Math 427, 527
Homework 2
2026-02-02

1. This problem is not to be handed in.

Let X be a topological space. Recall that $\pi_0(X)$ denotes the set of path components of X: the set of equivalence classes of points $x \in X$ where $x \sim y$ if there exists a path $\gamma : I \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

Write $\mathbb{Z}\pi_0(X)$ for the free abelian group with $\pi_0(X)$ as a basis. By constructing homomorphisms each way and checking they are inverses, prove there is an isomorphism between $\mathbb{Z}\pi_0(X)$ and $\mathrm{H}_0(X;\mathbb{Z})$.

We construct homomorphisms both ways, and check they are inverse to each other.

The group $S_0(X)$ is the free abelian group with basis consisting of maps $\Delta^0 \to X$. This basis is in obvious bijection with the set of points of X itself, all such maps being constant. Therefore we may identify $S_0(X)$ and $\mathbb{Z}X$, the free abelian group with X itself as a basis. There is a function $X \to \pi_0(X)$ given by sending x to its equivalence class. Therefore there is a homomorphism $\bar{f} : S_0(X) \to \mathbb{Z}\pi_0(X)$, extending the function between the bases. Note that $\bar{f}(x-y) = 0$ if x, y are points in the same path component of X.

If $\sigma \in \mathrm{Sin}_1(X)$, then σ is a path in X, and $d\sigma = y - x$ is the formal difference of the endpoints of this path. Since x, y lie in the same path component, $d\sigma$ is in the kernel of $\bar{f}$. The map $S_0(X) \to \mathbb{Z}\pi_0(X)$ therefore factors through the quotient of $S_0(X) = Z_0(X)$ by $\operatorname{im} d = B_0(X)$, giving us a homomorphism

$$f : \mathrm{H}_0(X;\mathbb{Z}) \to \mathbb{Z}\pi_0(X).$$

For any $x \in X$, the element $f(x + \operatorname{im} d)$ is the path component of x.

We can construct a homomorphism the other way as well. Pick a family of representative points $\{y_i\}$, one from each path component. Define a homomorphism $g : \mathbb{Z}\pi_0(X) \to \mathrm{H}_0(X;\mathbb{Z})$ by sending each path component to its corresponding $y_i + \operatorname{im} d$.

Applied to $Y \in \pi_0(X)$, the composite $f(g(Y))$ picks out the path component of a representative element of Y, i.e., Y itself. Therefore $f \circ g$ is the identity.

Applied to $x + \operatorname{im} d \in \mathrm{H}_0(X;\mathbb{Z})$, the composite $g(f(x + \operatorname{im} d))$ gives us $y_i + \operatorname{im} d$ where y_i is the representative element of the path component of $x \in X$. There is some path $\sigma : \Delta^1 \to X$ that starts at x and ends at y, so that $y - x \in \operatorname{im} d$, so that $x + \operatorname{im} d = y + \operatorname{im} d$. This says that $g \circ f$ is also the identity, which is what we needed. □

2. Suppose we are given a commutative diagram of abelian groups:

$$\begin{array}{ccccccc} A_4 & \xrightarrow{d} & A_3 & \xrightarrow{d} & A_2 & \xrightarrow{d} & A_1 \\ \downarrow f_4 & & \downarrow f_3 & & \downarrow f_2 & & \downarrow f_1 \\ B_4 & \xrightarrow{d} & B_3 & \xrightarrow{d} & B_2 & \xrightarrow{d} & B_1. \end{array}$$

Suppose further that the rows are exact sequences, the homomorphisms f_3 and f_1 are injective, and f_4 is surjective. Prove that f_2 is injective.

Suppose $a \in A_2$ is an element in the kernel of f_2. We will show $a = 0$.

Apply d to a and use $f_1 \circ d = d \circ f_2$ to deduce $d(a)$ is in the kernel of the injective map f_1. Therefore, $d(a) = 0$. As a consequence, there exists $a' \in A_3$ for which $d(a') = a$. Considering $d \circ f_3(a') = f_2 \circ d(a') = f_2(a) = 0$, we deduce that d annihilates $f_3(a')$. Therefore we can find $b \in B_4$ for which $d(b) = f_3(a')$, and using surjectivity of f_4 we find some $a'' \in A_4$ for which $d \circ f_4(a'') = f_3(a')$. Using commutativity of the leftmost square, we see that $f_3 \circ d(a'') = f_3(a')$, and using injectivity of f_3, we see that $d(a'') = a'$. This implies that $d(a') = d^2(a'') = 0$, which is what we wanted. □

3. This is part of Exercise 8.8 of Miller's notes. Suppose

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

is a short exact sequence of abelian groups. As we often do, we will identify A with its image under i. Show that the following are equivalent.

(a) There exists a homomorphism $s : C \to B$ such that $p \circ s = \mathrm{id}_C$.

(b) There exists a homomorphism $t : B \to A$ such that $t \circ i = \mathrm{id}_A$.

Prove that if s exists as above, then the homomorphism $f : A \oplus C \to B$ given by $f(a, c) = a + s(c)$ is an isomorphism.

Since $i(a) = a$, we generally drop i from the algebra.

Suppose s is as in 1. We define $t : B \to A$ by the formula $t(b) = b - s(p(b))$. We verify directly that applied to $a \in A$ this gives $i(a) - s(p(a)) = a$, as required by 2.

Suppose t is as in 2. We attempt to define a homomorphism $s : C \to B$ in the following way. For any $c \in C$, there exists at least one $b \in p^{-1}(c)$. Define $s(c) = b - t(b)$. At first glance, this appears to depend on the choice of $b \in p^{-1}(c)$, but if b' is some other choice, then $b - b' \in A$ so that

$$b - b' = t(b - b')$$

and rearranging gives

$$b - t(b) = b' - t(b').$$

Therefore s is well defined. Consider $p(s(c))$ for $c \in C$. This is $p(b) - p(t(b))$ for any $b \in p^{-1}(c)$. Since $p(t(b)) = 0$, we see that $p(s(c)) = p(b) = c$. Therefore s is of the form demanded by 1.

To prove that f is an isomorphism, we construct an inverse. First, define $t(b) = b - s(p(b))$. Then define $g(b) = (t(b), p(b)) = (b - s(p(b)), p(b))$. We calculate $f \circ g(b) = b - s(p(b)) + s(p(b)) = b$. The other direction we get

$$g \circ f(a,c) = g(a + s(c)) = (a + s(c) - s(p(a + s(c))), p(a + s(c))) = (a,c),$$

using $p(a) = 0$ and $p(s(c)) = c$ repeatedly. □

4. Recall that $[X,Y]$ denotes the set of homotopy classes of continuous functions $X \to Y$.

Let (X, x_0) be a space X with a chosen basepoint x_0 and endow S^1 with the basepoint $s_0 = (1,0)$. Recall that $\pi_1(X, x_0)$ is the set of equivalence classes of basepoint-preserving maps $(S^1, s_0) \to (X, x_0)$ where the homotopies also satisfy $h(s_0, t) = x_0$ for all $t \in [0,1]$. There is a natural transformation $\upsilon_X : \pi_1(X, x_0) \to [S^1, X]$ that forgets the basepoint-preserving nature of maps $S^1 \to X$ and homotopies between them.

Using homotopy invariance of $\pi_1(X, x_0)$ (as presented in e.g., [1, Prop. 1.5, Lem. 1.19]), prove that $\upsilon_X(\gamma) = \upsilon_X(\delta)$ implies that γ, δ are conjugate elements in $\pi_1(X, x_0)$.

In an abuse of notation, write $\gamma, \delta : S^1 \to X$ for functions that represent the elements of $\pi_1(X, x_0)$ with the same name. Write $1 \in \pi_1(S^1, s_0)$ for the class of the identity, so that $\pi_1(S^1, s_0)$ is identified with $\mathbb{Z}$.

The functions γ, δ induce homomorphisms $\gamma_*, \delta_* : \pi_1(S^1, s_0) \to \pi_1(X, x_0)$ for which $\gamma_*(1) = \gamma$ and $\delta_*(1) = \delta$.

Suppose $\upsilon_X(\gamma) = \upsilon_X(\delta)$. That is, assume there is a homotopy between the functions γ, δ, ignoring basepoints. Then [1, Lem. 1.19] implies the existence of a commutative diagram of groups:

$$\begin{array}{ccc} & & \pi_1(X, x_0) \\ & \overset{\gamma_*}{\nearrow} & \\ \mathbb{Z} & & \Big\downarrow \beta_h \\ & \underset{\delta_*}{\searrow} & \\ & & \pi_1(X, x_0), \end{array}$$

where β_h denotes conjugation by h, where h is in turn the path traced by the basepoint s_0 under the homotopy from γ to δ. Commutativity of the diagram implies that $h\gamma h^{-1} = \delta$, as required. □

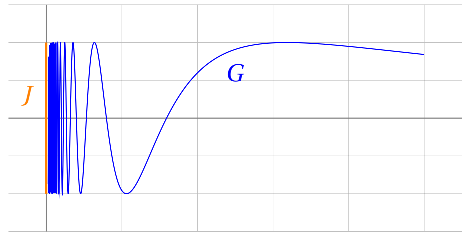

Figure 1: The topologist's sine curve, S.

5. The "topologist's sine curve" S is a closed subset of $\mathbb{R}^2$ defined as follows. Let G denote the graph of the function $f(x) = \sin(1/x)$ on the domain $(0,1]$, indicated in blue in Figure 1. Let J denote the interval $\{0\} \times [-1,1]$, denoted in orange in Figure 1. The space S is defined to be the union $G \cup J$. The space S is well known to be connected, but to have two path components, G and J. Note that J is a closed, contractible subspace of S.

There is a continuous function $f : S \to [0,1]$ defined by $f(x,y) = x$. Since $f(j) = 0$ for all $j \in J$, we know that f factors through the quotient map $p : S \to S/J$, i.e., we can write $f = \tilde{f} \circ p$ where $\tilde{f} : S/J \to [0,1]$ is a continuous function.

Prove that $\tilde{f}$ is a homeomorphism. Deduce that $p : S \to S/J$ is not a homotopy equivalence.

The space S is compact, being closed and bounded in $\mathbb{R}^2$. Therefore the quotient space S/J is compact.

The map $\tilde{f} : S/J \to [0,1]$ is a bijection, since for every $x \in (0,1]$ there is one and only one point $(x, \sin(1/x)) \in S$, while the unique point in S/J with preimage J maps to $\{0\}$ under $\tilde{f}$.

Therefore $\tilde{f}$ is a continuous bijection with compact source and Hausdorff target. It is a homeomorphism.

We know that $\mathrm{H}_0(S;\mathbb{Z}) = \mathbb{Z}\pi_0(S) \cong \mathbb{Z}^2$, whereas $\mathrm{H}_0(S/J;\mathbb{Z}) = \mathbb{Z}\pi_0(S/J) \cong \mathbb{Z}$. Since they have non-isomorphic homology, S and S/J are not homotopy equivalent. □

References

[1] Hatcher (2002) *Algebraic Topology*, Cambridge University Press, Cambridge.