $d\xi$ is also compactly supported.

Therefore there exists a (co)chain complex $C_c^*(X;R)$ of compactly supported singular cochains. The homology of this complex is the *compactly supported cohomology* of X, and is denoted $\mathrm{H}_c^*(X;R)$.

Definition 5.31. By a *directed system* of objects, we mean a diagram X_i indexed by a totally ordered set I in which there exists a unique morphism $X_i \to X_j$ whenever $i \leq j$. In practice, we draw the case of $I = \mathbb{N}$—the most important case.

Lemma 5.32. *Suppose given a directed system of short exact sequences of R-modules*

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d] & \dots \\ B_0 \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & \dots \\ C_0 \arrow[r] & C_1 \arrow[r] & \dots \end{tikzcd}$$

(the 0-s above and below having been omitted). Then the induced map diagram of colimits

$$0 \to \operatorname*{colim}_i A_i \to \operatorname*{colim}_i B_i \to \operatorname*{colim}_i C_i \to 0$$

is also short exact.

Proof. The proof is a sequence of diagram chases, and is not done in the notes. □

Corollary 5.33. *If $C_{0,*} \to C_{1,*} \to \dots$ is a sequence of chain complexes, then for all n,*

$$\operatorname*{colim}_i \mathrm{H}_n(C_i,*) = \mathrm{H}_n(\operatorname*{colim}_i C_{i,*})$$

Proposition 5.34. *Let X be a space and let $\{K_i\}$ denote a directed system of compact subspaces of X with the property that each compact $J \subseteq X$ is contained in some K_i. Then*

$$\operatorname*{colim}_{K_i} \mathrm{H}^*(X, X - K_i; R) = \mathrm{H}^*_c(X; R)$$

Proof. The system $C^*(X, X-K_i; R)$ is a directed system. Taking colimits first, we see that $\operatorname{colim}_i C^n(X, X-K_i; R)$ is the R-module of compactly supported n-cochains. Therefore $\mathrm{H}^n(\operatorname{colim}_i C^*(X, X-K_i); R) = \mathrm{H}^n_c(X; R)$. By the corollary above, this is naturally isomorphic to $\operatorname{colim}_{K_i} \mathrm{H}^n(X, X-K_i; R)$. □

Example 5.35. A specific example is given by $X = \mathbb{R}^n$ and the compact spaces $K_i = \overline{B(0,i)}$, the closed balls of radius i about the origin. We can calculate

$$\mathrm{H}^i(\mathbb{R}^n, \mathbb{R}^n - K_i; R) = \mathrm{H}^i(\mathbb{R}^n/(\mathbb{R}^n - B(0, i+1)); R) = \mathrm{H}^i(S^n; R)$$

and the maps between different groups are isomorphisms. In the colimit, therefore, we get

$$\mathrm{H}^i_c(\mathbb{R}^n; R) \cong \mathrm{H}^i(S^n; R).$$

In particular, $\mathrm{H}^i_c(\cdot; R)$ is not a homotopy invariant.

5.5 Orientations and Fundamental Classes

Definition 5.36. A *topological manifold of dimension n* is a topological space M^n satisfying the following properties

1. M^n is Hausdorff
2. For each point $x \in M^n$ there exists an open neighbourhood $U \ni x$ such that $U \approx \mathbb{R}^n$.

It should not be considered to be part of the definition, but we will also assume M^ns appearing are second countable (i.e., have a countable dense subspace). This is required to ensure that M^n embeds as a closed subset of $\mathbb{R}^N$ for some N. The manifolds without this property are considered pathological.

Remark 5.37. It's worth remarking that $\mathbb{R}^n$ is homeomorphic to any open ball $B(\vec{x};\epsilon) \subseteq \mathbb{R}^n$.

Remark 5.38. Topological mainfolds are the minimally structured examples of general manifolds.

Let M^n be a manifold and choose a set of open embeddings $\phi_i : \mathbb{R}^n \to M^n$ such that the sets $U_i := \phi_i(\mathbb{R}^n) \subset M^n$ cover M^n. These data will be called an *atlas* and the maps ϕ_i (or possibly the just the images) are called *charts*. Second-countability implies that we can assume the atlas is countable. Suppose one has two charts

$$\phi_i^{-1}(U_j) = \mathbb{R}^n \cap \phi_i^{-1}(U_j) \xrightarrow{\approx} U_i \cap U_j = U_j \cap U_i \xleftarrow{\approx} \phi_j^{-1}(U_i) \cap \mathbb{R}^n = \phi_j^{-1}(U_i)$$

Reading left-to-right, one has a homeomorphism $\phi_j^{-1} \circ \phi_i$ between two open subsets of $\mathbb{R}^n$. By imposing further conditions on these maps, we can define more highly-structured manifolds. For instance, we can require that they all be n-times or infinitely continuously differentiable, in which case we are led to the definition of a $\mathscr{C}^n$-smooth manifold. A smooth manifold is, however, not just a topological space (unlike the continuous manifold) but is also an equivalence class of atlases. That is, one can enquire whether certain topological manifolds admit inequivalent smooth-manifold structures. Strikingly, there are 28 inequivalent smooth structures on the topological space S^7.

Remark 5.39. Manifolds are all locally contractible and therefore locally path connected and locally simply connected. The connected components of a manifold are all manifolds themselves, and we will generally restrict our attention to connected manifolds.

Remark 5.40. By convention, a compact manifold as defined above is called a *closed manifold*. There is a notion of manifold-with-boundary, which generalizes that of manifold, and "closed" here indicates the absence of boundary. It is the case that a closed smooth manifold can be given a finite CW complex structure, a fact which we use but do not prove (Morse theory), and in dimensions other than 4 it is known that even a topological closed manifold can be given a finite CW complex structure. In [Hat10, Appendix A], it is proved that a compact manifold (even with boundary) has the homotopy type of a finite CW complex.

Example 5.41. A 0-manifold is a discrete space. A connected (second countable) 1-manifold is homeomorphic to $\mathbb{R}$ (noncompact) or to S^1 (compact). There exist non-second-countable connected 1-manifolds, for instance, the long line. The connected 2-manifolds form a much more interesting family: this contains the surfaces M_g of genus $g \geq 0$, with S^2 corresponding to the $g = 0$ case, it contains $\mathbb{R}P^2$ and it contains K, the Klein bottle. There is an operation $\sharp$, the connect sum, and all compact 2-manifolds can be obtained as connect sums from M_g, $\mathbb{R}P^2$ and K. The classification even of closed 3-manifold is beyond us.

Definition 5.42. Let M^n be a nonempty n-manifold, and let $x \in M^n$ be a point. We can find a chart $U_i \ni x$ and then we can calculate $\mathrm{H}^n(M^n|x;\mathbb{Z}) = \mathrm{H}^n(M^n, M^n - \{x\};\mathbb{Z}) = \mathrm{H}^n(U_i, U_i - \{x\};\mathbb{Z}) \cong \mathrm{H}^n(S^n;\mathbb{Z}) = \mathbb{Z}$. A *local orientation*) of M^n at x is a choice of generator for $\mathrm{H}^n(M^n|x;\mathbb{Z})$. Denote the set of local orientations at x by μ_x.

Construction 5.43. Let x and y be points in $\mathbb{R}^n$ and let V be an open ball containing both x and y. Then by use of excision and homotopy invariance, there are natural isomorphisms

$$\mathrm{H}_n(\mathbb{R}^n|x;\mathbb{Z}) = \mathrm{H}_n(\bar{V}|x;\mathbb{Z}) = \mathrm{H}_n(\bar{V}/\partial V;\mathbb{Z}) = \mathrm{H}_n(\bar{V}|y;\mathbb{Z}) = \mathrm{H}_n(\mathbb{R}^n|y;\mathbb{Z})$$

a short argument shows that the identification $\mathrm{H}_n(\mathbb{R}^n|x;\mathbb{Z}) = \mathrm{H}_n(\mathbb{R}^n|y;\mathbb{Z})$ doesn't change if we enlarge V, so that we obtain an identification of μ_x with μ_y. In fact, both are identified with a set of generators of $\mathrm{H}^n_c(\mathbb{R}^n;\mathbb{Z})$.

Construction 5.44. Let M^n be an n-manifold. Place a topology on the set of pairs $\tilde{M}^n = \{(x,u) | x \in M^n, u \in \mu_x\}$ as follows. Given a chart $U_i \ni x$, the orientation u gives rise to an orientation u_y for all $y \in U_i$ by the previous construction. For any open $V \subseteq U_i$ containing x, declare $\tilde{V}_u\{(y,u_y)|y \in V\} subseteq \tilde{M}^n$ to be an open subset. This gives a base for a topology on $\tilde{M}^n$ equipped with an obvious map $\tilde{M}^n \to M^n$. For each $(x,u) \in \tilde{M}^n$, moreover, we can find a neighbourhood $\tilde{U}_u$ that maps homeomorphically to $U \ni x$. That is, $\tilde{M}^n \to M^n$ is a 2-sheeted covering space, called the *orientation double cover* of M^n.

Definition 5.45. Let M^n be a manifold. If $\tilde{M}^n \approx M^n \times \{u_1, u_2\}$ is a split covering space, M^n is *orientable*. If not, it is *non-orientable*. If M^n is orientable, then either of the two sections of $\tilde{M}^n \to M^n$ is called a *orientation* of M^n.

Remark 5.46. An orientation amounts to a continuous choice of local orientations at all points of M^n.

Remark 5.47. There are two orientations on $\mathbb{R}^n$, all arising from generators of $\mathrm{H}_n(S^n;\mathbb{Z})$. One knows that the action of SO_n on $\mathbb{R}^n$ should therefore be orientation-preserving, while the action of $O_n - SO_n$ should be orientation-reversing. If one has a differentiable n-manifold M^n, allowing one to talk about a tangent bundle to M^n, then a local orientation of M^n at x amounts to an equivalence class of bases for the tangent bundle T_xM^n under the action of SO_n. An orientation of M^n amounts to a globally consistent choice of such bases at each point. This allows us to relate our general notion of “orientation” with the sorts of orientation used in multivariable calculus (or differential geometry or topology).

Construction 5.48. Let R be a ring. The orientation double cover $\tilde{M}^n$ is enlarged to a twisted form of R over M^n. This is our first view of a nontrivial sheaf of R-modules in this course. It appears as a covering space $\tilde{R} = \tilde{M}^n \times_{\mathbb{Z}/(2)} R$ where the action of $\mathbb{Z}/(2)$ on the left is the reversal of orientation, and that on the right is multiplication by -1.

If the ring R is of characteristic 2, then $+1 = -1$ in R and so $\tilde{R} \approx M \times R$ a space over M^n, but if $1 \neq -1$ in R, then the covering space $\tilde{R} \to M$ contains $\tilde{M}$ as a sub-covering-space, and may be nontrivial.

Definition 5.49. Given a space over M, i.e., a map $f : X \to M$, if $A \subseteq M$ is a subset, we define the set of *sections* of f on A to be

$$\Gamma(X; A) = \{s : A \to X \mid f \circ s = \mathrm{id}_A\}$$

In the case of $\tilde{R} \to M$, the set of sections over any subset is always an R-module, although it may be 0.

Remark 5.50. If A, B are both closed or both open, then the set of sections satisfies a *sheaf condition.* In our case, this means that the sequence of R-modules

$$0 \to \Gamma(\tilde{R}; A \cup B) \to \Gamma(\tilde{R}; A) \oplus \Gamma(\tilde{R}; B) \to \Gamma(\tilde{R}; A \cap B \tag{5.1}$$

is exact.

Over a point, the sections $\Gamma(\tilde{R}; x)$ form a free R-module of rank 1, and there are two distinguished R-generators u and v in $\Gamma(\tilde{R}; x)$, satisfying $u + v = 0$. Choosing a local orientation of M at x gives a choice of either u or v as the distinguished generator 1 in R.

Remark 5.51. Suppose M^n is an n-manifold, let R be a ring, and let $x \in M^n$ be a point. We can identify

$$\mathrm{H}_n(M, M - \{x\}; \mathbb{Z}) \cong \Gamma(\tilde{\mathbb{Z}}, x)$$

since both are the infinite cyclic abelian groups generated by the local orientations of M at x. We can promote this to the tensor products

$$\mathrm{H}_n(M, M - \{x\}; R) = R \otimes_{\mathbb{Z}} \mathrm{H}_n(M, M - \{x\}; \mathbb{Z}) = R \otimes_{\mathbb{Z}} \Gamma(\tilde{\mathbb{Z}}; x) = \Gamma(\tilde{R}; x)$$

Definition 5.52. If R is a ring and M an n-manifold, then we will say M is R*-orientable* if $\tilde{R}$ is a trivial covering space. This happens exactly when M is orientable or when $2 = 0 \in R$.

Theorem 5.53. *Let M^n be a closed connected n-manifold and let $A \subseteq M^n$ be a compact subset.*

1. $\mathrm{H}_i(M^n|A; R) = 0$ *for all $i > n$.*
2. *If M^n is R-orientable then the maps* $\mathrm{H}_n(M^n|A; R) \to \mathrm{H}_n(M^n|x; R)$ *are isomorphisms for all $x \in A$.*
3. *If M^n is not R-orientable, then the maps* $\mathrm{H}_n(M^n|A; R) \to \mathrm{H}_n(M^n|x; R)$ *are injections having image equal to the* 2*-torsion in* $\mathrm{H}_n(M|x; R)$.

By taking $A = M$, we see that $\mathrm{H}_i(M;R) = 0$ for $i > n$ in all cases. We see that $\mathrm{H}_n(M;\mathbb{Z}) \cong \mathbb{Z}$ if and only if M is orientable, otherwise it is 0. Finally, when $R = \mathbb{F}_2$, every manifold appears orientable.

Proof. This proof is a variation of [Hat10, Theorem 3.26].

The main trick in the proof of the theorem is to show that there are natural isomorphisms

$$\Gamma(\tilde{R};A) \xrightarrow{\cong} \mathrm{H}_n(M^n, M^n - A; R)$$

whenever A is a compact subset of M^n. The sheaf condition (C.1) and the Mayer–Vietoris sequence allow one to reduce the proof of this claim to the case of compact sets within the charts, which are homeomorphic to $\mathbb{R}^n$, and then eventually to convex compact sets in the charts. In the case of a convex compact set B in a chart, since B is convex, it is contractible, and covering space theory says elements of $\Gamma(\tilde{R};B)$ (i.e., sections of $\tilde{R}$ over B) are determined by their values at any single point x of B, i.e., to $\Gamma(\tilde{R};x)$. Similarly, homotopy invariance means that $\mathrm{H}_n(M, M-B;R)$ is naturally isomorphic to $\mathrm{H}_n(M, M-\{x\};R)$, but we have identified (i.e., produced a natural isomorphism between) these two R-modules.

Now to prove the actual theorem. For the vanishing, we again use a Mayer–Vietoris argument. The vanishing statement is known when A is a contractible compact subset, and we can build any A up out of these. At each stage, will have a sequence

$$0 \to \mathrm{H}_{n+1}(M|A\cup B;R) \to \mathrm{H}_n(M|A\cap B;R) \to \mathrm{H}_n(M|A;R) \oplus \mathrm{H}_n(M|B;R) \to \mathrm{H}_n(M|A\cap B;R) \to \dots$$

and the sheaf condition applies to show that the higher homology vanishes.

The other two claims follow from the natural isomorphism.

□

Definition 5.54. In particular $\mathrm{H}_n(M^n;\mathbb{Z}) \cong \mathbb{Z}$ if M^n is orientable and a choice of local orientation at any point (equivalently, a choice of orientation of M^n) yields a choice of distinguished generator $[M^n] \in \mathrm{H}_n(M^n;\mathbb{Z})$. This is called *a fundamental class*. In fact, for any coefficients $\mathrm{H}_n(M^n;R) \cong R = \mathrm{H}_n(M^n;\mathbb{Z}) \otimes_{\mathbb{Z}} R$, and the absence of any Tor-term here implies that $\mathrm{H}_{n-1}(M^n;\mathbb{Z})$ is torsion-free. Since it is known to be finitely generated, it must be a free abelian group.

If M^n is not orientable, then $\mathrm{H}_n(M^n;\mathbb{Z}) = 0$ while $\mathrm{H}_n(M^n;\mathbb{F}_2) = \mathbb{F}_2$. A similar argument to the orientable case and universal coefficients shows that $\mathrm{H}_{n-1}(M^n;\mathbb{Z}) \cong \mathbb{Z}/(2) \oplus \mathbb{Z}^a$ for some a.

Remark 5.55. The orientable closed 2-manifolds are precisely the genus-g-surfaces. The Klein bottle and $\mathbb{R}P^2$ are not orientable.

Remark 5.56. The fundamental classes of oriented manifolds allow us to get a geometric view of certain homology classes. Let R be a ring. Consider arbitrary X, and a map $f : N^d \to X$ from an R-oriented manifold, giving $f_*([N]) \in \mathrm{H}_d(X;R)$. It is a theorem that all homology classes arise this way with $\mathbb{Z}/(2)$-coefficients. With $\mathbb{Z}$ coefficients, in dimensions $d \geq 7$, not all integral homology classes are representable this way.

If $X = M^n$ is an differentiable manifold itself, and $\alpha = f_*[N]$ is a homology class obtained from a map $f : N^d \to M^d$, and if $d < n/2$ then we may actually obtain α as the class of a embedded submanifold $P^d \subseteq M^n$. Moreover, all rational homology classes may be obtained this way, without restriction on the dimension.

All this was proved by René Thom in his seminal paper [Tho54], in which he introduced the theory of (co)bordism.

5.6 Poincaré Duality

Construction 5.57. Let M be an R-oriented n-manifold and let $A \subseteq M$ be a compact subset, and let $x \in A$. Recall that the cap product gives us a map

$$\mathrm{H}^k(M|A;R) \times \mathrm{H}_n(M|A;R) \to \mathrm{H}_{n-k}(M;R)$$

Recall that there are natural isomorphisms $\mathrm{H}_n(M|A;R) = \mathrm{H}_n(M|x;R) = R$. Write $[M_A]$ for the generator of $\mathrm{H}^k(M|A;R)$ determined by the orientation. If $A = M$ then this is just the fundamental class again, and in general it should be thought of as the "fundamental class with support on A". Write $D_{M|A} : \mathrm{H}^k(M|A;R) \to \mathrm{H}_{n-k}(M;R)$.

If $i : A \subseteq B$ is an inclusion of compact sets, then there is a compatibility diagram

$$\begin{tikzcd}
\mathrm{H}^k(M|A;R) \arrow[dd, "i_*"] & \times & \mathrm{H}_n(M|A;R) \arrow[dr] & \\
& & & \mathrm{H}_{n-k}(M|B;R) \\
\mathrm{H}^k(M|B) & \times & \mathrm{H}_n(M|B) \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

so that $D_{M|A})(i_*\xi) = D_{M|B}(\xi)$. Therefore in the colimit, there is a well defined map $D_M : \mathrm{H}^k_c(M;R) \to \mathrm{H}_{n-k}(M;R)$. When M is compact, $D_M : \mathrm{H}^k(M;R) \to \mathrm{H}_{n-k}(M;R)$ is given by $D_M(\xi) = \xi \frown [M]$— and in general, it wants to be this, but you have to restrict classes ξ supported on a compact A because $[M]$ doesn't always exist, only $[M|_A]$.

Theorem 5.58. *Let R be a ring. Let M be an R-oriented n-manifold. Then the map*

$$D_M : \mathrm{H}^k_c(M;R) \to \mathrm{H}_{n-k}(M;R)$$

given by

$$D_M(\xi) = \xi \frown [M]$$

is an isomorphism.

Proof. We give only an idea of the proof here, which should be enough to explain why this the result is true. There is a major technical step omitted.

The simplest kind of manifold is $\mathbb{R}^n$, and one observes that the result holds for $\mathbb{R}^n$. We already calculated $\mathrm{H}_c^k(\mathbb{R}^n;R)$ and $\mathrm{H}_{n-k}(\mathbb{R}^n;R)$—both vanish except when $k=n$, and in each case the modules are free of rank 1—in fact, in the process of establishing the existence of a fundamental class for oriented manifolds, we asserted there was a natural isomorphism between the R-duals of these two modules. The $\frown$ product $\mathrm{H}^n(\mathbb{R}^n|\bar{B};R)\times\mathrm{H}_n(\mathbb{R}^n|\bar{B};R)\to R$ is actually just the evaluation pairing, and so there is not just an abstract isomorphism between $\mathrm{H}_c^n(\mathbb{R}^n;R)$ and $\mathrm{H}_n(\mathbb{R}^n;R)$, but a specific isomorphism depending only on the orientation.

Now we would like to build a general M up as a union of coordinate charts. Most importantly, we would like to build the result up for Ms that are unions of finitely many spaces, each homeomorphic to an oriented $\mathbb{R}^n$. To that end, we would like a commuting diagram of Mayer–Vietoris sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}_c^k(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & \mathrm{H}_c^k(U)\oplus\mathrm{H}_c^k(V) \arrow[r] \arrow[d, "D_U\oplus -D_V"] & \mathrm{H}_c^k(U\cup V) \arrow[r] \arrow[d, "D_{U\cup V}"] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^{n-k}(U\cap V) \arrow[r] & \mathrm{H}^{n-k}(U)\oplus\mathrm{H}^{n-k}(V) \arrow[r] & \mathrm{H}^{n-k}(U\cup V) \arrow[r] & \cdots
\end{tikzcd}$$

It is the case that this exists, and so one can build up the result for any finite union of coordinate charts in M.

One peculiarity of the sequence is the variance in the cohomology: if A is a compact subset of U and B a compact subset of V, then we get

$$\mathrm{H}^k(U\cap V|A\cap B)=\mathrm{H}^k(M|A\cap B)\to\mathrm{H}^k(M|A)\oplus\mathrm{H}^k(M|B)=\mathrm{H}^k(U|A)\oplus\mathrm{H}^k(V|B)$$

from excision. Passing to colimits over a directed system pairs of $A\subseteq U$ and $B\subseteq V$ where A and B are compact, we get the required maps on compactly supported cohomology. This gives the slightly peculiar cohomology long exact sequence.

The hard part of the construction is showing the diagram commutes. We don't prove that here.

This suffices to establish Poincaré duality for all manifolds with finite atlases. For the general case, we let $\mathscr{U}$ denote the collection of open subsets of M for which the duality map is an isomorphism. This is partially ordered by inclusion. Let $\{U_i\}$ denote a totally ordered subset, i.e., a chain. We know that homology commutes with direct limits of chain complexes, so we can write

$$\mathrm{H}_c^k(\bigcup U_i)=\operatorname{colim}\mathrm{H}_c^k(U_i)\cong\operatorname{colim}\mathrm{H}_{n-k}(U_i)=\mathrm{H}_{n-k}(\bigcup U_i)$$

and so the union of the U_i is also in $\mathscr{U}$. Therefore, by Zorn's lemma, there must be a maximal open set for which the result holds. Using the Mayer–Vietoris argument again, we see this maximal open set must be M itself. $\square$

The two cases to pay attention to here are when M is oriented and R is arbitrary, or when M is arbitrary and $R = \mathbb{F}_2$.

Note that if M itself is closed, then $\mathrm{H}^*_c(M;R) = \mathrm{H}^*(M;R)$

Remark 5.59. This is an astonishing result, because the isomorphism is quite different from the relation between homology and cohomology one obtains from universal coefficients.

Corollary 5.60. *Let M^n be a closed n-manifold where n is odd. Then $\chi(M) = 0$.*

Proof. It is possible to calculate $\chi(M)$ as the Euler characteristic of $\mathrm{H}_*(M;\mathbb{F}_2)$, since M has the homotopy type of a finite CW complex.

Then $\mathrm{H}_i(M;\mathbb{F}_2) \cong \mathrm{H}^i(M;\mathbb{F}_2)$ by universal coefficients—these are dual vector spaces. By Poincaré duality, however, $\mathrm{H}^i(M;\mathbb{F}_2) \cong \mathrm{H}_{n-i}(M;\mathbb{F}_2)$. Writing h_i for the dimension of the i-th homology (the i-th Betti number), we see that $h_i = h_{n-i}$, and so $\sum_{i=0}^n (-1)^i h_i = 0$. □

Construction 5.61. Suppose we have a map of oriented closed manifolds, $f : M^a \to N^b$. We can produce an *Umkehr* map in homology:

$$\mathrm{H}_i(N;\mathbb{Z}) \xleftarrow{D_N} \mathrm{H}^{b-i}(N;\mathbb{Z}) \xrightarrow{f^*} \mathrm{H}^{b-i}(M;\mathbb{Z}) \xrightarrow{D_M} \mathrm{H}_{i+(a-b)}(M;\mathbb{Z})$$

Construction 5.62. Let M be a closed oriented differentiable n-manifold, let A and B be closed oriented differentiable submanifolds of dimensions $a = n - i$ and $b = n - j$ respectively, and assume $A \cap B$ intersect transversely.

We produce an orientation on $A \cap B$ by the following procedure. If $A \cap B$ is empty, there is nothing to do. Pick a point $p \in A \cap B$, and observe that there is a short exact sequence of tangent spaces at p

$$0 \to T_p(A \cap B) \to T_p(A) \oplus T_p(B) \to T_p(X) \to 0$$

Pick an oriented basis for T_p of the form $(u_1, \dots, u_{n-i-j}, v_1, \dots, v_j, w_1, \dots w_i)$ so that the ordered basis $(u_1, \dots, v_j)$ is an oriented basis for $T_p A$ and $(u_1, \dots, u_{n-i-j}, w_1, \dots, w_i)$ is an oriented basis for $T_p B$. Then declare $(u_1, \dots, u_{n-i-j})$ to be an oriented basis for $T_p(A \cap B)$—i.e., a local orientation.

Now we can speak meaningfully of the fundamental classes $[M]$, $[A]$, $[B]$ and $[A \cap B]$. We have defined $[A] \in \mathrm{H}_{n-i}(A;R)$. In this case, let $[A]^* \in \mathrm{H}^i(M;R)$ denote the D_M-dual of the image of $[A]$ in $\mathrm{H}_{n-i}(M;R)$, and similarly for $[B]^* \in \mathrm{H}^j(M;R)$ and $[A \cap B]^* \in \mathrm{H}^{i+j}(M;R)$. We assert that

$$[A]^* \smile [B]^* = [A \cap B]^*.$$

This gives a geometric interpretation of the cup product.

Example 5.63. Let $M = \mathbb{C}P^n$. This is an oriented manifold (or, at the very least, orientable) of dimension $2n$. We know from the cell structure that

$$\mathrm{H}^{2i}(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}Y_{2i} \quad 0 \le i \le n$$

for some class X_i. We asserted previously that the relations were so as to make $\mathrm{H}^*(\mathbb{C}P^n;\mathbb{Z})$ into a truncated polynomial ring, but we don't know why.

We establish the ring structure by induction. It is true in the case $n = 1$, because $\mathbb{C}P^1 \approx S^2$, and there is no interesting cup product for dimensional reasons.

We know by looking at the cell structure that the inclusion $\mathbb{C}P^{n-1} \to \mathbb{C}P^n$ induces an isomorphism on $\mathrm{H}^i(\cdot;\mathbb{Z})$ when $i \le 2n-1$, and is the 0-map in $\mathrm{H}^{2n}(\cdot;\mathbb{Z})$. So we can say that

$$\mathrm{H}^*(\mathbb{C}P^n;\mathbb{Z}) = (\mathbb{Z}[x_2] \oplus \mathbb{Z}Y_{2n})/I$$

where the subscripts denote the degrees of classes, and where the ideal I contains x_2^{n+1} and $x_2 Y_{2n}$ and Y_{2n}^2 (for dimensional reasons) and some relation $x_2^n = aY_{2n}$ for some $a \in \mathbb{Z}$. The hard thing to determine is a.

The class x_2 is Poincaré dual to a generator of $\mathrm{H}_{2n-2}(\mathbb{C}P^n;\mathbb{Z})$, since x_2 generates H^2. Another way of getting a generator of $\mathrm{H}_{2n-2}(\mathbb{C}P^n;\mathbb{Z})$ is as the fundamental class of the embedded closed submanifolds $\mathbb{C}P^{n-1} \to \mathbb{C}P^n$ embedded as hyperplanes (all these embeddings are homotopic to each other). Let $P_1,\dots,P_{n-1},P_n$ be $n-1$-embedded hyperplanes, and assume the P_i are all transverse. Say, P_i is given by the vanishing of the i-th projective coordinate. Then, by our geometric identification of cup product, the class of x_2^n is the class of the dual of $[P_1 \cap P_2 \cap \cdots \cap P_n]$. This intersection is a point, so the class of x_2^n is (up to sign, at least) the poincare dual to $[*] \in \mathrm{H}_0(\mathbb{C}P^n;\mathbb{Z})$. But this is $[\mathbb{C}P^n]$ itself.

Example 5.64. The same story works the same way for $\mathbb{R}P^n$, except you have to use $\mathbb{F}_2$ coefficients because the manifolds appearing are not $\mathbb{Z}$-orientable.

Example 5.65 (Alexander duality). Let X be a compact subspace of S^n. A point-set argument says that because S^n is locally contractible, X is the limit of a directed system of open neighbourhoods $U_i \supset X$, all of which deformation retract onto X.

Then $\mathrm{H}_k(S^n - X;R) \cong \mathrm{H}_c^{n-k}(S^n - X;R)$, by Poincaré duality. Further (R coefficients used throughout),

$$\begin{aligned}
\mathrm{H}_c^{n-k}(S^n - X) &= \operatorname*{colim}_i \mathrm{H}^{n-k}(S^n - X|(S^n - U_i) = \\
&= \operatorname*{colim}_i \mathrm{H}^{n-k}(S^n - X, U_i - X) \\
&= \operatorname*{colim}_i \mathrm{H}^{n-k}(S^n, U_i) \\
&= \mathrm{H}^{n-k}(S^n, X) \\
&= \tilde{\mathrm{H}}^{n-k-1}(X)
\end{aligned}$$

where the last step comes from the long exact sequence in cohomology associated to the pair (S^n, X) and requires $k \ge 1$.

In the exceptional case of $k = 0$, use functoriality for the map $(S^n, \emptyset) \to (S^n, X)$ to see that

$$\tilde{\mathrm{H}}_k(S^n - X;R) \cong \tilde{\mathrm{H}}^{n-k-1}(X;R)$$

for all values of k.

Appendix A

Category Theory

A.1 Categories, Functors and Natural Transformations

We generally disregard problems of size, viz. whether or not something is a set or not.

Definition A.1. A category $\mathbf{C}$ consists of a collection of objects, $\mathrm{ob}\mathbf{C}$ and a collection of morphisms $\mathrm{Mor}\,\mathbf{C}$, such that

1. Every morphism has a *source* in $\mathrm{ob}\mathbf{C}$ and a *target* in $\mathrm{ob}\mathbf{C}$. A morphism f is often written $f : X \to Y$ or $X \xrightarrow{f} Y$, where X is the source and Y is the target.

2. For any two objects X and Y, there is a set $\mathrm{Mor}_{\mathbf{C}}(X, Y)$ or $\mathbf{C}(X, Y)$, consisting of precisely those morphisms of $\mathbf{C}$ having source X and target Y.

3. For any three objects X, Y, Z of $\mathbf{C}$, there is a composition of morphisms

$$\circ : \mathbf{C}(X, Y) \times \mathbf{C}(Y, Z) \to \mathbf{C}(X, Z)$$

   and this composition is associative in that $f \circ (g \circ h) = (f \circ g) \circ h$ whenever these composites are defined.

4. For each object X of $\mathbf{C}$, there exists an *identity morphism* $\mathrm{id}_X \in \mathbf{C}(X, X)$ such that $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_X \circ g = g$ whenever these composites are defined.

Remark A.2. An easy and standard argument proves that id_X is the unique morphism $X \to X$ with the stated property.

Notation A.3. There are categories $\mathbf{Set}, \mathbf{Gr}, \mathbf{Ab}$, of sets, groups, abelian groups, and many other similar categories of objects commonly studied in mathematics. These are generally *large* categories, in that the collection of objects does not form a set.

Example A.4. There are also *small* categories, where the collection of objects forms a set, and therefore the collection of morphisms also forms a set (under our hypotheses). For instance, given any partially ordered set S, one can construct a category, also called S, where one regards 'element of' and 'object of' as synonymous, and then declares that $S(a,b) = \emptyset$ if $b < a$ and that $S(a,b)$ consists of one morphism if $a \leq b$.

It is often possible to depict such small categories diagrammatically. It is customary to draw only a subset of all morphisms, and to leave out morphisms that can be inferred from the morphisms and objects drawn. In particular, identity morphisms are seldom drawn.

1. The standard span:

$$\bullet \longleftarrow \bullet \longrightarrow \bullet$$

2. The standard cospan:

$$\bullet \longrightarrow \bullet \longleftarrow \bullet$$

3. The category $\mathbb{N}$ (with the usual order)

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots$$

Example A.5. There is a category **Top** of topological spaces where the objects are topological spaces and morphisms are continuous functions. There is also a category of *pointed spaces*, $\mathbf{Top}_\bullet$, where the objects are pairs (X, x_0) where X is a topological space and $x_0 \in X$. The morphisms are the based maps, i.e., $\mathrm{Top}_\bullet((X,x_0),(Y,y_0))$ is the set of continuous $f : X \to Y$ such that $f(x_0) = y_0$.

Definition A.6. Given a category $\mathbf{C}$, a *subcategory*, $\mathbf{D}$ of $\mathbf{C}$ consists of a subcollection $\mathrm{ob}\mathbf{D}$ of $\mathrm{ob}\mathbf{C}$ and a subcollection $\mathrm{Mor}\,\mathbf{D}$ of $\mathrm{Mor}\,\mathbf{C}$, containing id_X for all objects X in $\mathrm{ob}\mathbf{D}$, such that $\mathrm{Mor}\,\mathbf{D}$ is closed under composition.

Example A.7. There are many examples of subcategories that arise by restricting the class of objects, but not restricting the morphisms between the objects. For instance, **Ab** is the subcategory of **Gr** where the groups considered are required to be abelian, but given any two abelian groups G, H, one has $\mathbf{Gr}(G,H) = \mathbf{Ab}(G,H)$. In this situation, **Ab** is a *full* subcategory of **Gr**.

Example A.8. At the other extreme, it is possible to form subcategories where one considers all the objects, but strictly fewer morphisms. For instance, given a field k, one might consider the category having as objects the collection of finite-dimensional k vector spaces, but where the morphisms are restricted to be isomorphisms. This is a subcategory of the usual category of finite-dimensional k vector spaces and all k linear maps, and it appears in some definitions of algebraic K-theory.

Definition A.9. Given two categories, $\mathbf{C}$ and $\mathbf{D}$, it is possible to form a *product category* $\mathbf{C}\times\mathbf{D}$. The objects in this category are ordered pairs (X,Y) where X is an object of $\mathbf{C}$ and Y is an object of $\mathbf{D}$. The morphisms are also ordered pairs, $(f,g) : (X,Y) \to (Z,W)$ is a morphism in the product category if $f : X \to Z$ is a morphism in $\mathbf{C}$ and $g : Y \to W$ is a morphism in $\mathbf{D}$.

Definition A.10. If **C** is a category, and $f : X \to Y$ is a morphism in this category, then we say that f is an *isomorphism* if there exists a morphism $f^{-1} : Y \to X$ such that $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$. It is immediate that id_X is an isomorphism.

Remark A.11. An isomorphism in Top or a related category is generally called a *homeomorphism*

Definition A.12. If **C** is a category, and $f : X \to Y$ is a morphism in this category, then we say that f is

1. a *monomorphism* if, whenever $g, h : Z \to X$ are morphisms, the statement $f \circ g = f \circ h$ implies $g = h$. That is, f is *left cancellable*,
2. an *epimorphism* if, whenever $g, h : Y \to Z$ are morphisms, the statement $g \circ f = h \circ f$ implies $g = h$. That is, f is *right cancellable*,
3. a *bimorphism* if it is both a monomorphism and an epimorphism.

Definition A.13. If **C** is a category, and $f : X \to Y$ is a morphism in this category, then we say that f is

1. a *split monomorphism* if there exists a morphism $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$.
2. a *split epimorphism* if there exists a morphism $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$.

Exercises

1. Suppose $f : X \to Y$ is an isomorphism. Prove that f^{-1} is uniquely determined by f.
2. Prove that the class of isomorphisms in a category has the *two-out-of-three* property, namely: if
$$A \xrightarrow{f} B \xrightarrow{g} C$$
are composable morphisms such that two of f, g and $g \circ f$ are isomorphisms, then so too is the third.
3. Prove that the class of isomorphisms in a category has the *two-out-of-six* property, namely: if
$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} F$$
are composable morphisms such that $g \circ f$ and $h \circ g$ are isomorphisms, then so too are f, g, h and $h \circ g \circ f$.
4. Determine the monomorphisms, epimorphisms and bimorphisms in the category of sets.
5. Give an example in **Top** of a bimorphism that is not an isomorphism.
6. Let **Haus** denote the full subcategory of Hausdorff topological spaces. Give an example in **Haus** of an epimorphism that is not surjective.

A.2 Functors and Natural Transformations

Definition A.14. Given two categories $\mathbf{C}$ and $\mathbf{D}$, a *(covariant) functor* $F : \mathbf{C} \to \mathbf{D}$ consists of an assignment
$$F : \mathrm{ob}\mathbf{C} \to \mathrm{ob}\mathbf{D}$$
and for every pair of objects X, Y in ob$\mathbf{C}$, a function
$$F : \mathbf{C}(X,Y) \to \mathbf{D}(F(X), F(Y))$$
such that

1. $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all object X of $\mathbf{C}$ and
2. $F(f \circ g) = F(f) \circ F(g)$ wherever $f \circ g$ is defined.

Example A.15. Given any category $\mathbf{C}$, there is an identity functor $\mathrm{id}_{\mathbf{C}}$.

Definition A.16. Given two categories $\mathbf{C}$ and $\mathbf{D}$, a *contravariant functor* $F : \mathbf{C} \to \mathbf{D}$ consists of an assignment
$$F : \mathrm{ob}\mathbf{C} \to \mathrm{ob}\mathbf{D}$$
and for every pair of objects X, Y in ob$\mathbf{C}$, a function
$$F : \mathbf{C}(X,Y) \to \mathbf{D}(F(Y), F(X))$$
such that

1. $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all object X of $\mathbf{C}$ and
2. $F(f \circ g) = F(g) \circ F(f)$ wherever $f \circ g$ is defined.

Remark A.17. Warning: contravariant functors reverse the direction of morphisms. Failure to keep adequate track of the variance of functors is the category-theoretical analogue of a sign error in arithmetic. These errors are minor, frustrating and common.

Notation A.18. Given a category $\mathbf{C}$, there is an *opposite category*, $\mathbf{C}^{\mathrm{op}}$ having the same collection of objects, but where
$$\mathbf{C}^{\mathrm{op}}(X,Y) = \mathbf{C}(Y,X).$$
One may view a contravariant functor $F : \mathbf{C} \to \mathbf{D}$ as a covariant functor $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$.

Example A.19. There are many functors in mathematics that consist largely of forgetting structures. Such functors are often called "forgetful", but it is difficult to give a precise definition of what this means. Common examples include:

1. $V : \mathrm{Top}_\bullet \to \mathrm{Top}$, forgetting the basepoint.

2. $V : \text{Top} \to \mathbf{Set}$, forgetting the topology.

3. $V : \mathbf{Ab} \to \mathbf{Grp}$, forgetting that the group is abelian.

Example A.20. There is a canonical functor $\eta : \mathbf{C}^{\text{op}} \times \mathbf{C} \to \mathbf{Set}$ given by $\eta(X, Y) = \mathbf{C}(X, Y)$. Fixing either X or Y gives rise to functors

1. $\eta_X : \mathbf{C} \to \mathbf{Set}$,

2. $\eta^Y : \mathbf{C}^{\text{op}} \to \mathbf{Set}$.

Definition A.21. Let $F : \mathbf{C} \to \mathbf{D}$ be a functor. We say F is

1. *full* if, for any two objects X, Y of $\mathbf{C}$, the function $F : \mathbf{C}(X, Y) \to \mathbf{D}(F(X), F(Y))$ is surjective.

2. *faithful* if, for any two objects X, Y of $\mathbf{C}$, the function $F : \mathbf{C}(X, Y) \to \mathbf{D}(F(X), F(Y))$ is injective.

3. *essentially surjective* if, for any object Z of $\mathbf{D}$, one can find an object X of $\mathbf{C}$ such that there exists an isomorphism $Z \to F(X)$.

Definition A.22. Given two (covariant) functors $F, G : \mathbf{C} \to \mathbf{D}$, a *natural transformation* $\Psi : F \to G$ consists of a collection of morphisms $\Psi_X : F(X) \to G(X)$, one for each object X of $\mathbf{C}$, such that for any morphism $h : X \to Y$ in the category $\mathbf{C}$, the square

$$\begin{tikzcd} F(X) \arrow[r, "\Psi_X"] \arrow[d, "F(h)"] & G(X) \arrow[d, "G(h)"] \\ F(Y) \arrow[r, "\Psi_Y"] & G(Y) \end{tikzcd}$$

commutes, which is to say: $G(h) \circ \Psi_X = \Psi_Y \circ F(h)$.

Remark A.23. A similar definition of *natural tranformation* can be made if F and G are both contravariant. The details are left to the reader.

The word “natural” is often applied to morphisms between objects in categories. It should be used only to apply to a morphism that is part of a, possibly implicit, natural transformation. If the morphisms Ψ_X are all of a certain type, for instance all isomorphisms or all inclusions, then Ψ_X may be said to be a natural isomorphism or a natural inclusion as appropriate.

Example A.24. Fix a field k. Let $k\mathbf{Vect}$ denote the category of k vector spaces and all linear maps between them. Then there is a contravariant functor sending $f : V \to W$ to $f^* : W^* \to V^*$, where $V^* = \text{Hom}_k(V, k)$ and f^* is the evident map $\text{Hom}_k(W, k) \to \text{Hom}_k(V, k)$ given by postcomposing with f.

There is a covariant functor sending $f : V \to W$ to $f^{**} : V^{**} \to W^{**}$ given by applying V^* twice. That is, V^{**} is the k vector space of linear functionals on the k vector space of linear functionals on V. There is a natural transformation $e : \text{id}_{\mathbf{Vect}} \to (\cdot)^{**}$ given by a collection of

k-linear maps $e_V : V \to V^{**}$ given by defining $e_V(x)$, where $x \in V$, to be the functional sending $\psi \in V^*$ to $\psi(x)$.

At least if one assumes the Axiom of Choice, the map $e_V : V \to V^{**}$ defined above is a natural inclusion. If one restricts to the full subcategory of finite dimensional k vector spaces, then e is a natural isomorphism, but if V is not finite dimensional, then $e_V : V \to V^{**}$ is not an isomorphism.

Definition A.25. If $F : \mathbf{C} \to \mathbf{D}$ is a functor, then we say F is an *equivalence of categories* if there exists a functor $G : \mathbf{D} \to \mathbf{C}$ and natural isomorphisms $\Phi : G \circ F \to \mathrm{id}_{\mathbf{C}}$ and $\Psi : F \circ G \to \mathrm{id}_{\mathbf{D}}$.

Remark A.26. In contrast to the case of isomorphisms, the functor F is not sufficient to determine G, Ψ and Φ uniquely. The notion of "isomorphism of categories", where $G \circ F$ and $F \circ G$ are required to be identity functors, is not particularly common.

Remark A.27. In the presence of a sufficiently strong version of the Axiom of Choice, a functor is an equivalence of categories if an and only if it is full, faithful, and essentially surjective.

Example A.28. Let **Fin** denote the category of finite sets. This category is not small. Let **N** denote the full subcategory of sets $\{\emptyset, \{1\}, \{1,2\}, \ldots\}$. Then $\mathbf{N} \to \mathbf{Fin}$ is an equivalence of categories. In this situation, one says that **N** is a *small skeleton* for **Fin**.

Remark A.29. If one restricts attention to small categories, then one can define a "category of categories", but as we have remarked, the notion of isomorphism one gets is not generally useful. It is better to incorporate the natural transformations and form a "2-category" of small categories, a structure having objects (categories), morphisms (functors), and morphisms of morphisms (natural tranformations). We will not pursue this further here.

Exercises

1. Let $F : \mathbf{C} \to \mathbf{D}$ be a functor. Show that F preserves isomorphisms and split mono- and epimorphisms. Show by example that it need not preserve monomorphisms or epimorphisms that are not split.

A.3 Adjoint Functors

Definition A.30. Given two functors $L : \mathbf{C} \to \mathbf{D}$ and $R : \mathbf{D} \to \mathbf{C}$, we say L is *left adjoint* to R and R is *right adjoint* to L if, for any object X of **C** and Y of **D**, there exists a bijection

$$\Psi_{X,Y} : \mathbf{D}(L(X), Y) \to \mathbf{C}(X, R(Y))$$

and such that the bijection Ψ is a natural isomorphism of functors $\mathbf{C}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{Set}$. More explicitly, if $f : X \to X'$ and $g : Y' \to Y$ are morphisms in the appropriate categories, then the square of sets

$$\begin{tikzcd} \mathbf{D}(L(X'), Y') \arrow[r, "\Psi_{X',Y'}"] \arrow[d] & \mathbf{C}(X', R(Y')) \arrow[d] \\ \mathbf{D}(L(X), Y) \arrow[r, "\Psi_{X,Y}"] & \mathbf{C}(X, R(Y)) \end{tikzcd}$$

commutes.

Example A.31. Forgetful functors often have one or both kinds of adjoint. For instance, the forgetful functor $V : \mathbf{Top} \to \mathbf{Set}$ has both a left- and a right-adjoint. The forgetful functor $V : \mathbf{Ab} \to \mathbf{Set}$ has a left adjoint, but no right adjoint.

Example A.32. A very important family of adjunctions is modelled on the following one: fix a set X. This gives rise to two functors $\mathbf{Set} \to \mathbf{Set}$; the cartesian product functor $Y \mapsto Y \times X$, and the mapping space functor $Z \mapsto Z^X$, where Z^X is notation for the set of functions $X \to Z$. That these are indeed functors $\mathbf{Set} \to \mathbf{Set}$ is left as an exercise. We assert that they form an adjoint pair, in that there is a natural bijection

$$\mathbf{Set}(Y \times X, Z) \to \mathbf{Set}(Y, Z^X).$$

Verifying this is left to the reader.

Example A.33. The previous example has a variant for topological spaces, provided some additional hypothesis is placed on the spaces appearing. For instance, if X is a locally compact Hausdorff space, then there is a natural bijection

$$\mathbf{Top}(Y \times X, Z) \to \mathbf{Top}(Y, \mathscr{C}(X, Z))$$

where $\mathscr{C}(X, Z)$ is the space of continuous functions $X \to Z$ given the compact–open topology.

Definition A.34. Given an adjoint pair of functors $L : \mathbf{C} \to \mathbf{D}$ and $R : \mathbf{D} \to \mathbf{C}$, we can define two natural transformations.

1. The *unit* of the adjunction $\epsilon : \mathrm{id}_{\mathbf{C}} \to R \circ L$
2. The *counit* of the adjunction $\eta : L \circ R \to \mathrm{id}_{\mathbf{D}}$.

The unit is formed by letting $\eta_X : X \to R(L(X))$ be the element of $\mathbf{C}(X, R(L(X)))$ corresponding to $\mathrm{id}_{L(X)} \in \mathbf{D}(L(X), L(X))$ under the natural isomorphism of the adjunction. The counit is formed similarly.

Remark A.35. We continue with the notation of the previous definition. The unit and counit have certain universal properties. In the case of the unit, suppose that there is a morphism $f : X \to R(Y)$ in $\mathbf{C}$. Since L and R are adjoint, the morphism f is equivalent to a unique morphism $g : L(X) \to Y$. This morphism can be written, tautologically, as $\mathrm{id}_{L(X)} \circ g : L(X) \to L(X) \to Y$, which, by adjunction, is equivalent to a factorization $f = R(g) \circ \epsilon_X : X \to R(L(X)) \to R(Y)$.

Dually, any morphism $h : L(X) \to Y$ factors uniquely as $\eta_Y \circ L(i) : L(X) \to L(R(Y)) \to Y$.

Remark A.36. If $L : \mathbf{C} \to \mathbf{D}$ and $M : \mathbf{D} \to \mathbf{E}$ are two functors, each left adjoint to functors R and S respectively, then $M \circ L$ is left adjoint to $R \circ S$.

Proposition A.37. *Suppose $L, L' : \mathbf{C} \to \mathbf{D}$ are two naturally isomorphic functors and R, R' are right adjoints to L and L'. Then R and R' are naturally isomorphic.*

This result applies in particular in the case where $L = L'$.

A.4 Diagrams, Limits and Colimits

Notation A.38. If $\mathbf{I}$ is a small category and $\mathbf{C}$ is a category, then a functor $D : \mathbf{I} \to \mathbf{C}$ may be called a *diagram*. If, for any morphism $f : i \to j$ in the category $\mathbf{I}$, the morphism $D(f)$ depends only on i and j, then we say the diagram is *commutative*.

Example A.39. Not all commonly occurring diagrams are commutative. For instance, pairs of parallel morphisms $X \rightrightarrows Y$ appear often but form a commutative diagram only when the two morphisms agree.

Definition A.40. Given a small category $\mathbf{I}$ and a category $\mathbf{C}$, one can define a category $\mathbf{Fun}(\mathbf{I}, \mathbf{C})$ of $\mathbf{I}$-shaped diagrams. The objects are the functors $D : \mathbf{I} \to \mathbf{C}$, and the morphisms are the natural transformations between them.

Definition A.41. Give a small category $\mathbf{I}$, a category $\mathbf{C}$ and an object X of $\mathbf{C}$, we can form the *constant* $\mathbf{I}$*-shaped diagram with value* X by $\mathrm{const}_{\mathbf{I}}(X) : \mathbf{I} \to \mathbf{C}$ by sending all objects to X and all morphisms to id_X. In fact, $\mathrm{const}_{\mathbf{I}}$ is a functor $\mathrm{const}_{\mathbf{I}} : \mathbf{C} \to \mathrm{Fun}(\mathbf{I}, \mathbf{C})$.

Definition A.42. Let $\mathbf{I}$ be a small category and $\mathbf{C}$ a category.

Given an $\mathbf{I}$-shaped diagram D in $\mathbf{C}$, a *limit* of D is an object $\lim D$ of $\mathbf{C}$ and a natural transformation $\Phi : \mathrm{const}_{\mathbf{I}}(\lim D) \to D$ such that for any object X of $\mathbf{C}$ equipped with a natural transformation $\Psi : \mathrm{const}_{\mathbf{I}}(X) \to D$, there is a unique map $u : X \to \lim D$ such that $\Psi = \Phi \circ \mathrm{const}(u)$.

Dually, a *colimit* of an $\mathbf{I}$-shaped diagram D is an object $\operatorname{colim} D$ of $\mathbf{C}$ and a natural transformation $\Phi : D \to \mathrm{const}_{\mathbf{I}}(\operatorname{colim} D)$ such that for any object X of $\mathbf{C}$ equipped with a natural transformation $\Psi : D \to \mathrm{const}_{\mathbf{I}}(X)$, there is a unique map $u : \operatorname{colim} D \to X$ such that $\Psi = \mathrm{const}(u) \circ \Phi$.

Remark A.43. Strictly speaking a limit or colimit of a diagram encompasses both the object and the natural transformation of functors—which is to say, the morphisms. In practice, one often refers to the object as the limit or colimit, leaving the morphisms implicit.

Remark A.44. It follows easily from a standard argument that if L and L' are two limits of the same diagram $D : \mathbf{I} \to \mathbf{C}$, then there is a unique isomorphism $f : L \to L'$ in $\mathbf{C}$ such that the diagram

$$\begin{tikzcd} \mathrm{const}_{\mathbf{I}} L \arrow[rr, "\mathrm{const}_{\mathbf{I}}(f)"] \arrow[dr] & & \mathrm{const}_{\mathbf{I}} L' \arrow[dl] \\ & D & \end{tikzcd}$$

commutes. A dual statement applies to colimits.

Since they are unique up to unique isomorphism, one often abuses terminology and speaks of "the limit" or "the colimit" of a diagram.

Remark A.45. There is another view of limits and colimits that is sometimes useful. Suppose the functor $\mathrm{const}_{\mathbf{I}}$ has a right adjoint ℓ. Then a limit of D is given by the object $\ell(D)$ and the counit map $\mathrm{const}_{\mathbf{I}}\,\ell(D) \to D$.

Dually, if $\mathrm{const}_{\mathbf{I}}$ has a right adjoint colim, the *colimit* of D is the unit map $D \to \mathrm{const}_{\mathbf{I}}\,\mathrm{colim}(D)$.

Example A.46. The language used above is technical. In practice, the idea is simple. Let us consider as a category I the standard cospan

$$\begin{tikzcd} \bullet \arrow[r] & \bullet & \bullet \arrow[l] \end{tikzcd}$$

Let $\mathbf{C} = \mathbf{Top}$ be the category of topological spaces. Then the data of an I-shaped diagram D consists of three spaces and two continuous functions $X \to Y \leftarrow Z$.

The constant-diagram functor takes a space W and produces $W \to W \leftarrow W$, where the morphisms are identities. A natural transformation $\mathrm{const}(W) \to D$ is the data of continuous functions $f : W \to X$, $g : W \to Y$ and $h : W \to Z$ such that

$$\begin{tikzcd} W \arrow[r, equal] \arrow[d, "f"] & W \arrow[r, equal] \arrow[d, "g"] & W \arrow[d, "h"] \\ X \arrow[r] & Y & Z \arrow[l] \end{tikzcd}$$

commutes, or, more succinctly

$$\begin{tikzcd} W \arrow[drr, "h"] \arrow[ddr, "f"'] \arrow[ddrr, dotted, "g"] & & \\ & & X \arrow[d] \\ & Y \arrow[r] & Z. \end{tikzcd} \tag{A.1}$$

Note further that the dotted arrow is determined by either f or h, and may be omitted.

The space $\lim D$ and the natural transformation amounts to an objcet and morphisms fitting in the following diagram

$$\begin{tikzcd} \lim D \arrow[r, "h"] \arrow[d, "f"] & X \arrow[d] \\ Y \arrow[r] & Z. \end{tikzcd} \tag{A.2}$$

This diagram has the property that if W is as in Diagram (A.1), then there exists a unique map $W \to \lim D$ such that Diagram (A.3) commutes.

$$
\begin{tikzcd}
W \arrow[dr, dashed, "g"] \arrow[drr, "h"] \arrow[ddr, "f"'] & & \\
& \lim D \arrow[r] \arrow[d] & X \arrow[d] \\
& Y \arrow[r] & Z.
\end{tikzcd}
\tag{A.3}
$$

This particular kind of limit is called a *fibre product* and is written $X \times_Y Z$. While our definition specifies the limit only up to unique isomorphism, we can easily construct an explicit model for $X \times_Y Z$ in the category of topological spaces. Most usually, let $X \times_Y Z$ consist of the subset of pairs $(x, z) \in X \times Z$ such that the image of x and of z in Y agree. Then endow $X \times_Y Z$ with the coarsest topology (fewest open sets) such that the evident projection maps $X \times_Y Z \to X$ and $X \times_Y Z \to Z$ are both continuous.

It is instructive to consider $X \times_Y Z$ in the following cases:

1. When Y is a singleton space.
2. When $X \to Y$ is the inclusion of a subspace.

Remark A.47. By uniqueness of adjoints and of unit or counit maps, if a limit or colimit of a diagram exists, it is unique up to unique isomorphism.

Notation A.48. A category in which all limits can be constructed is *complete* and one in which all colimits can be constructed is *cocomplete*. The following categories are all complete and cocomplete:

1. **Set**.
2. **Top** and $\mathbf{Top}_\bullet$.
3. R-**Mod**.

The full subcategory **Haus** of Hausdorff topological spaces is complete but not cocomplete.

Notation A.49. If D is a diagram in $\mathbf{C}$ consisting of a family of objects $\{X_i\}_{i \in I}$ and no nonidentity arrows, then a limit of D is called a *product* of $\{X_i\}_{i \in I}$ and a colimit of D is called a *coproduct* of $\{X_i\}_{i \in I}$. The product of topological spaces is an example of a categorical product, and the disjoint union of topological spaces is an example of a categorical coproduct.

Notation A.50. If D is a diagram in $\mathbf{C}$ of the form

$$
\begin{tikzcd}
& A \arrow[d] \\
B \arrow[r] & C
\end{tikzcd}
$$

then a limit of D is called a *pullback* of D, and often denoted $A \times_C B$.

The dual concept is the *pushout*, a colimit of.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ C & \end{tikzcd}$$

Proposition A.51. *Suppose $F : \mathbf{C} \to \mathbf{C}$ is a functor between complete categories such that F has a left adjoint, L. Suppose further that D is a diagram in $\mathbf{C}$. Let $\lim D$ be a limit of D. Then $F(\lim D)$ is a limit of $F(D)$.*

Dually, suppose $F : \mathbf{C} \to \mathbf{C}$ is a functor between cocomplete categories such that F has a right adjoint, R. Suppose further that D is a diagram in $\mathbf{C}$. Let $\operatorname{colim} D$ be a limit of D. Then $F(\operatorname{colim} D)$ is a colimit of $F(D)$.

Remark A.52. Let $\mathbf{C}$ be a category. Consider the empty diagram D. If $\lim D$ exists, then it is an object $*$ such that all objects X of $\mathbf{C}$ are equipped with a unique morphism $X \to *$. Such an object $*$ is called a *terminal* object of $\mathbf{C}$. Any two terminal objects are isomorphic by a unique isomorphism.

Dually, the colimit of an empty diagram is called an *initial* object; such an object may often be denoted $\emptyset$. If an object is both initial and terminal, then it is called a *zero object.*

Exercises

1. The forgetful functor $V : \mathbf{Ab} \to \mathbf{Set}$ has a left adjoint, L. Describe the unit map $\epsilon : S \to V(L(S))$.

2. Show that $V : \mathbf{Ab} \to \mathbf{Set}$ does not preserve colimits. For instance, consider the colimit of a diagram consisting of two nonzero abelian groups and no nontrivial arrows. Therefore V does not have a right adjoint.

3. Let R be a ring and let $\mathbf{M}$ denote the category of R-modules and R-linear maps, and let $f : M \to N$ be a morphism in $\mathbf{M}$. Describe the limit of the diagram

$$\begin{tikzcd} & 0 \arrow[d] \\ M \arrow[r, "f"] & N. \end{tikzcd}$$

   Express the cokernel of f as the colimit of a diagram.

4. Consider the forgetful functor $V : \mathbf{Top}_{\bullet} \to \mathbf{Top}$. Describe a left adjoint to this functor. Prove that V does not have a right adjoint.

5. Let X be a locally compact Hausdorff space, and consider the adjunction between $\times X$ and $\mathscr{C}(X, \cdot)$ in $\mathbf{Top}$. Describe the counit of this adjunction.

Appendix B

Tensor Products, Tor and Ext

B.1 Tensor Products

Fix a commutative ring R and consider two R modules A and B. In more generality, R can be noncommutative and A need only be a right R module and B a left R module, but we will not need this level of generality.

Definition B.1. A map $f : A \times B \to M$ of R-modules is *bilinear* if

1. $f(ra + r'a', b) = rf(a,b) + r'f(a',b)$ and

2. $f(a, rb + r'b') = rf(a,b) + r'f(a,b')$

hold for all choices of element.

Remark B.2. The tensor product $A \otimes_R B$ is a universal R-module for R-bilinear maps $f : A \times B \to M$. That is, $A \otimes_R B$ is an R-module, there is a canonical R-bilinear map $A \times B \to A \otimes_R B$, and given any R-bilinear map $f : A \times B \to M$, there is a unique factorization of f through $A \times B \to A \otimes_R B$.

Remark B.3. We can study the tensor product using the images of elements $(a,b) \in A \times B$ in $A \times_R B$. The image of such an element is called an *elementary tensor* and is written $a \otimes b$. The elementary tensors satisfy bilinear relations:

1. $a \otimes (b + b') = a \otimes b + a \otimes b'$

2. $(a + a') \otimes b = a \otimes b + a' \otimes b$

3. $r(a \otimes b) = (ra) \otimes b = a \otimes rb$.

You can use the above to produce the tensor product: It is the R-module generate by all elementary tensors subject to these relations. In fact, to generate $A \otimes_R B$, it's sufficient to take elementary tensors $a \otimes b$ as a and b range over some sets of generators of A and B.

Remark B.4. It follows from the above that there is a natural *twist* isomorphism $A \otimes_R B \cong B \otimes_R A$. It is also the case that $(A \otimes_R B) \otimes_R C = A \otimes_R (B \otimes_R C)$. This can be worked out directly from the universal property.

Example B.5. If F_1 and F_2 are free R-modules with bases $\{b^1_\alpha\}_A$ and $\{b^2_\gamma\}_C$, then $F_1 \otimes_R F_2$ is a free R-module with basis $\{b^1_\alpha \otimes b^2_\gamma\}_{A\times C}$.

Remark B.6. Similarly to the above, there is a natural isomorphism

$$\left(\bigoplus_{i\in I} A_i\right) \otimes_R B \cong \bigoplus_{i\in I} (A_i \otimes_R B)$$

Remark B.7. The data of a homomorphism $\phi : A \otimes_R B \to C$ is equivalent to a bilinear map $A \times B \to C$, which is in turn equivalent to an R-linear map $A \to \mathrm{H}_R(B,C)$. That is,

$$\mathrm{Hom}_R(A \otimes_R B, C) = \mathrm{Hom}_R(A, \mathrm{Hom}_R(B,C))$$

so that $\cdot \otimes_R B$ is left adjoint to $\mathrm{Hom}_R(B,\cdot)$.

From this we can derive the very useful fact that if

$$0 \to X \to Y \to Z \to 0$$

is a short exact sequence of R modules, then

$$X \otimes_R B \to Y \otimes_R B \to Z \otimes_R B \to 0$$

is an exact sequence of R-modules. We say that $\cdot \otimes_R B$ is a *right exact* functor. The same also holds for the isomorphic construction $B \otimes_R \cdot$.

B.2 Examples

Example B.8. If I is an ideal of R, then there is a short exact sequence

$$0 \to I \to R \to R/I \to 0$$

and if M is an R-module, then applying $\cdot \otimes_R M$ gives us

$$I \otimes_R M \xrightarrow{\phi} R \otimes_R M = M \to (R/I) \otimes_R M \to 0.$$

Identifying $R \otimes_R M = M$ see that ϕ sends an elementary tensor $i \otimes m$ to im. Therefore, $\mathrm{im}\,\phi = IM$, precisely the submodule of M generated by multiplying all elements of M by all elements of I. We deduce $M/IM \cong (R/I) \otimes_R M$.

Example B.9. The previous example gives us a calculation of all tensor products of f.g. abelian groups. Each such group decomposes as a direct sum of cyclic groups. $\mathbb{Z} \otimes_{\mathbb{Z}} A = A = A \otimes_{\mathbb{Z}} \mathbb{Z}$, so all that remains is the calculation of tensor products

$$\mathbb{Z}/(m) \otimes_{\mathbb{Z}} \mathbb{Z}/(n)$$

but this is just $\mathbb{Z}/(n)/m\mathbb{Z}/(n) \cong \mathbb{Z}/\gcd m, n$.

Remark B.10. The following is a list of properties of the tensor product over R.

1. $\cdot \otimes_R \cdot$ is a functor of both variables (a bifunctor).
2. $\cdot \otimes_R A$ is additive.
3. $\cdot \otimes_R A$ preserves arbitrary direct sums.
4. $\cdot \otimes_R A$ preserves quotients.
5. $\cdot \otimes_R R$ is naturally isomorphic to the identity functor.

Almost everything we do involving tensor products can be deduced from these.

Remark B.11. If $\phi : R \to S$ is a homomorphism of commutative rings, then ϕ makes S into an R-module, allowing us to define a functor $S \otimes_R \cdot$. If M is an R-module, then $S \otimes_R M$ (ostensibly an R-module) can be endowed with an S-module structure by $s'(s \otimes m) = (s's) \otimes m$ and extending to sums of elementary tensors.

B.3 The Tor functor

We continue our digression into homological algebra. Suppose we are given a short exact sequence of R-modules:

$$0 \to A \to B \to C \to 0$$

and another R-module M. We apply $\otimes_R M$ and get a sequence

$$A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$$

(this much is guaranteed to be exact).

In certain cases, the sequence is actually short exact.

Definition B.12. Let M be an R-module. We say M is *flat* if the functor $\cdot \otimes_R M$ preserves short exact sequences, or equivalently, preserves injections.

Example B.13. The R-module R is flat. A direct sum of flat modules is flat. A free module is flat. If R is a field, then all R-modules (R-vector spaces) are free, therefore flat.

Example B.14. The $\mathbb{Z}$-module $\mathbb{Z}/(n)$ is not flat (if $n \geq 2$). For instance, take the sequence

$$0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/(n) \to 0$$

and apply $\cdot \otimes_{\mathbb{Z}} \mathbb{Z}/(n)$ to get the exact sequence

$$\mathbb{Z}/(n) \xrightarrow{0} \mathbb{Z}/(n) \to \mathbb{Z}/(n) \to 0$$

Example B.15. (As-yet unproved assertion) An abelian group, that is, a $\mathbb{Z}$-module, is flat if and only if it is nontorsion (all nonzero elements have infinite order).

Construction B.16. Let M be an R-module. It is possible to write a long exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$$

in which the F_i are free R-modules. This is called a *free resolution* of M.

Suppose N is an R-module. Then we can make a chain complex (if N is not flat, then it need not be exact):

$$\cdots \to F_2 \otimes_R N \to F_1 \otimes_R N \to F_0 \otimes_R N \to 0.$$

Define the i-th homology of this complex to be

$$\mathrm{H}_i(F_* \otimes_R N) = \mathrm{Tor}_R^i(M, N).$$

It is the case that $\mathrm{Tor}_R^i(M,N)$ does not depend on the free resolution of M we used. We will not prove this in full generality, however.

Remark B.17. Given a map $f : M_1 \to M_2$ of R-modules, one can produce compatible free resolutions, $C_{*,1} \to M_1$ and $C_{*,2} \to M_2$, and therefore by functoriality of $\otimes_R N$ and of homology, we can produce a map $f_* : \mathrm{Tor}_*(M_1, N) \to \mathrm{Tor}_*(M_2, N)$. In this way Tor_i becomes a functor for all i.

Example B.18.

$$\mathrm{Tor}_R^0(M,N) = M \otimes_R N$$

We will concentrate on the case of Tor over a principal ideal domain R, most importantly, $\mathbb{Z}$. In this setting we establish some of the unproved assertions above, at least for Tor_R^1.

Lemma B.19. *Let R be a PID and M, N be R-modules. Then* $\mathrm{Tor}_R^{\geq 2}(M,N) = 0$.

Proof. If you believe the above statement about independence of the resolution, then this follows from the existence of a resolution of length 2:

$$0 \to F_1 \to F_0 \to M \to 0.$$

□

Whether or not you're willing to believe the unproved assertion, we use this lemma to justify ignoring $\mathrm{Tor}_R^{\geq 2}$. In fact, from now on, we restrict attention to free resolutions of length 2.

Remark B.20. If M is a free R-module, then $\mathrm{Tor}^1(M,N) = 0$. The reason is that any free resolution will take the form $0 \to F_1 \to F_1 \oplus M \to M \to 0$, and applying $\cdot \otimes_R N$ results in an exact sequence again.

Proposition B.21. *Let R be a PID. Let M and N be R-modules, let $0 \to F_1 \xrightarrow{\theta} F_0 \xrightarrow{\phi} M \to 0$ and $0 \to G_1 \xrightarrow{\omega} G_0 \xrightarrow{\psi} M \to 0$ be two free resolutions of M. Then*

$$\ker(\theta \otimes \mathrm{id}_N) \cong \ker(\omega \otimes \mathrm{id}_N)$$

and either can be used as a construction of $\mathrm{Tor}^1_R(M,N)$. *In summary,* Tor^1_R *is well defined.*

Proof. For the sake of making the construction more easily, identify G_1 and F_1 with their images. Construct a diagram of R-modules:

$$\begin{array}{ccccc} G_1 & \longrightarrow & G_1 \oplus F_0 & \longrightarrow & F_0 \\ \downarrow & & \downarrow & & \downarrow \\ G_1 & \longrightarrow & G_0 & \longrightarrow & M \end{array}$$

Most of the arrows in this diagram are self-evident. The exception is the map $F_0 toG_0$. This is constructed as a lift of the map $\phi : F_0 \to M$ to a map $\tilde{\phi} : F_0 \to G_0$, such that $\psi \circ \tilde{\phi} = \phi$.

We claim the map $G_1 \oplus F_0 \to G_0$ is surjective. Suppose $g \in G_0$ is some element, and choose $f \in F_0$ mapping to $\psi(g) \in M$. Then $g - \tilde{\phi}(f)$ lies in the kernel of ψ, and therefore in the image of $G_1 \to G$.

Now we determine the kernel of $G_1 \oplus F_0 \to G_0$. Suppose $(g, f) \mapsto 0$. Then $\tilde{\phi}(f) = -g$. Applying ψ, we see that $\phi(f) = 0$, so f must lie in $F_1 = \ker(\phi)$. Then $\tilde{\phi}(f) \in G_1$, and so g is uniquely determined, $g = -\tilde{\phi}(f)$.

The upshot is that there is a commutative diagram with split short exact sequences for rows

$$\begin{array}{ccccc} 0 & \longrightarrow & F_1 & \longrightarrow & F_0 \\ \downarrow & & \downarrow & & \downarrow \\ G_1 & \longrightarrow & G_1 \oplus F_0 & \longrightarrow & F_0 \end{array}$$

Since the rows are split exact, applying $\cdot \otimes_R N$ gives a diagram where the rows are still short exact

$$\begin{array}{ccccc} 0 & \longrightarrow & F_1 \otimes_R N & \longrightarrow & F_0 \otimes_R N \\ \downarrow & & \downarrow & & \downarrow \\ G_1 \otimes_R N & \longrightarrow & (G_1 \oplus F_0) \otimes_R N & \longrightarrow & F_0 \otimes_R N \end{array}$$

Observe that the middle column here is a free resolution of G_0, and therefore the inclusion $F_1 \to G_0 \oplus F_0$ is actually split. Applying the snake lemma gives us an exact sequence

$$0 \to \ker(\theta \otimes \mathrm{id}_N) \to G_1 \otimes_R N \to G_0 \otimes_R N \to 0$$

thus establishing the result. ∎

Proposition B.22.

$$\operatorname{Tor}\left(\bigoplus_{i\in I} M_i, N\right) \cong \bigoplus_{i\in I} \operatorname{Tor}(M_i, N)$$

and similarly in the other variable.

The proof is immediate

We now concentrate on the case $R = \mathbb{Z}$, i.e., on abelian groups.

Lemma B.23.

$$\operatorname{Tor}(\mathbb{Z}, A) = 0$$

and

$$\operatorname{Tor}(A, \mathbb{Z}) = 0$$

Since $\mathbb{Z} \otimes_{\mathbb{Z}} \cdot$ and $\cdot \otimes_{\mathbb{Z}} \mathbb{Z}$ are naturally isomorphic to the identity functor, this is immediate.

Lemma B.24.

$$\operatorname{Tor}(\mathbb{Z}/(n), A) = \{a \in A \mid na = 0\}$$

Proof. Look at the obvious resolution. □

That is, $\operatorname{Tor}(\mathbb{Z}/(n), A)$ is the set of n-torsion elements in A. This is sometimes written ${}_nA$. This is what gives the Tor functor its name.

Example B.25. These three results allow us to calculate $\operatorname{Tor}(A, B)$ for all f.g. abelian groups. The case we have not yet covered is $\operatorname{Tor}(\mathbb{Z}/(m), \mathbb{Z}/(n))$, which is the m-torsion in $\mathbb{Z}/(n)$, a group isomorphic to $\mathbb{Z}/\gcd m, n$.

Remark B.26. We observe that in all cases, $\operatorname{Tor}(M, N) \cong \operatorname{Tor}(N, M)$. This is a general theorem, but we will not prove it.

Proposition B.27. *Let* $0 \to M_1 \to M_2 \to M_3 \to 0$ *be a short exact sequence of modules over a PID* R. *Let* N *be an* R*-module. Then there is a long exact sequence*

$$0 \to \operatorname{Tor}^R(M_1, N) \to \operatorname{Tor}^R(M_2, N) \to \operatorname{Tor}^R(M_3, N) \to M_1 \otimes_R N \to M_2 \otimes_R N \to M_3 \otimes_R N \to 0$$

That is, $\operatorname{Tor}_1^R(M, \cdot)$ is measuring the failure of $M \otimes_R \cdot$ to be exact.

B.4 Ext

Remark B.28. Let R be a (commutative) ring and let B be an R-module. Let

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

be a short exact sequence of R-modules. We remark that $\operatorname{Hom}_R(\cdot, B)$ is a contravariant functor, and that

$$0 \to \operatorname{Hom}_R(A_3, B) \to \operatorname{Hom}_R(A_2, B) \to \operatorname{Hom}_R(A_1, B)$$

is exact.

Remark B.29. The functor $\mathrm{Hom}_R(\cdot, B)$ does preserve split short exact sequences. Moreover, $\mathrm{Hom}_R(\bigoplus_{i\in I} A_i, B) = \prod_{i\in I} \mathrm{Hom}_R(A_i, B)$, even in the case of infinite I.

Definition B.30. We define $\mathrm{Ext}^i_R(A,B)$ analogously to Tor. That is, we take a free R-resolution $F_\bullet \to A$, then apply $\mathrm{Hom}(\cdot, B)$, then finally take the homology of the possibly inexact chain complex $\mathrm{Hom}(F_\bullet, B)$.

Remark B.31. In all cases, $\mathrm{Ext}^0_R(A,B) = \mathrm{Hom}_R(A,B)$. Note that, in contrast to $\otimes_R$, the functor $\mathrm{Hom}_R(A,B)$ is not symmetric. For instance, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/(n)) \cong \mathbb{Z}/(n)$ whereas $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(n), \mathbb{Z}) \cong 0$. The same goes for all the Ext groups—they are not symmetric in general.

Remark B.32. $\mathrm{Ext}^i_R(A_1 \oplus A_2, B) = \mathrm{Ext}^i_R(A_1, B) \oplus \mathrm{Ext}^i_R(A_2, B)$ and $\mathrm{Ext}^i_R(A, B_1 \oplus B_2) = \mathrm{Ext}^i_R(A, B_1) \oplus \mathrm{Ext}_R(A, B_2)$.

Remark B.33. If A is a free R-module, then $\mathrm{Ext}^i_R(A,B) = 0$ for $i \geq 1$.

Remark B.34. As with Tor, if the ring R is a PID, then the resolution has length 2, and therefore only Ext^0 and Ext^1 are nonzero. We will not show here that $\mathrm{Ext}^1_R(A,B)$ is independent of the free resolution, but the argument is similar to that used for Tor.

Proposition B.35. *Over a PID, there is a long exact sequence*

$$\begin{aligned} 0 \to \mathrm{Hom}_R(A_3, B) \to \mathrm{Hom}_R(A_2, B) \to \mathrm{Hom}_R(A_1, B) \to \\ \to \mathrm{Ext}^1(A_3, B) \to \mathrm{Ext}^1(A_2, B) \to \mathrm{Ext}^1(A_3, B) \to 0 \end{aligned}$$

Example B.36. We can carry out the following calculations:

1. Over a field k, all higher Ext groups vanish.

2. $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/(n), A) \cong A/nA$.

Remark B.37. Suppose

$$0 \to B \to X \to A \to 0$$

is a short exact sequence of abelian groups, and $0 \to F_1 \to F_0 \to A \to 0$ is a free resolution, then we can lift the map $F_0 \to A$ to a map $F_0 \to X$. Then the composite map $F_1 \to F_0 \to X \to A$ is 0, so it follows that there is a map $F_1 \to B$, i.e., an element $\phi \in \mathrm{Hom}_R(F_1, B)$. It turns out that the lift we chose affects ϕ, but only up to an element of $\mathrm{Hom}_R(F_0, B)$. The short exact sequence therefore determines an element of $\mathrm{Ext}^1_{\mathbb{Z}}(A,B)$. There is a bijection between $\mathrm{Ext}^1_{\mathbb{Z}}(A,B)$ and isomorphism classes of short exact sequences. We do not prove this here, or pursue the idea further.

Appendix C

Solutions

C.1 Homework 2

Exercise C.1. Classify up to isomorphism all abelian groups A that can appear in a short exact sequence

$$0 \longrightarrow \mathbb{Z}/(p^a) \xrightarrow{f} A \xrightarrow{g} \mathbb{Z}/(p^b) \longrightarrow 0 \ .$$

Solution C.2. The isomorphism classes that can arise are $A \cong \mathbb{Z}/(p^k) \oplus \mathbb{Z}/(p^{a+b-k})$ where $k \leq \min\{a, b\}$.

We give three solutions, in increasing order of sophistication.

1. The group A, by coset-counting, must have order p^{a+b}.

   Identify $\mathbb{Z}/(p^a)$ with its image in A. Choose an element $x \in A$ such that $g(x)$ is a generator of $\mathbb{Z}/(p^b)$. We know that $p^b x \in \mathbb{Z}/(p^a)$, and we may therefore find a generator y of $\mathbb{Z}/(p^a)$ such that
   $$p^b x = p^k y \tag{C.1}$$
   for some $k \in \{0, \ldots, a-1\}$. This equation implies that $p^{b+a-k} x = 0$.

   We claim that $\{x, y\}$ form a set of generators for A. Let $a \in A$, then there is some m such that $mg(x) = g(a)$, so that $mx - a$ is a multiple of y.

   Now we use (C.1) to give a presentation of A. We split this into two cases.

   (a) $k \leq b$. In this case, define $y' = -p^{b-k} x + y$. The group A is generated by $\{x, y'\}$, and these are p^{a+b-k}- and p^k-torsion respectively, so they induce a surjective homomorphism
   $$\mathbb{Z}/(p^{a+b-k}) \oplus \mathbb{Z}/(p^k) \to A.$$
   By order considerations, this is necessarily also an injection, so an isomorphism.

In order to prove that this case actually arises, define a map $\mathbb{Z}/(p^{a+b-k}) \oplus \mathbb{Z}/(p^k) \to \mathbb{Z}/(p^b)$ sending $(1,0) \mapsto 1$ and $(0,1) \mapsto p^{b-k}$. The kernel is cyclic of order p^a, generated by $(p^{b-k}, -1)$.

(b) $k > b$. In this case, define $x' = x - p^{k-b}y$. The elements $\{x', y\}$ generate A, and they are p^b- and p^a-torsion. As in the previous case, this establishes an isomorphism $\mathbb{Z}/(p^a) \oplus \mathbb{Z}/(p^b) \to A$.

2. We know that A is a finitely generated abelian group of order p^{a+b}. Let $x \mapsto 1$ under g and let y be a generator of $\ker(g)$. Then $\{x, y\}$ generates A, as in the previous solution.

   By the structure theorem of f.g. abelian groups, $A \cong \mathbb{Z}/(p^{c_1}) \oplus \mathbb{Z}/(p^{c_2}) \oplus \cdots \oplus \mathbb{Z}/(p^{c_m})$, where each $c_i > 0$. Calculating the order gives $\sum_{i=1}^m c_i = a + b$. We wish to show that $m \le 2$. We claim that if $m \ge 3$, then A cannot be generated by 2 elements. This is not trivial, but one can argue as follows:
$$\frac{A}{pA} \cong \frac{\mathbb{Z}}{(p)} \oplus \frac{\mathbb{Z}}{(p)} \oplus \cdots \oplus \frac{\mathbb{Z}}{(p)}$$
   with m summands. Any generating set of A descends to give a generating set of A/pA, but A/pA is a vector space of dimension m over $\mathbb{Z}/(p)$, so can have no basis of cardinality less than m, and therefore $m \le 2$.

   It follows that $A \cong \mathbb{Z}/(p^{a+b-k}) \oplus \mathbb{Z}/(p^k)$. It remains to ascertain which values of k can actually arise. Without loss of generality, let $k \le a+b-k$. Then the existence of a surjection $A \to \mathbb{Z}/(p^b)$ implies that at least one element of A has order p^b or more. The existence of an injection $\mathbb{Z}/(p^a) \to A$ implies that at least one element of A has order p^a. Therefore $a + b - k \ge \max\{a, b\}$, which is equivalent to $k \le \min\{a, b\}$.

   As in the previous answer, it is possible to construct A for each value of k by means of $(1,0) \mapsto 1$ and $(0,1) \mapsto p^{b-k}$.

3. (Sketch of solution). There exists a functor $\mathrm{Ext} : \mathbf{Ab} \times \mathbf{Ab} \to \mathbf{Ab}$, the *Ext-functor*. Given two abelian groups, the elements of $\mathrm{Ext}(A, B)$ are in bijection with isomorphism classes of extensions of B by A, that is, the elements are in bijection with exact sequences
$$0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0$$
   where two such exact sequences are equivalent if there is a commutative diagram
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\cong"] & B \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r] & X' \arrow[r] & B \arrow[r] & 0. \end{tikzcd}$$
   The general theory of Ext can be found in [Wei94, Chapters 2 & 3].

   It should be noted that two extensions $0 \to A \to X \to B \to 0$ and $0 \to A \to X' \to B \to 0$ might satisfy the condition that $X \cong X'$ without being equivalent. For instance, if $B =$

$\mathbb{Z}/(3)$ and $X = X' = \mathbb{Z}$, there are two different maps $X \to B$, given by $1 \mapsto \pm 1$. These two maps yield inequivalent extensions $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/(3) \to 0$. Since we are interested in isomorphism classes of groups A fitting in $0 \to \mathbb{Z}/(p^a) \to A \to \mathbb{Z}/(p^b) \to 0$, this will be a concern for us.

Our first calculation is that $\mathrm{Ext}(\mathbb{Z}, B) = 0$ for any abelian group A. This is because the group $\mathbb{Z}$ is free, and any short exact sequence

$$0 \to B \to X \to \mathbb{Z} \to 0$$

splits: $X \cong B \oplus \mathbb{Z}$.

The functor $\mathrm{Ext}(A, \cdot)$ is a derived functor of $\mathrm{Hom}(A, \cdot)$ and $\mathrm{Ext}(\cdot, B)$ is a derived functor of $\mathrm{Hom}(\cdot, B)$. This means, in our situation, that there is an exact sequence for calculating $\mathrm{Ext}(\mathbb{Z}/(p^b), \mathbb{Z}/(p^a))$. Start with the usual presentation $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/(p^b)$, then use the derived functor property to obtain an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(\mathbb{Z}/(p^b), \mathbb{Z}/(p^a)) \longrightarrow \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/(p^a)) \xrightarrow{\times p^b} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/(p^a)) \longrightarrow \\ \longrightarrow \mathrm{Ext}(\mathbb{Z}/(p^b), \mathbb{Z}/(p^a)) \longrightarrow \mathrm{Ext}(\mathbb{Z}, \mathbb{Z}/(p^a)) = 0 \end{aligned}$$

We deduce from this exact sequence that

$$\mathrm{Ext}(\mathbb{Z}/(p^b), \mathbb{Z}/(p^a)) \cong \left(\frac{\mathbb{Z}}{(p^a)} \Big/ \mathrm{im}(\times p^b) \right)$$

so

$$\mathrm{Ext}(\mathbb{Z}/(p^b), \mathbb{Z}/(p^a)) \cong \mathbb{Z}/(p^{\min\{a,b\}}).$$

This is great as far as it goes, but it classifies isomorphism classes of extensions, rather than simply of the group A. To classify isomorphism classes of the group A, we observe that we can take any diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}/(p^a) \arrow[r] \arrow[d, dashed, "\cong"] & A \arrow[r] \arrow[d, "\cong"] & \mathbb{Z}/(p^b) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{Z}/(p^a) \arrow[r] & A \arrow[r] & \mathbb{Z}/(p^b) \arrow[r] & 0 \end{tikzcd}$$

without the dashed isomorphism and produce the dashed arrow. This implies that any two extensions with isomorphic groups A are related by an automorphism of $\mathbb{Z}/(p^a)$. Therefore isomorphism classes of groups A are classified by equivalence classes of elements of $\mathbb{Z}/(p^{\min\{a,b\}})$ up to an induced action of $\mathrm{Aut}(\mathbb{Z}/(p^a))$. That is, they are in bijection with $\{1, p, p^2, \ldots, p^{\min\{a,b\}}\}$, a set of cardinality $\min\{a, b\}$. We have already constructed a set of isomorphism classes of such A, so we have found all isomorphism classes.

In principle, it is possible to use the calculation of $\mathrm{Ext}(\mathbb{Z}/(p^a), \mathbb{Z}/(p^b))$ to construct the different isomorphism classes of extension, but this is no more direct than simply writing down the extensions in this case.

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