Math 427, 527
Homework 3
2026-02-17

1. This problem is not to be handed in. Prove the snake lemma to your own satisfaction.

No solution given. □

2. Let $f : X \to Y$ be a map of spaces. Construct a space M_f as the quotient of the disjoint union of $X \times I$ and Y by the relation $(x,0) \sim f(x)$. There are continuous functions $i : X \to M_f$ given by $i(x) = (x,1)$ and $p : M_f \to Y$ given by $p(x,t) = f(x)$ for all t (including the case $t = 0$) and $p(y) = y$. You do not have to prove these are continuous, or that $p \circ i = f$.

(a) Prove that $p : M_f \to Y$ is a homotopy equivalence.

(b) Let C_f denote the quotient of M_f given by collapsing the subspace $X \times \{1\}$ to a point. Construct a long exact sequence in homology:

$$\cdots \to \mathrm{H}_n(X;\mathbb{Z}) \xrightarrow{f_*} \mathrm{H}_n(Y;\mathbb{Z}) \to \bar{\mathrm{H}}_n(C_f;\mathbb{Z}) \xrightarrow{\partial} \mathrm{H}_{n-1}(X;\mathbb{Z}) \to \cdots.$$

(c) If $Y = \mathrm{pt}$, so that f is the unique map $f : X \to \mathrm{pt}$, then C_f is also known as the *(unreduced) suspension* of X, and may be denoted SX. Give a formula relating $\bar{\mathrm{H}}_*(X)$ and $\bar{\mathrm{H}}_*(SX)$.

(d) **Not to be handed in:** Check that the constructions of M_f, C_f extend to give functors to **Top** from the category whose objects are maps $f : X \to Y$ and where a morphism $v : f \to f'$ is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "v_X"] & Y \arrow[d, "v_Y"] \\ X' \arrow[r, "f'"] & Y'. \end{tikzcd}$$

Deduce that suspension S gives a functor **Top** $\to$ **Top**.

(a) Observe that Y is embedded as a closed subset of M_f by the quotient map $(X \times I) \coprod Y \to M_f$. We will show that p is the retracting map, i.e., $h(-,1)$ of a deformation retraction. This implies that p is a homotopy equivalence, since it is homotopy inverse to the inclusion of a deformation retract.

Define $h : M_f \times I \to M_f$ by the formulas

$$h(x,t,s) = (x, t(1-s)), \forall (x,t) \in X \times I, \quad h(y,s) = y, \forall y \in Y$$

valid for all $s \in I$. Superficially, this is defined only on $(X \times I) \coprod Y$ but if $t = 0$, then $h(x,0,s) = (x,0) = f(x) = h(f(x),s)$ for all $x \in X$ and all $s \in I$, so the map h is well defined.

Then $h(z,0) = z$ for all $z \in M_f$, whereas $h(z,1) = p(z)$ for all $z \in M_f$. This is what we needed to show.

(b) The closed subspace $i(X) = X \times \{1\} \subset M_f$ is a deformation retract of the open subspace $X \times (1/2, 1]$. Therefore we can replace the homology of the pair $(M_f, i(X))$ by the homology of the pair $(M_f / i(X), *) = (C_f, *)$ and get a long exact sequence in homology

$$\cdots \to \mathrm{H}_n(X) \xrightarrow{i_*} \mathrm{H}_n(M_f) \to \bar{\mathrm{H}}_n(C_f) \to \mathrm{H}_{n-1}(X) \to \ldots.$$

Replacing $\mathrm{H}_n(M_f)$ by $\mathrm{H}_n(Y)$ using the isomorphism p_* and exploiting $p_* \circ i_* = f_*$ finishes the solution.

(c) The long exact sequence in reduced homology is an exact sequence

$$\cdots \to \bar{\mathrm{H}}_n(\mathrm{pt}) \to \bar{\mathrm{H}}_n(SX) \xrightarrow{\partial} \bar{\mathrm{H}}_{n-1}(X) \to \bar{\mathrm{H}}_{n-1}(\mathrm{pt}) \to \cdots.$$

Since $\bar{\mathrm{H}}_*(\mathrm{pt}) = 0$, we deduce that $\bar{\mathrm{H}}_n(SX) \cong \bar{\mathrm{H}}_{n-1}(X)$ for all n.

(d) No solution given.

□

3. Suppose X is a topological space and $g \in X$ is a point with the property that $\overline{\{g\}} = X$ (alternatively, g is an element of all nonempty open sets of X). Such a g is called a *generic point* of X.

(a) Define $f : X \to X$ by $f(x) = g$ for all x. Prove that f is homotopic to the identity function.

(b) Let $n \geq 1$ be an integer. Let $s \in S^n$ be a point. Prove that $\bar{\mathrm{H}}_*(S^n/(S^n - \{s\}))$ and $\mathrm{H}_*(S^n, S^n - \{s\})$ are not isomorphic.

(a) Define $h : X \times I \to X$ by the following formula: $h(x,0) = x$ and $h(x,t) = g$ if $t > 0$. If this is continuous, then it is a homotopy between id_X and f.

We verify that h is continuous. Let $U \subseteq X$ be an open set. We will show that $h^{-1}(U)$ is open. If $U = \emptyset$, this is trivial. If U is not empty, then $h^{-1}(U) = (X \times (0,1]) \cup (U \times [0,1])$ is the union of two open sets in the product. Therefore $h^{-1}(U)$ is indeed open and h is continuous.

(b) First observe that $S^n/(S^n - \{s\})$ consists of two points: $\{s', g\}$, where s' is the image of s and g is the image of $S^n - \{s\}$ under the quotient map. Because $S^n - \{s\}$ is open in S^n, the set $\{g\}$ is open in $S^n/(S^n - \{s\})$, whereas $\{s\}$ is not open, so $\{s'\}$ is also not open. In particular, g is a generic point for $S^n/(S^n - \{s\})$ and so the previous part of this question tells us $S^n/(S^n - \{s\})$ is contractible.

Then we have $\bar{\mathrm{H}}_*(S^n/(S^n - \{s\})) = 0$, whereas $\mathrm{H}_n(S^n, S^n - \{s\}) \cong \bar{\mathrm{H}}_n(S^n) \cong \mathbb{Z}$ using the long exact sequence and the contractibility of $S^n - \{s\}$.

□

4. When $X = Y = S^n$, the unreduced suspension functor, $S(-)$ from question 2, may be replaced by an explicit geometric construction, as we outline here.

Recall that $S^n \subset \mathbb{R}^{n+1}$, and let $\|-\|$ denote the usual norm on $\mathbb{R}^{n+1}$. Starting with a continuous function $f : S^{n-1} \to S^{n-1}$, define a continuous function

$$S'f : S^n \to S^n$$

by the formula

$$S'f(x_0, \dots, x_{n-1}, x_n) = \left(\sqrt{1 - x_n^2}\, f\left(\frac{1}{\sqrt{1 - x_n^2}} (x_0, \dots, x_{n-1}) \right), x_n \right).$$

(a) Prove that $\deg(S'f) = \deg(f)$. It may be helpful either to assert that $S'f$ is homeomorphic to Sf (no proof required) or to use the Mayer–Vietoris sequence.

(b) Suppose $f : S^n \to S^n$ is a reflection of the sphere across a hyperplane through $\mathbf{0} \in \mathbb{R}^{n+1}$. Prove that $\deg(f) = -1$.

(a) We adopt the Mayer–Vietoris approach suggested. Let $U = S^n - \{(0, \dots, 0, 1)\}$ and $V = S^n - \{(0, \dots, 0, -1)\}$. Since $S'f$ preserves the x_n-coordinate, we get restrictions $S'f|_U : U \to U$ and $S'f|_V : V \to V$.

Take two copies of the Mayer–Vietoris sequence in reduced homology for the cover U, V of S^n, and use $S'f$ to set up maps between the groups concerned. The resulting diagram is a commutative ladder of long exact sequences

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \bar{\mathrm{H}}_q(U \cap V) & \xrightarrow{\begin{bmatrix} j_{U*} \\ -j_{V*} \end{bmatrix}} & \bar{\mathrm{H}}_q(U) \oplus \bar{\mathrm{H}}_q(V) & \xrightarrow{[i_U, i_V]} & \bar{\mathrm{H}}_q(S^n) & \xrightarrow{\partial} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \bar{\mathrm{H}}_q(U \cap V) & \xrightarrow[\begin{bmatrix} j_{U*} \\ -j_{V*} \end{bmatrix}]{} & \bar{\mathrm{H}}_q(U) \oplus \bar{\mathrm{H}}_q(V) & \xrightarrow{[i_U, i_V]} & \bar{\mathrm{H}}_q(S^n) & \xrightarrow{\partial} & \cdots
\end{array} \tag{1}$$

where all vertical maps are induced by $S'f$.

Both U and V are homeomorphic to $\mathbb{R}^n$, and therefore contractible, so that $\bar{\mathrm{H}}_*(U) = \bar{\mathrm{H}}_*(V) = 0$.

One explicit homeomorphism $\phi : U \to \mathbb{R}^n$ is given by (the inverse of) stereographic projection:

$$\phi(x_0, x_1, \dots, x_n) = \left(\frac{x_0}{1 - x_n}, \frac{x_1}{1 - x_n}, \dots, \frac{x_{n-1}}{1 - x_n} \right).$$

The image of $U \cap V$ is $\mathbb{R}^n - \{0\}$, which admits the unit sphere $S^{n-1} \subset \mathbb{R}^n$ as a deformation retract. Since ϕ is a homeomorphism, $\phi^{-1}(S^{n-1})$ is a deformation retract of $U \cap V$. But $\phi^{-1}(S^{n-1})$ is exactly the image of the equatorial embedding $e_n : S^{n-1} \hookrightarrow S^n$ given by setting $x_n = 0$.

If we restrict $S'f$ to the equatorial S^{n-1}, i.e., set $x_n = 0$, then we recover f. In summary, this square commutes and the horizontal arrows are homotopy equivalences:

$$\begin{tikzcd} S^{n-1} \arrow[r, "e_n", "\simeq"'] \arrow[d, "f"] & U \cap V \arrow[d, "S'f|_{U\cap V}"] \\ S^{n-1} \arrow[r, "e_n", "\simeq"'] & U \cap V. \end{tikzcd}$$

The leftmost pictured arrow in (1) can be replaced by $f_* : \mathrm{H}_q(S^{n-1}) \to \mathrm{H}_q(S^{n-1})$. Therefore, the groups in (1) are 0 everywhere except for one commutative square:

$$\begin{tikzcd} \bar{\mathrm{H}}_n(S^n) \arrow[r, "\partial"] \arrow[d, "S'f_*"] & \bar{\mathrm{H}}_{n-1}(S^{n-1}) \arrow[d, "f_*"] \\ \bar{\mathrm{H}}_n(S^n) \arrow[r, "\partial"] & \bar{\mathrm{H}}_{n-1}(S^{n-1}). \end{tikzcd}$$

in which the groups appearing are all free abelian of rank 1 and ∂ is an isomorphism. Pick a generator $\iota \in \bar{\mathrm{H}}_n(S^n)$, so that $\partial(\iota)$ is also a generator. Commutativity of the square tells us

$$\deg(f)\partial(\iota) = f_*(\partial(\iota)) = \partial(S'f_*(\iota)) = \partial(\deg(S'f)\iota) = \deg(S'f)\partial(\iota),$$

i.e., $\deg(f) = \deg(S'f)$, as required.

(Alternatively, in a less pedantic way: identify $\mathbb{Z} = \bar{\mathrm{H}}_n(S^n)$ using a chosen generator, and use ∂ to identify $\mathbb{Z} = \bar{\mathrm{H}}_{n-1}(S^{n-1})$. Then $\deg(S'f) = S'f_*(1) = f_*(1) = \deg(f)$.)

(b) Let f be given by rotation across $L \subseteq \mathbb{R}^n$. There is some linear rotation $T : \mathbb{R}^n \to \mathbb{R}^n$ (i.e., a linear transformation by an orthogonal matrix of determinant 1) around $\mathbf{0}$ which takes L to the hyperplane $x_0 = 0$. We observe T, being a rotation, preserves S^n. Some algebra shows that

$$\deg(T^{-1}fT) = \deg(T^{-1})\deg(f)\deg(T) = \deg(f)\deg(T^{-1}T) = \deg(f)$$

so that the precise plane L is not important: we may assume it is $x_0 = 0$. Reflection across this plane is $S'f_{n-1}$ where $f_{n-1} : S^{n-1} \to S^{n-1}$ is reflection across the plane $x_0 = 0$. In this way, we reduce the question to $\deg(f_1)$ where $f_1 : S^1 \to S^1$ is the function $(x, y) \mapsto (-x, y)$. This has degree -1 by consideration of $\pi_1(S^1)$.

□

5. The notation $\mathrm{GL}(n;\mathbb{R})$ denotes the group of invertible $n \times n$ matrices with entries in $\mathbb{R}$. Give it a topology as a subspace of $\mathbb{R}^{n\times n}$. You may assume that matrix operations such as multiplication and inversion are continuous.

By a *path* in $\mathrm{GL}(n;\mathbb{R})$ between two matrices B and C, we mean a continuous function $A : [0,1] \to \mathrm{GL}(n;\mathbb{R})$ so that $A(0) = B$ and $A(1) = C$.

(a) Give a path in $\mathrm{GL}(2;\mathbb{R})$ between $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and the identity matrix. Beyond giving the path, no further argument is required.

(b) Let n be a positive integer, let $a \in \mathbb{R}$ and $i, j \in \{1, \dots, n\}$ be indices satisfying $i \neq j$. The *elementary matrix* $E_{ij}(a)$ differs from the identity matrix only in the i, j-position, where it has the value a. Give a path in $\mathrm{GL}(n;\mathbb{R})$ from $E_{ij}(a)$ to the identity matrix. Beyond giving the path, no further argument is required.

(c) Let $A \in \mathrm{GL}(n;\mathbb{R})$ be a matrix. It is well known that A may be brought to reduced row-echelon form by elementary row operations. A variant is this: there exists a list of elementary matrices (of the kind defined above) $E_1, \dots, E_n$ such that

$$E_1 E_2 \cdots E_n A$$

is a diagonal matrix.

Prove that there is a path in $\mathrm{GL}(n;\mathbb{R})$ from A to a diagonal matrix whose diagonal entries are elements of $\{-1, 1\}$.

(d) For the same A, sketch a proof that there is a path from A either to I_n, the $n \times n$ identity matrix, or to J, the matrix that agrees with I_n everywhere except for a -1 in the $1,1$-position.

(e) If $A \in \mathrm{GL}(n;\mathbb{R})$, then A gives us a function $f_A : S^{n-1} \to S^{n-1}$ by the formula

$$f_A(\mathbf{v}) = \frac{1}{\|A\mathbf{v}\|} A\mathbf{v}.$$

You may assume this depends continuously on $\mathbf{v}$ and on the entries of A.

Prove

$$\deg(f_A) = \frac{\det(A)}{|\det(A)|}.$$

(a) There are many possible answers. For instance, the function

$$A(t) = \begin{bmatrix} -\cos(\pi t) & \sin(\pi t) \\ -\sin(\pi t) & -\cos(\pi t) \end{bmatrix}$$

gives $A(0) = -I_2$ and $A(1) = I_2$. For all t, we have $\det(A(t)) = 1$, so this always lies in $\mathrm{GL}(2;\mathbb{R})$.

(b) The easiest answer is to define $A(t) = E_{i,j}((1-t)a)$. The matrices $E_{i,j}(a)$ are upper- or lower-triangular, with 1s on the diagonal, so they all satisfy $\det(E_{i,j}(a)) = 1$.

(c) First we prove there is a path from A to a diagonal matrix. Write down elementary matrices E_i for which

$$E_1 E_2 \dots E_n A = D,$$

where $D = \operatorname{diag} d_1, \dots, d_n$ is an $n \times n$ diagonal matrix whose diagonal entries are $d_1, \dots, d_n$ in that order.

Then define, for each E_i, a path $E_i(t)$ in $\mathrm{GL}(n;\mathbb{R})$ such that $E_i(0) = I_n$, the $n \times n$ identity matrix and $E_i(1) = E_i$. Consider

$$A(t) = E_1(t)E_2(t)\dots E_n(t)A$$

which gives us $A(0) = A$ and $A(1) = D$. Since matrix multiplication is continuous, $A(t)$ is at least a path in the space of matrices and a determinant calculation shows that $\det(A(t)) = \det(A)$ for all t, so it is actually a path in $\mathrm{GL}(n;\mathbb{R})$.

Next, define a path from $\mathrm{diag}(d_1,\dots,d_n)$ by the formula

$$\mathrm{diag}\left(\frac{d_1}{t|d_1|+1-t}, \frac{d_2}{t|d_2|+1-t}, \dots, \frac{d_n}{t|d_n|+1-t}\right)$$

A calculation shows that the denominators here are never 0 when $t \in [0,1]$. This path ends at $\mathrm{diag}(d_1/|d_1|, d_2/|d_2|, \dots, d_n/|d_n|)$, where each value is either 1 or -1.

Paths can be concatenated using the usual formula from Math 426 or equivalent, so this completes the proof.

(d) Let i, j be two different values in $\{1,\dots,n\}$. Using a variation of the matrix in answer (a), we can find a path $K_{i,j}(t)$ from the identity matrix I_n to a matrix agreeing with I_n except in the i-th and j-th diagonal positions, where it is -1. Multiplying by $K_{i,j}(t)$ gives a path from any given matrix to the matrix where two rows have been multiplied by -1.

Applying this to a diagonal matrix with -1s and 1s on the diagonal, we can find a path to any other such matrix where we have made an even number of sign changes. In particular, every such matrix can be connected by a path either to I or J. Concatenating this with the path from the previous part, we can bring any A to either I or J.

(e) First, we remark that $\det : \mathrm{GL}(n;\mathbb{R}) \to \mathbb{R} - \{0\}$ is a continuous function, so that any path $A(t)$ in $\mathrm{GL}(n;\mathbb{R})$ induces a path $\det(A(t))$ in $\mathbb{R} - \{0\}$. This space is not path connected, and in particular, if $\det(A) < 0$, then there is no path from A to a matrix of positive determinant, e.g., I_n. A similar result applies to if $\det(A) > 0$: there is no path from A to J because $\det(J) < 0$.

The previous parts of the question have constructed a path from A to either I or J, and we have just observed that the sign of $\det(A)$ determines which of I or J we can reach.

A path $A(t)$ in $\mathrm{GL}(n;\mathbb{R})$ from A to $I = I_n$ or to J gives us a homotopy from f_A to f_I, which is the identity map, or to f_J respectively. Therefore $\deg(f_A) = \deg(f_I)$ or $\deg(f_A) = \deg(f_J)$ according to whether $\det(A)$ is positive or negative.

It remains to check that $\deg(f_I) = 1$ and $\deg(f_J) = -1$. The first of these is immediate: f_I is the identity function. The transformation f_J is actually relflection across the plane $x_0 = 0$, which is known to have degree -1 by the answer to a previous question.

Remark 0.1. If the matrix A is an orthogonal matrix, i.e., it satisfies $A^T A = I_n$, then $\mathbf{v} \mapsto A\mathbf{v}$ actually preserves distances in $\mathbb{R}^n$, and so preserves S^{n-1}. This includes all rotations and reflections of the sphere. The formula $f_A(\mathbf{v})$ simplifies to $f_A(\mathbf{v}) = A\mathbf{v}$, because lengths are preserved by A. Since A is orthogonal, $\det(A) = \pm 1$.

This exercise, in particular, shows that for rigid transformations $v \mapsto A\mathbf{v}$ of S^{n-1}, the degree is simply the determinant of the matrix A.

□