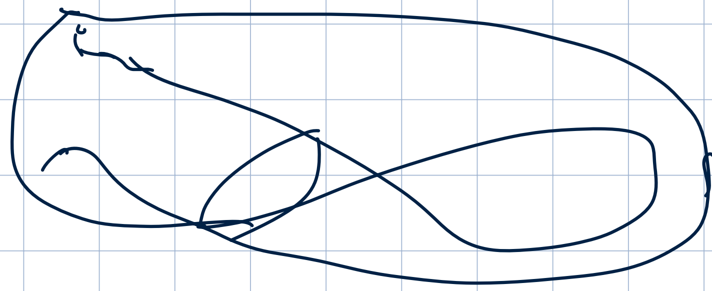
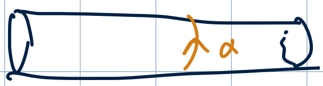
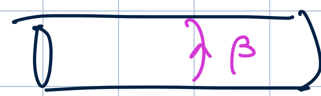


K



U



V

U ∩ V

$j_{U*}: \gamma \mapsto \alpha \qquad \varepsilon \mapsto \alpha$

$j_{V*}: \gamma \mapsto \beta \qquad \varepsilon \mapsto -\beta$ — orientation reversed

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = j_*$$

$$0 \to H_2(K) \xrightarrow{\partial} H_1(U \cap V) \xrightarrow{j_*} H_1(U) \oplus H_1(V) \xrightarrow{i_*} H_1(K) \xrightarrow{\partial}$$

$$\to H_0(U \cap V) \xrightarrow{j_*} H_0(U) \oplus H_0(V) \to H_0(K)$$

$H_0(U\cap V) = \mathbb{Z}[p_{U\cap V}] \oplus \mathbb{Z}q$, $\quad H_0(U) = \mathbb{Z}[p_U]$, $\quad H_0(V) = \mathbb{Z}[p_V]$

0

Let's simplify

$$0 \to H_1(U \cap V) \xrightarrow{j_*} H_1(U) \oplus H_1(V) \xrightarrow{i_*} H_1(K) \to$$

$$\to H_0(U \cap V) \to \dots$$

We can factor i_* as

$$\begin{tikzcd}
\mathbb{Z}\alpha \oplus \mathbb{Z}\beta \arrow[rr, hook, "i_*"] \arrow[dr, two heads] & & H_1(K) \\
& \mathrm{Im}(i_*) \arrow[ur, hook] &
\end{tikzcd}$$

$$\mathrm{Im}(i_*) \cong (\mathbb{Z}\alpha \oplus \mathbb{Z}\beta)/(\ker i_*) \cong (\mathbb{Z}\alpha \oplus \mathbb{Z}\beta)/\mathrm{Im}\, j_*$$

j_* is given by $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$j_*(\gamma) = \alpha - \beta \quad ; \quad j_*(\varepsilon) = \alpha + \beta$

$$\mathrm{Im}\, j_* = \mathbb{Z}(\alpha - \beta) \oplus \mathbb{Z}(\alpha + \beta)$$
$$= \mathbb{Z}(2\alpha) \oplus \mathbb{Z}(\alpha + \beta)$$

Whereas $\mathbb{Z}\alpha \oplus \mathbb{Z}\beta = \mathbb{Z}\alpha \oplus \mathbb{Z}(\alpha + \beta)$

so $(\mathbb{Z}\alpha \oplus \mathbb{Z}\beta)/\mathrm{Im}\, j_* = \mathbb{Z}/(2)\ \bar{\alpha}$

image of α in quotient.

So we get

$$0 \to \underset{\operatorname{Im} i_*}{\underset{\parallel}{\mathbb{Z}/(2)\,\bar{\alpha}}} \longrightarrow H_1(K) \xrightarrow{\partial} H_0(U\cap V) \xrightarrow{j_*} H_0(U)\oplus H_0(V)$$

We also apply these factorization ideas to ∂

$$H_1(K) \xrightarrow{\text{surj.}} \underset{\ker j_*}{\underset{\cong}{\operatorname{Im}\partial}} \hookrightarrow H_0(U\cap V)$$

To calculate $\ker j_*$ we use the geometry

$$\underset{\mathbb{Z}[p_{U\cap V}]\,\oplus\,\mathbb{Z}[q]}{\underset{\parallel}{H_0(U\cap V)}} \xrightarrow{\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = j_*} \underset{\mathbb{Z}[p_U]}{\underset{\parallel}{H_0(U)}} \oplus \underset{\mathbb{Z}[p_V]}{\underset{\parallel}{H_0(V)}}$$

kernel is $\mathbb{Z}([p_{U\cap V}] - [q])$

We have a short exact sequence for $H_1(K)$

$$0 \to \mathbb{Z}/(2)\,\bar{\alpha} \xrightarrow{i_*} H_1(K) \xrightarrow{\partial} \mathbb{Z}([p_{U\cap V}] - [q]) \to 0$$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/(2) \arrow[r, "i_*"] & H_1(K) \arrow[r, "\partial"] & \mathbb{Z} \arrow[r] \arrow[l, bend left, dashed, "s"] & 0
\end{tikzcd}$$

Determining $H_1(K)$ is an <u>extension problem</u>

Any short exact sequence of abelian groups

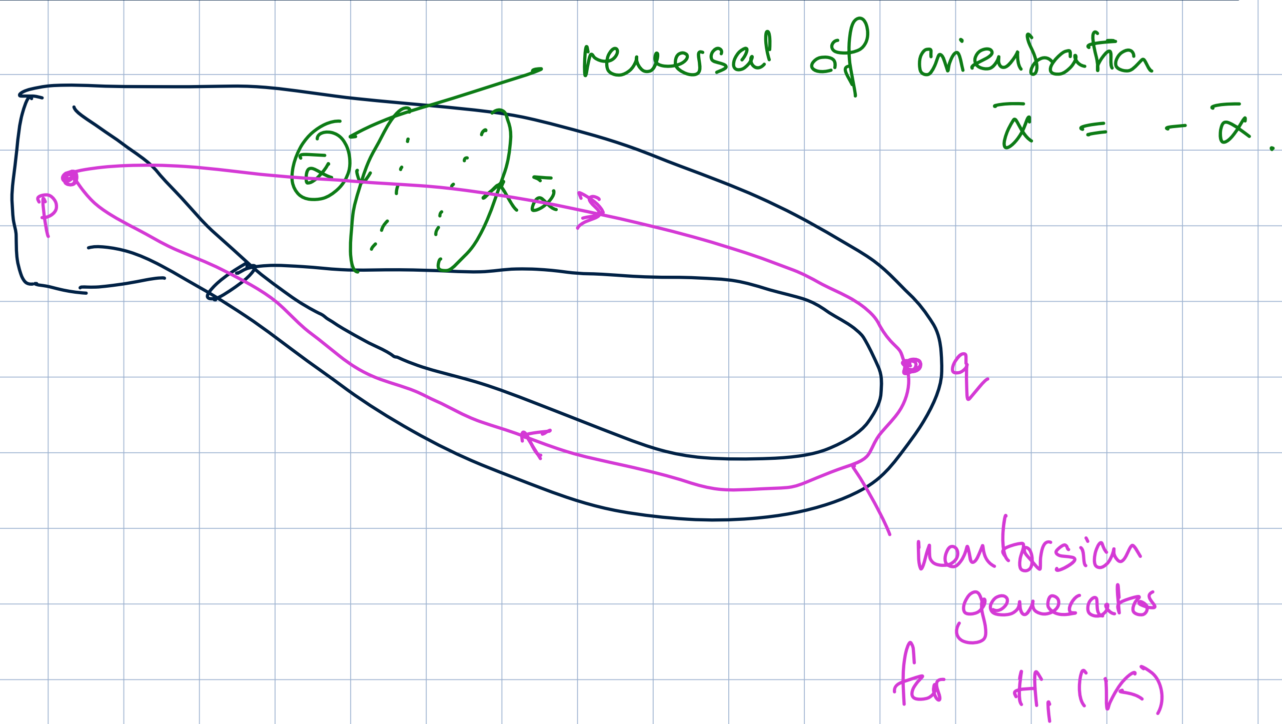
$$0 \to A \to B \to F \to 0 \qquad (F \text{ free},\ s: F \to B)$$

free

is split: $B \cong F \oplus A$.

We see $H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}/(2)$.

$H_0(K) = \mathbb{Z}[p]$



Eilenberg - Steenrod Axioms

A homology theory is a sequence of functors

$$E_n : \underline{\mathrm{Top}}_2 \longrightarrow \underline{\mathrm{Ab}} \qquad n \in \mathbb{Z}$$

pairs of spaces

(the notation $E_n(X) = E_n(X, \emptyset)$)

and natural transformations

$$\partial : E_n(X, A) \longrightarrow E_{n-1}(A, \emptyset) = E_{n-1}(A)$$

satisfying

☆ Homotopy invariance : if $f_1, f_2 : (X, A) \to (Y, B)$ such that

$$f_1 \simeq f_2 : X \to Y$$
$$f_1 \simeq f_2 : A \to B$$

then $(f_1)_* = (f_2)_* : E_*(X, A) \to E_*(Y, B)$.

☆ Excision $E_*(X - U, A - U) \xrightarrow{\cong} E_*(X, A)$

for excisive triples (X, A, U).

★ : L/E/S

$$\cdots \to E_n(A) \to E_n(X) \to E_n(X,A) \xrightarrow{\partial} E_{n-1}(A) \to \cdots$$

★ (Milnor) $X_i \hookrightarrow \coprod_{i \in I} X_i$ induces

$$\bigoplus E_n(X_i) \xrightarrow{\cong} E_n\Big(\coprod_{i\in I} X_i\Big)$$

★ $E_n(pt) = 0$ unless $n = 0$.

(dimension)

A theory satisfying all these is an <u>ordinary homology theory</u>.

If it satisfies the first four axioms but not the dimension axiom, then it is an <u>extraordinary homology theory</u>

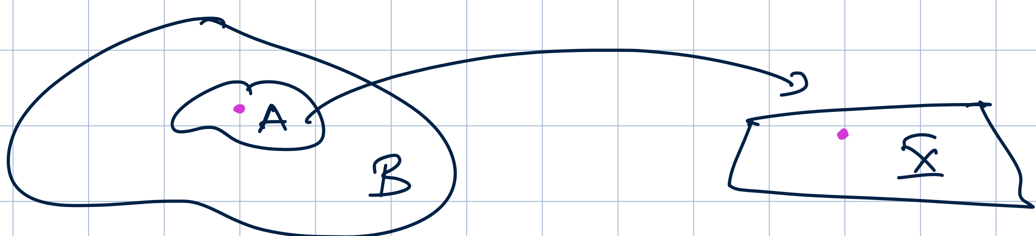
CW Complexes

Pushouts of spaces:

If $A \subseteq B$ are spaces, and $\alpha : A \to X$

we define the pushout

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d] & X \arrow[d] \\ B \arrow[r] & X \cup_{\alpha} B = (X \amalg B)/_{y \sim \alpha(y)\ \forall y \in A} \end{tikzcd}$$



Pushouts have a universal property.

If there is a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, hook, "inc"'] & X \arrow[d] \arrow[ddr, bend left, "g"] & \\
B \arrow[r] \arrow[drr, bend right, "f"'] & X \cup_\alpha B \arrow[dr, dashed, "\exists! \, h"'] & \\
 & & Z
\end{tikzcd},$$

$(f \circ inc = g \circ \alpha)$

then there exists a unique c'ts

$h : X \cup_\alpha B \longrightarrow Z$

making everything commute.

The universal property specifies $X \cup_\alpha B$ up to unique homeomorphism.

$$
\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d] & X \arrow[d] \arrow[ddr, bend left] \arrow[dddrr, bend left] & & \\
B \arrow[r] \arrow[drr, bend right] \arrow[ddrrr, bend right] & X \cup_\alpha B \arrow[dr, "\phi", "\exists!"'] \arrow[ddrr, bend right, "\mathrm{id}"'] & & \\
 & & U \arrow[dr, "\psi", "\exists!"'] & \\
 & & & X \cup_\alpha B
\end{tikzcd}
$$

$\psi \circ \phi = \mathrm{id}_{X \cup_\alpha B}$ — by uniqueness of maps making diagram.

similarly

$\phi \circ \psi = \mathrm{id}_U$

Examples

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & * \arrow[d] \\
B \arrow[r] & * \cup_A B =: B/A
\end{tikzcd}
$$

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, hook] & X \arrow[d, hook] \\ B \arrow[r] & X \amalg B \end{tikzcd}$$

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, hook] & * \arrow[d, hook] \\ B \arrow[r] & * \amalg B \end{tikzcd} = B/\emptyset$$

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, hook] & * \arrow[d, hook] \\ \emptyset \arrow[r] & * \amalg \emptyset \end{tikzcd} = * = \emptyset/\emptyset.$$

Recall $D^n = \{ \vec{x} \in \mathbb{R}^n \mid \|\vec{x}\| \leq 1 \}$

$S^{n-1} = \{ \vec{x} \in \mathbb{R}^n \mid \|\vec{x}\| = 1 \}$

$$S^{n-1} \hookrightarrow D^n$$

$D^n \setminus S^{n-1} = B^n = \{ \vec{x} \in \mathbb{R}^n \mid \|\vec{x}\| < 1 \}$

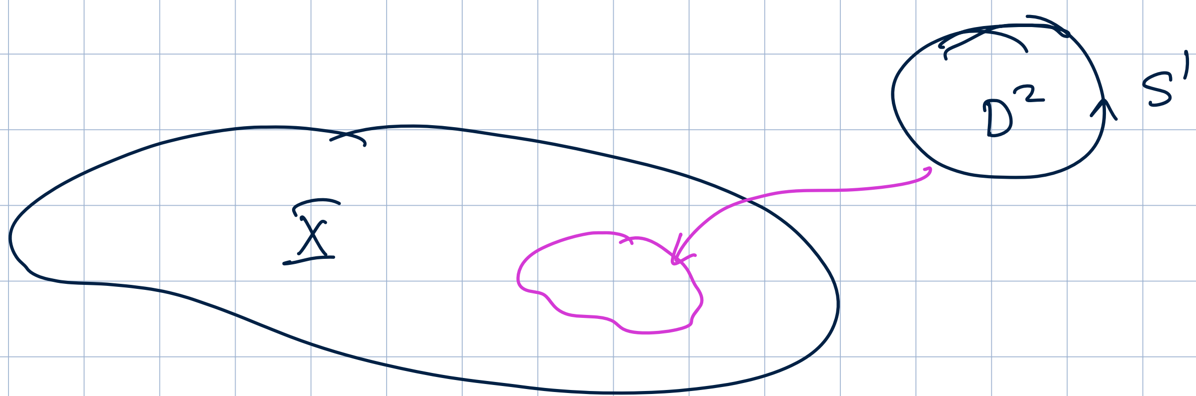
$\approx \mathbb{R}^n.$

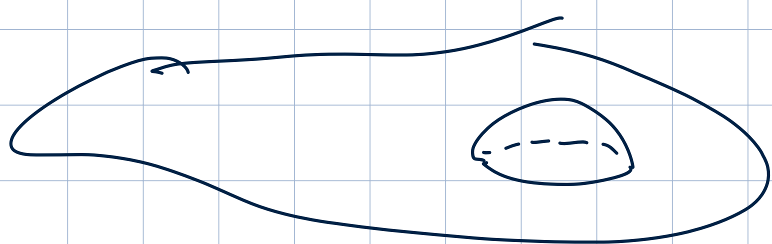
Attaching an n-cell to X:

$$\begin{tikzcd} S^{n-1} \arrow[r, "\alpha"] \arrow[d] & X \arrow[d] \\ D^n \arrow[r, "j"'] & X \cup_\alpha D^n \end{tikzcd}$$

α: attaching map

j: characteristic map.





$X \cup_\alpha D^2$

The image $j(B^n) = j(D^n \setminus S^{n-1})$ is $\approx B^n \approx \mathbb{R}^n$. $j(B^n)$ is the <u>cell</u> that we attached.

Attaching Multiple n-Cells

$$\begin{tikzcd} \coprod_{i\in I_n} S_i^{n-1} \arrow[r, "\alpha"] \arrow[d, hook] & \underline{X} \arrow[d, hook] \\ \coprod_{i\in I_n} D_i^n \arrow[r] & \underline{X} \cup_\alpha \coprod_{i\in I} D_i^n . \end{tikzcd}$$

Def 14.5 : A CW-complex $\underline{X}$ is a space equipped with a sequence of subspaces (the skeleta)

$$\emptyset = Sk_{-1}\underline{X} \subseteq Sk_0\underline{X} \subseteq Sk_1\underline{X} \subseteq \cdots \subseteq \underline{X}$$

s.t. 1. $\underline{X}$ is the colimit (union) of the skeleta.

2. for all n there is a pushout

$$\begin{tikzcd} \coprod_{i\in I_n} S_i^{n-1} \arrow[r, "\alpha_n"] \arrow[d, hook] & Sk_{n-1}\underline{X} \arrow[d] \\ \coprod_{i\in I_n} D_i^n \arrow[r, "\beta_n"] & Sk_n\underline{X} \approx Sk_{n-1}\underline{X} \cup_{\alpha_n} \coprod D^n \end{tikzcd}$$

—

When we say "colimit" in 1, we mean

$$\bar{X} = \bigcup_n Sk_n \bar{X}$$

and a set $U \subseteq \bar{X}$ is open / closed if $U \cap Sk_n \bar{X}$ is open / closed $\forall n$.