Math 427, 527
Homework 4
2026-03-02

1. This question is not to be handed in. Let X be a CW complex.

(a) Let Z be a set of points such that each cell e of X contains exactly one point from Z. Let $f_Z : Z \to \mathbb{R}$ be a function. Prove by induction on the skeleta of X that there exists a continuous function $f : X \to \mathbb{R}$ with the property that $f(z) = f_Z(z)$ for all $z \in Z$. You may assume the Tietze extension theorem.

(b) Deduce that if $K \subseteq X$ is a compact subset, then K has nonempty intersection with only finitely many cells of X.

(c) Suppose $A \subseteq X$ is a closed subset. Prove there exists a continuous function $f : X \to \mathbb{R}$ with the property that $f^{-1}(\{0\}) = A$. Deduce X is Hausdorff. You may assume points are closed.

(a) We first define f on the skeleta of X. The case of $\mathrm{Sk}_{-1}(X)$ is vacuous. Suppose a continuous function $f_{n-1} : \mathrm{Sk}_{n-1} X \to \mathbb{R}$ has been defined with the property that $f_{n-1}(z_i) = i$ for all $x_i \in S \cap \mathrm{Sk}_{n-1}(X)$. We will extend this definition to $\mathrm{Sk}_n(X)$.

Set up the pushout diagram

$$\begin{tikzcd}
\coprod_{i \in I_n} S^{n-1} \arrow[r, "\alpha"] \arrow[d] & \mathrm{Sk}_{n-1}(X) \arrow[d] \\
\coprod_{i \in I_n} D^n \arrow[r, "\beta"] & \mathrm{Sk}_n(X).
\end{tikzcd}$$

Let Z_n denote $Z \cap (\mathrm{Sk}_n(X) - \mathrm{Sk}_{n-1}(X))$. For each disk D^n in the disjoint union, the set $\beta|_{D^n}^{-1}(Z)$ consists of 1 point in the interior, by hypothesis. Denote this point by w. The set $\{w\} \cup S^{n-1} \subset D^n$ is closed and in two connected components. Using the Tietze extension theorem, we can define a continuous function $D^n \to \mathbb{R}$ that agrees with $f_{n-1} \circ \alpha$ on S^{n-1} and that agrees with $f_Z \circ \beta$ on $\{w\}$. Repeat this for all disks D^n in the disjoint union to get a function $F : \coprod D^n \to \mathbb{R}$ that agrees with $f_{n-1} \circ \alpha$ on $\coprod S^{n-1}$ and with f_Z on $\beta^{-1}(Z_n)$.

The universal property of the coproduct applied to the two maps $F : \coprod D^n \to \mathbb{R}$ and $f_{n-1} : \mathrm{Sk}_{n-1}(X) \to \mathbb{R}$ yields $f_n : \mathrm{Sk}_n(X) \to \mathbb{R}$, which agrees with f_Z on Z.

The universal property of X as a colimit of the skeleta means that the functions f_n defined on each Sk_n assemble to give a continuous $f : X \to \mathbb{R}$ that agrees with f_Z on Z.

(b) Suppose that $U \subset X$ is a subset that meets an infinite set of cells. Let $\{u_n\}_{n \in \mathbb{N}}$ be an $\mathbb{N}$-indexed of points in U, each in a different cell of X. For each other cell $e \subset X$, pick a point $w \in e$. Let W denote the set of all these points. Define a function $f_Z : \{u_n\}_{n \in \mathbb{N}} \cup W \to \mathbb{R}$ by setting $f_Z(u_n) = n$ and $f_Z(w) = 0$ for all $w \in W$. Use the previous part of the question to extend f_Z to a continuous function $f : X \to \mathbb{R}$. Note that the image $f(U)$ contains every natural number, so is unbounded. Therefore U is not compact.

(c) We will establish this result with an even stronger conclusion: the function f can be assumed to be nonnegative, i.e., $f : X \to [0,\infty)$.

First, we handle the case of a finite dimensional X, which we do by induction on the dimension. The case of dimension 0 is trivial, since X is discrete and all functions $X \to \mathbb{R}$ are continuous.

Next, we suppose that $X = \mathrm{Sk}_n X$ and the result has been established for $n-1$-dimensional CW-complexes. Let $A' = A \cap \mathrm{Sk}_{n-1} X$, which is a closed subset of $\mathrm{Sk}_{n-1} X$. We may find a continuous function $f_{n-1} : \mathrm{Sk}_{n-1}(X) \to [0,\infty)$ for which $f_{n-1}^{-1}(0) = A'$.

Now consider the pushout diagram (without the dashed or dotted arrows)

$$\begin{tikzcd}
\coprod S^{n-1} \arrow[r, "\alpha"] \arrow[d] & \mathrm{Sk}_{n-1} X \arrow[d] \arrow[ddr, bend left, "f_{n-1}"] & \\
\coprod D^n \arrow[r, "\beta"] \arrow[drr, dashed, bend right, "\tilde{g}"'] & \mathrm{Sk}_n X \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dotted, "f"] & \\
& & {[0,\infty)}
\end{tikzcd}$$

Define a set $A'' = \coprod S^{n-1} \cup \beta^{-1}(A)$, which is closed in $\coprod D^n$. We already know the value we want $f \circ \beta$ to take on A'': it should agree with $f_{n-1} \circ \alpha$ on $\coprod S^{n-1}$ and it should give the value 0 on $\beta^{-1}(A)$. On the intersection, i.e., on $\alpha^{-1}(A')$, these two functions agree by construction of f_{n-1}. Therefore we already have a continuous definition of a function $g : A'' \to \mathbb{R}$ that agrees with what we want from $f \circ \beta$.

Since each D^n is a metric space, we can use the Tietze extension theorem to extend g to give a continuous function $\tilde{g} : \coprod D^n \to \mathbb{R}$ that agrees with $f_{n-1} \circ \alpha$ on $\coprod S^{n-1}$ and that gives the value 0 on $\beta^{-1}(A)$. In principle, $\tilde{g}(x)$ might be 0 for some x outside of $\beta^{-1}(A)$, but we can add to $\tilde{g}$ a function $\varepsilon : D^n \to [0,1]$ that is 0 on and only on A''. In this way, we can ensure $\tilde{g}(x) = 0$ when and only when $x \in \beta^{-1}(A)$.

Using the universal property of the pushout, therefore, we can define a function $f : \mathrm{Sk}_n X = X \to \mathbb{R}$ that agrees with f_{n-1} on Sk_{n-1} and gives the value 0 on and only on A. This completes the induction step.

For the case of an infinite-dimensional CW complex X, we use the finite-dimensional result to produce an infinite family of continuous functions $f_n : \mathrm{Sk}_n(X) \to [0,\infty)$, one for each $n \in \mathbb{N}$ so that

i. f_n agrees with the restriction of f_{n+d} to $\mathrm{Sk}_n X$ whenever $d > 0$.
ii. $f_n(x) = 0$ if and only if $x \in \mathrm{Sk}_n X \cap A$.

The topology on $X = \bigcup \mathrm{Sk}_n X$ is such that if we define $f(x) = f_n(x)$ if $x \in \mathrm{Sk}_n(X)$, then f is continuous. This function $f : X \to [0,\infty)$ has exactly the property we want.

Finally, we can use this result to prove that X is Hausdorff. Let $x, y \in X$ be two points. There are continuous functions $f_x : X \to [0,\infty)$ and $f_y : X \to [0,\infty)$ with the property that $f_x(t) = 0$ if and only if $t = x$, and similarly for f_y. Then form the continuous function

$$g : X \to [0,\infty), \quad g(t) = \frac{f_x(t)}{f_x(t) + f_y(t)}$$

for which $g(t) = 0$ if and only if $t = x$ and $g(t) = 1$ if and only if $t = y$. Then $g^{-1}(0,1/2)$ and $g^{-1}(1/2,1)$ are disjoint open neighbourhoods of x and y respectively.

□

2.

(a) **This part is not to be handed in** Establish a homotopy equivalence between $S^1 \vee S^1$ and the pair-of-pants space p illustrated in Figure 4. The space p is homeomorphic to S^2 with three disjoint open disks removed.

(b) Calculate the homology of the torus-with-two-open-ends space in Figure 5. This is homeomorphic to the torus with two open balls removed. In your presentation of the homology, what is the homology class corresponding to the indicated loop under the Hurewicz map?

(c) Calculate the homology of the genus-2 surface in Figure 6.

(a) A deformation retraction is indicated in Figure 1.

(b) There are several methods to do this. Call the space X We can decompose X as the union of two subspaces, U, V, each homeomorphic to the pair-of-pants P, intersecting in two cylinders. This gives a Mayer–Vietoris sequence:

$$\cdots \longrightarrow \bar{\mathrm{H}}_q(S^1 \cup S^1) \xrightarrow{j_*} \bar{\mathrm{H}}_q(P) \oplus \bar{\mathrm{H}}_q(P) \xrightarrow{i_*} \bar{\mathrm{H}}_q(X) \longrightarrow \bar{\mathrm{H}}_{q-1}(S^1 \cup S^1) \longrightarrow \cdots$$

Only $\bar{\mathrm{H}}_2(X)$ and $\bar{\mathrm{H}}_1(X)$ might potentially be nonzero. Once we insert the known groups (up to isomorphism) we get:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & \bar{\mathrm{H}}_2(X) \arrow[dll, "\partial"] \\
\mathbb{Z}^2 \arrow[r, "j_*"] & \mathbb{Z}^2 \oplus \mathbb{Z}^2 \arrow[r, "i_*"] & \bar{\mathrm{H}}_1(X) \arrow[dll, "\partial"] \\
\mathbb{Z} \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

The calculation now depends on the calculation of $j_* = \begin{bmatrix} j^U_* \\ -j^L_* \end{bmatrix}$ where j^U and j^L denote the inclusions of the intersection into the upper and lower pairs-of-pants. Reference to part (a) gives us

$$j_* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix},$$

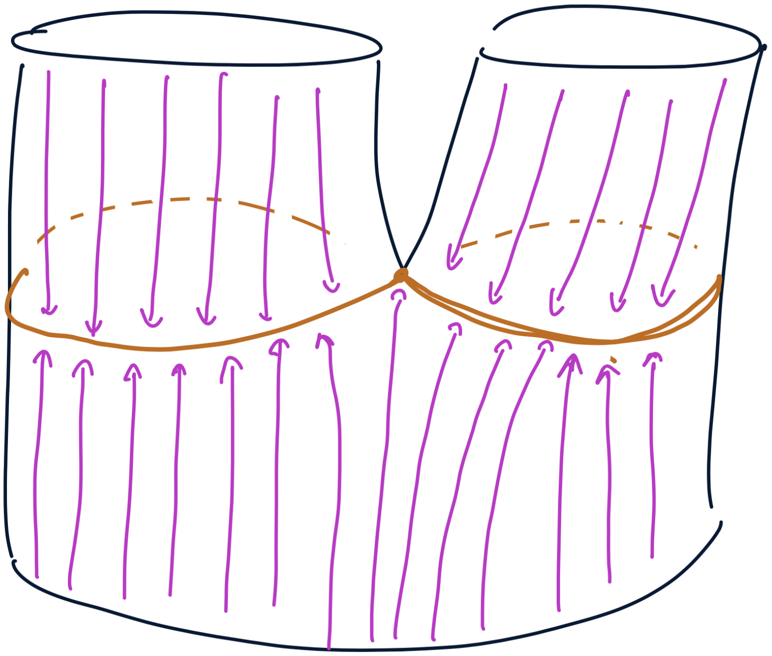


Figure 1: A deformation retraction of the pair of pants onto $S^1 \vee S^1$.

which is a split injective homomorphism. We deduce that $\mathrm{H}_2(X) \cong 0$ and that there is a short exact sequence

$$0 \to \mathbb{Z}^2 \xrightarrow{\bar{\imath}_*} \mathrm{H}_1(X) \to \mathbb{Z} \to 0$$

which is necessarily split since the quotient is a free group, giving us $\mathrm{H}_1(X) \cong \mathbb{Z}^3$.

It is possible to draw loops on the surface X representing the three generators. This will be useful for the next part of the question, so we do so in Figure 2. The nature of the generator γ can be deduced from a snake lemma argument, but we do not need it, so we do not give the argument here.

(c) Again, there are a number of ways to proceed. Call the genus-2 surface Y. In order to calculate $\bar{\mathrm{H}}_*(Y)$, we first calculate the homology of the torus with one open end, Z

The homology of Z can be calculated by means of a Mayer–Vietoris sequence arising from covering Z by a space homeomorphic to X from part (b) and a disk, intersecting in a cylinder C. This is pictured in Figure 3. We get a sequence in homology of the form

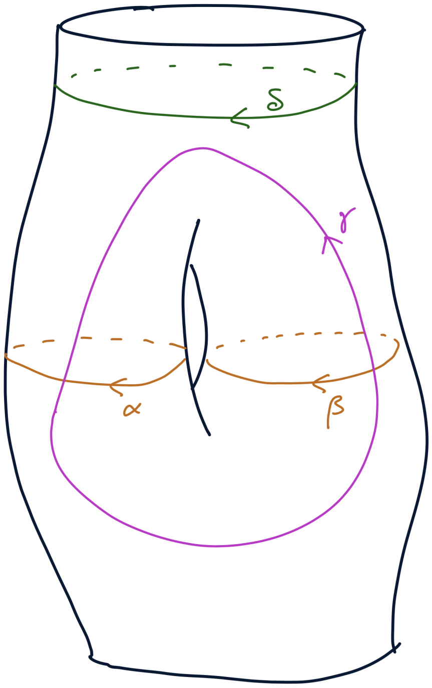


Figure 2: The surface X and homology generators.

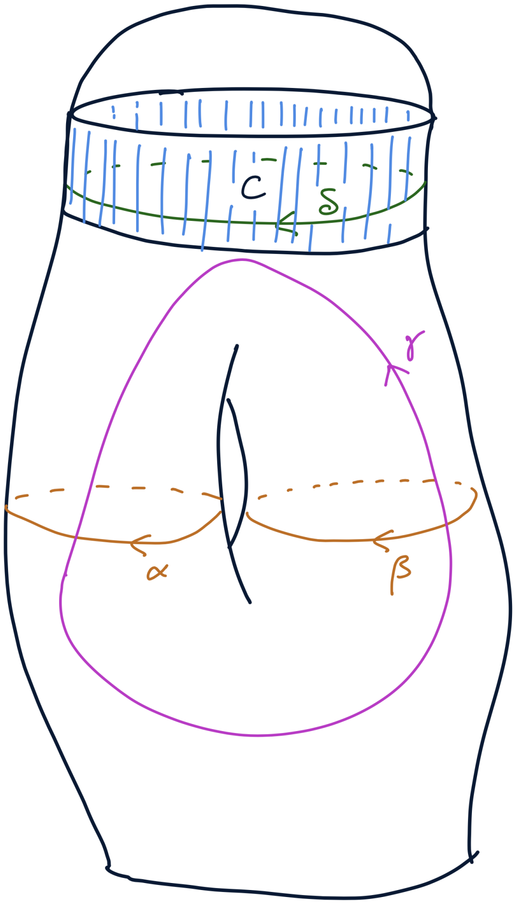


Figure 3: A cover of the space Z.

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & \bar{\mathrm{H}}_2(Z) \arrow[dll, "\partial"] \\
\mathbb{Z} \arrow[r, "j_*"] & \mathbb{Z}^3 \arrow[r] & \mathrm{H}_1(Z) \arrow[dll, "\partial"] \\
0 \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

Again, the calculation depends on the nature of j_*. The homology of X is generated by three classes α, β, γ, indicated in Figure 2. The generator H_1 of the cylinder C in Figure 3 is δ, which maps maps to $\alpha+\beta$ in $\mathrm{H}_1(X)$. This gives us the calculation

$$\bar{\mathrm{H}}_q(Z) \cong \begin{cases} \mathbb{Z}^2 & \text{if } q = 2; \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore a loop around the surface near the opening, e.g., ε in Figure 3, represents 0 in homology. This is because it is equivalent to $\alpha+\beta$, which also represents 0.

Now we may take two copies of Z to cover the genus-2 surface Y, intersecting in a cylinder. Applying the Mayer–Vietoris sequence again, we have a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & \bar{\mathrm{H}}_2(Y) \arrow[dll, "\partial"] \\
\mathbb{Z} \arrow[r, "j_*"] & \mathbb{Z}^2 \oplus \mathbb{Z}^2 \arrow[r, "i_*"] & \bar{\mathrm{H}}_1(Y) \arrow[dll, "\partial"] \\
0 \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

Here, the map j_* sends a generator of $\mathrm{H}_1(S^1)$ to the class represented by δ, i.e., to 0. Therefore $j_* = 0$, and we arrive at

$$\bar{\mathrm{H}}_q(Y) \cong \begin{cases} \mathbb{Z} & \text{if } q = 2; \\ \mathbb{Z}^4 & \text{if } q = 1; \\ 0 & \text{otherwise.} \end{cases}$$

□

3. Produce a CW complex as follows. Start with a solid cube, with the "obvious" CW structure: there are eight 0-cells at the vertices, twelve 1-cells, being the edges, six 2-cells being the faces and a single interior 3-cell. Now form the quotient CW complex X by identifying opposite (closed) faces, but with a twist. For each pair of opposite closed faces A and B, identify A with B by twisting A by 90° in the positive direction from the point of view of someone looking at this face from outside the

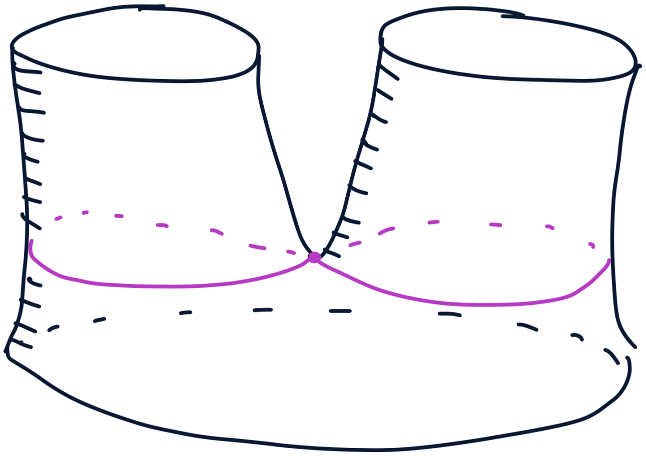


Figure 4: The “pair of pants” space P, and an embedded copy of $S^1 \vee S^1$.

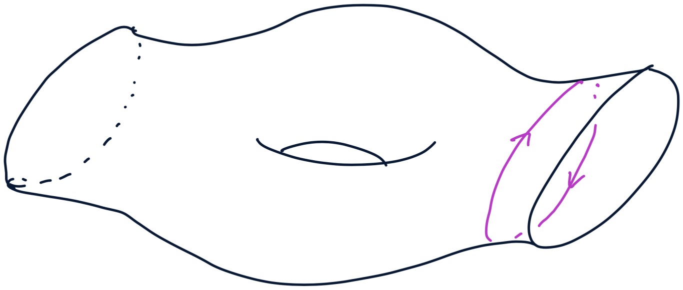


Figure 5: Torus with two open ends, and a marked embedded circle.

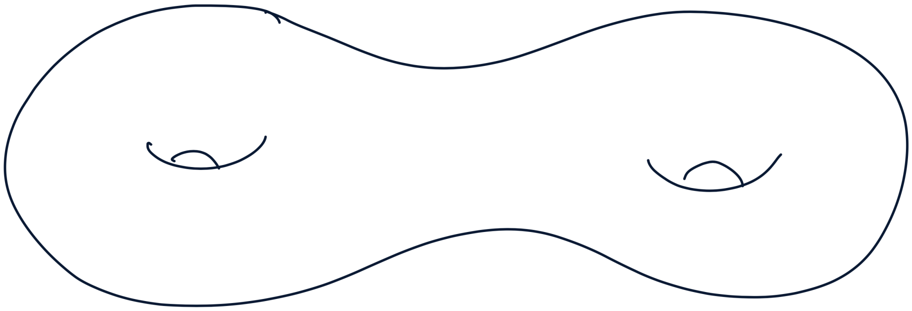


Figure 6: A genus-2 surface.

cube, and then projecting to the opposite face, B. (Observe that the identification specified by this procedure is the same if the roles of A and B are reversed.)

The resulting CW complex X has one 3-cell, three 2-cells and some number of 1- and 0-cells. Determine the number of cells of X and calculate $\mathrm{H}_*(X;\mathbb{Z})$.

The identifications are illustrated in Figure 7. One should imagine the smallest square face in the middle as being closest to the viewer. The cube has been deformed so that four of the remaining five faces are also visible: the top face on top, the bottom face on the bottom, the left face to the left and the right face to the right. Only the rear face is invisible in the picture.

After the identifications, there remains 1 3-cell, the interior of the cube. There are 3 faces (one representative of each is labelled A, B, C in 7a). There are 4 edges, labelled $1,2,3,4$ in Figure 7a. Let us call these edges e_1, e_2, e_3, e_4. There are 2 vertices, or 0-cells. These are labelled with orange squares or blue disks in Figure 7b. The cellular chain complex takes the form:

$$0 \longrightarrow Z \xrightarrow{d_3} \mathbb{Z}^3 \xrightarrow{d_2} \mathbb{Z}^4 \xrightarrow{d_1} \mathbb{Z}^2 \longrightarrow 0$$

Where, for definiteness, we give the group in degree 2 the ordered bases $\{A, B, C\}$. These cells are oriented in the way indicted by the green arrows in Figure 7a (strictly, these arrows indicate an orientation on the boundaries of the attached 2-disks). The group in degree 1 is given the ordered basis $\{e_1, e_2, e_3, e_4\}$, whose orientations are specified by the arrows along the edges. In this basis, the transformation d_2 is given by the matrix

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix},$$

where a 1 indicates that the attaching of the 2-cell to the edge is in an orientation-preserving direction, and a -1 indicates the opposite. For instance, the second column of this matrix indicates $d_2(B) = -e_1 - e_2 + e_3 + e_4$.

If we use the ordered basis $\{A, A+B, A+C\}$ instead, the matrix is

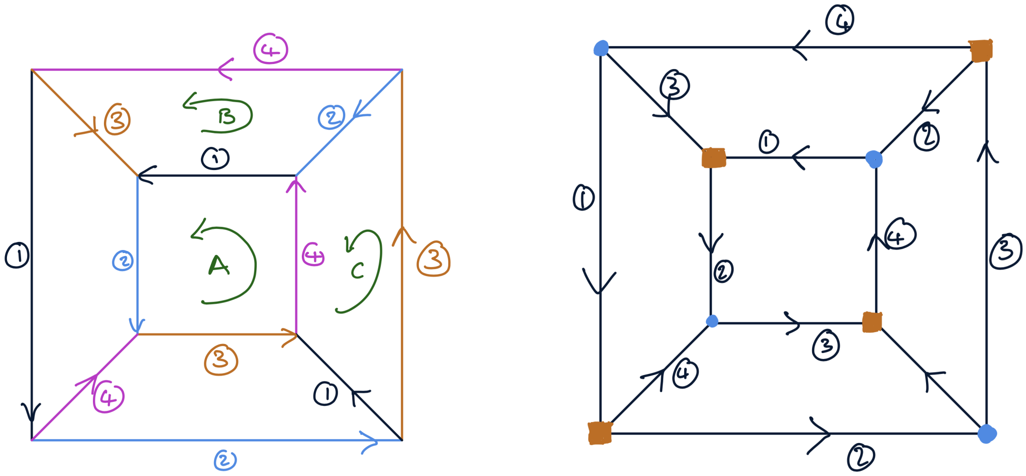
$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 0 \end{bmatrix},$$

which is the result of carrying out two elementary column operations on the original matrix. Now also use the ordered basis $\{e_1 + e_2 + e_3 + e_4, e_3 + e_4, e_2 + e_3, e_4\}$ for the target. We have

$$\begin{aligned} d_2(A) &= e_1 + e_2 + e_3 + e_4 \\ d_2(A+B) &= 2(e_3 + e_4) \\ d_2(A+C) &= 2(e_2 + e_3), \end{aligned}$$

so that the matrix of d_2 with respect to these ordered bases is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$



(a) The identification of the edges.

(b) The identifications of the vertices

Figure 7: Identifications of the cube

We see that d_2 is injective. In particular, $\mathrm{H}_2(X)$, which is a quotient of $\ker(d_2)$, must be 0. The identity $d_2 \circ d_3 = 0$ in the chain complex now tells us that $\mathrm{im}(d_3) = 0$ as well, so that $\ker(d_3) = \mathbb{Z}$. Since $\mathrm{im}(d_4) = 0$, we deduce $\mathrm{H}_3(X) \cong \mathbb{Z}$.

Give $C_0(X)$ the ordered basis consisting of the orange square first, then the blue dot. We can calculate the matrix of d_1 with the choices of ordered bases made so far:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

We see that $\ker(d_1) = \langle e_1 + e_2 + e_3 + e_4, e_3 + e_4, e_2 + e_3 \rangle$.

Having calculated $\ker(d_1)$ and d_2 (and therefore $\mathrm{im}(d_2)$) in the same basis, we deduce that $\mathrm{H}_1(X) \cong \mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$, generated by the classes of $e_3 + e_4$ and of $e_2 + e_3$.

Finally, it is easy to calculate $\mathrm{H}_0(X) \cong \mathbb{Z}$, generated by the difference of the two vertices (alternatively, one could argue that X is a path-connected space, being a quotient of the cube, which is contractible). We see $\mathrm{H}_0(X) \cong \mathbb{Z}$.

In particular, we have the following homology calculation:

$$\mathrm{H}_q(X) \cong \begin{cases} \mathbb{Z} \text{ if } q = 0, 3; \\ \mathbb{Z}/(2) \oplus \mathbb{Z}/(2) \text{ if } q = 1; \\ 0 \text{ otherwise.} \end{cases}$$

□

4. Let n be a positive integer. Suppose A is an abelian group that appears in a short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{i} A \xrightarrow{f} \mathbb{Z}/(n) \to 0.$$

(a) Prove that A is generated by 2 elements.

(b) Prove that A is generated by 2 elements, one of which is torsion of order dividing n. Hint: if x, y generate A, so also do m_1x+m_2y, n_1x+n_2y whenever m_1, m_2, n_1, n_2 are integers satisfying $m_1n_2 - m_2n_1 = 1$. This, combined with Bézout's lemma, may be useful.

(c) Prove that the two generators obtained in the previous step actually generate direct summands, so that $A \cong \mathbb{Z} \oplus \mathbb{Z}/(d)$ where d is a positive integer dividing n.

(d) For any positive integer d dividing n, construct a short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Z} \oplus \mathbb{Z}/(d) \xrightarrow{f} \mathbb{Z}/(n) \to 0.$$

(a) Let $x = i(1)$ and let $y \in A$ be an element such that $f(y) = 1$. If $a \in A$ is an element, then we can find an integer m for which $f(a) = m1 = f(my)$. Then $a - my = nx$ for some integer x, giving us $a = my + nx$. Since a was arbitrary, we see that x, y generate A.

(b) Consider the same x, y as in the previous part. We know that $f(ny) = 0$, so that $ny = bx$ for some integer b. Let d denote the greatest common divisor of n and b. Dividing n and b by g gives n' and b', two relatively prime integers. Using the hint, we can find a pair of generators $x' = b'x - n'y$, $y' = m_1x+m_2y$ for A, where $m_2b' - m_1n' = 1$ (such m_1, m_2 exist by the Bézout lemma in elementary number theory). But $dx' = d(b'x - n'y) = 0$, so one of these generators is torsion of order dividing d, which divides n in turn.

(c) We continue using the notation of the previous part. Since A is not finite, it cannot be generated by two torsion elements, so y' is of infinite order.

It suffices to prove that the intersection between the subgroups of A generated by x' and y' is $\{0\}$. Any element in the intersection can be written as $m_1x' = m_2y'$ some integers m_1, m_2. Since x' is torsion, $m_1x' = m_2y'$ is torsion, but y' is not torsion, so $m_2 = 0$. Therefore the only element in the intersection is 0.

(d) Define $f(1,0) = 1$ and $f(0,1) = n/d$. Certainly $df(0,1) = 0$, so that this homomorphism is well defined. Since $f(1,0) = 1$, it is surjective.

By direct calculation $f(n/d, -1) = 0$. We claim that $(n/d, -1) \in \mathbb{Z} \oplus \mathbb{Z}/(d)$ generates the kernel.

Suppose $f(a, b) = 0$ for some integers a, b. That is

$$a + b\frac{n}{d} \equiv 0 \pmod{n}, \tag{1}$$

which implies that n/d divides a. Write $a' = a/(n/d)$ The division formula for dividing (1) by n/d gives us

$$a' \equiv -b \pmod{d}.$$

That is, $a'(n/d,-1) = (a'n/d,-a') = (a,b)$ since the second integer is reduced modulo d. This concludes a proof that $(n/d,-1)$ generates the kernel of f, which must therefore be an infinite cyclic group.

□

5. Let $n \geq 2$ be an integer. Let $f : [0,1] \to S^n$ be a closed embedding (i.e., a closed continuous injective function).

(a) Let $[c] \in \bar{\mathrm{H}}_q(S^n - f([0,1]);\mathbb{Z})$ be a homology class where $q \geq 0$. Let $0 = i_0 < i_1 < i_2 < \cdots < i_{m-1} < i_m = 1$. Use the Mayer–Vietoris sequence to prove that $[c] = 0$ if and only if the images of $[c]$ in $\bar{\mathrm{H}}_q(S^n - f([i_{j-1}, i_j]);\mathbb{Z})$ are 0 for all $j \in \{1,\ldots,m\}$.

(b) Observe that $f([0,1])$, being a compact set, is some positive distance away from the image of each singular n-simplex σ_k appearing in a q-chain $c = \sum_k m_k \sigma_k$. Hence or otherwise, prove that if $c \in Z_q(S^n - f([0,1]))$, then $[c] = 0 \in \bar{\mathrm{H}}_q(S^n - f([0,1]);\mathbb{Z})$.

(c) Suppose $g : S^1 \to S^n$ is a continuous injective function. Calculate $\bar{\mathrm{H}}_*(S^n - g(S^1);\mathbb{Z})$.

(a) We may generalize the problem slightly by replacing $[0,1]$ by an arbitrary closed interval $[a,b]$ where $a < b$ are two real numbers, and where $a = i_0 < i_1 < \cdots < i_m = b$. The original problem is a special case of this one, and the general case can be deduced from the case of $[0,1]$ by a linear reparameterization of the interval.

We now proceed by induction on m. The base case of $m = 1$ is trivially true. Suppose the result has been established for a subdivision of the interval into $m-1$ parts. There is an open cover of the contractible space $S^n - \{f(i_{m-1})\}$ by two sets: $U = S^n - f([a, i_{m-1}])$ and $V = S^n - f([i_{m-1}, b])$. The Mayer–Vietoris sequence for reduced homology implies that the map

$$\bar{\mathrm{H}}_q(S^n - f([a,b])) \overset{\begin{bmatrix} j_*^U \\ -j_*^V \end{bmatrix}}{\longrightarrow} \bar{\mathrm{H}}_q(S^n - f([a, i_{m-1}])) \oplus \mathrm{H}_q(S^n - f([i_{m-1}, b]))$$

is an isomorphism, where j^U, j^V are the inclusions of $S^n - f([a,b])$ into U, V respectively.

Consequently, $[c] = 0$ if and only if $j_*^U([c]) = 0$ and $j_*^V([c]) = 0$. The second of these is the image of $[c]$ under the inclusion induced by $S^n - f([0,1]) \hookrightarrow S^n - f([i_{m-1}, i_m])$. The induction hypothesis tells us that $j_*^U([c]) = 0$ if and only if the image of $[c]$ in each of $\mathrm{H}_q(S^n - f([i_j, i_{j+1}]))$ under the inclusion is 0 for $j \in \{0,\ldots,m-2\}$. Therefore $[c] = 0$ if and only if the image of $[c]$ is 0 in all of the groups $\mathrm{H}_q(S^n - f([i_j, i_{j+1}]))$ for $j \in \{0,\ldots,m-1\}$.

(b) Suppose c is a q-cycle in $S^n - f([0,1])$. We may write $c = \sum m_k \sigma_k$ as a finite $\mathbb{Z}$-linear combination of q-simplices σ_k. The image of each σ_k is a compact subset of S^n disjoint from the compact subset $f([0,1])$. Therefore there is some positive distance $\varepsilon_k > 0$ for which $\mathrm{dist}(\sigma_k, f([0,1])) > \varepsilon_k$. There being only finitely many σ_k, we can find $\varepsilon > 0$ for which $\mathrm{dist}(\sigma_k, f([0,1])) > \varepsilon$ for all σ_k.

We claim that we may decompose $[0,1]$ into subintervals $[i_0 = 0, i_1], [i_1, i_2], \cdots [i_{m-1}, 1 = i_m]$ with the property that $f([i_j, i_{j+1}]) \subset S^n$ is contained within an open ball of radius ε centred at one of the points of $f([i_j, i_{j+1}])$. There are many ways to establish this: for instance, consider the set of all balls of radius ε with centres on $f([0,1])$. Take the inverse image of this cover, which is a cover of the compact set $[0,1]$. Apply the Lebesgue covering lemma to find a positive radius r so that all subintervals $[a,b] \subset [0,1]$ of length $< r$ are contained in at least one of the open sets of the cover. Then find an integer $N > 1/r$ and divide $[0,1]$ into N equal-length closed subintervals, which constitute the $[i_j, i_{j+1}]$. By construction $f([i_j, i_{j+1}])$ is fully contained in one of the open balls of radius ε centred at a point of $f([0,1])$. This establishes the claim.

Having decomposed $[0,1]$ into subintervals, we invoke the previous part of the problem. This tells us that $[c] = 0 \in \bar{\mathrm{H}}_q(S^n - f([0,1]))$ if and only if the image of $[c]$ vanishes in $\bar{\mathrm{H}}_q(S^n - f([i_j, i_{j+1}]))$ for all $j \in \{0, \ldots, m-1\}$.

Fix one such j. Let B denote an open ball of radius ε centred at a point of $f([0,1])$ such that $f([i_j, i_{j+1}]) \subset B$. The image of $[c]$ in $\bar{\mathrm{H}}_q(S^n - f([i_j, i_{j+1}]))$ is represented by the cycle $\sum m_k \sigma_k$, where each σ_k is actually contained in the contractible space $S^n - B$. Therefore $[c]$ in the image of the map induced by inclusion $\bar{\mathrm{H}}_q(S^n - B) \to \bar{\mathrm{H}}_q(S^n - f([i_j, i_{j+1}]))$. Since $S^n - B$ is contractible, this implies $[c] = 0$, as required.

(c) Cover S^1 by two closed semicircles, $X = \{(x,y) \in S^1 \mid x \geq 0\}$ and $Y = \{(x,y) \in S^1 \mid x \leq 0\}$. These are both homeomorphic to $[0,1]$, and their intersection consists of two points $(\pm 1, 0)$.

Now take $S^n - f(X)$ and $S^n - f(Y)$ to be an open cover of a space $S^n - f(X \cap Y)$ with intersection $S^n - f(X \cup Y) = S^n - f(S^1)$. Here we exploit the fact that f is an injective closed function. Observe that $S^n - f(X \cap Y) \approx \mathbb{R}^n - \mathrm{pt} \simeq S^{n-1}$.

There is a Mayer–Vietoris sequence in reduced homology

$$\cdots \longrightarrow \bar{\mathrm{H}}_{q+1}(S^{n-1}) \xrightarrow{\partial} \bar{\mathrm{H}}_q(S^n - f(S^1)) \xrightarrow{j_*} \bar{\mathrm{H}}_q(S^n - f(X)) \oplus \bar{\mathrm{H}}_q(S^n - f(Y)) \xrightarrow{i_*} \bar{\mathrm{H}}_q(S^{n-1}) \longrightarrow \cdots$$

Since $X \approx [0,1] \approx Y$, the previous part of the problem assures us that $\bar{\mathrm{H}}_q(S^n - f(X)) \cong 0 \cong \bar{\mathrm{H}}_q(S^n - f(Y))$. Therefore the connecting homomorphism ∂ is an isomorphism $\bar{\mathrm{H}}_{q+1}(S^{n-1}) \xrightarrow{\cong} \bar{\mathrm{H}}_q(S^n - f(S^1))$. Therefore the answer is that

$$\bar{\mathrm{H}}_q(S^n - f(S^1)) \cong \begin{cases} \mathbb{Z} \text{ if } q = n-2; \\ 0 \text{ otherwise.} \end{cases}$$

Remark. The importance of this result follows from the weakness of the hypothesis on f. For instance, $f(S^1)$ could exhibit fractal behaviour as in the case of the Koch snowflake in $\mathbb{R}^2$. More regular, but very interesting, are knotted embeddings of S^1 in S^3 as studied in knot theory.

We state a corollary of the main result of this exercise.

Theorem (The Jordan curve theorem). *Let $f : S^1 \to S^2$ be a closed embedding of S^1 in S^2. Then $S^2 - f(S^1)$ consists of two path components.*

Proof. We can calculate $\mathrm{H}_0(S^2 - f(S^1)) \cong \mathbb{Z} \oplus \bar{\mathrm{H}}_0(S^2 - f(S^1)) \cong \mathbb{Z} \oplus \mathbb{Z}$ from the exercise. Therefore $\pi_0(S^2 - f(S^1))$ consists of two elements. □

This result is famous for being intuitively obvious, but difficult to prove in full generality.

One might also wonder whether the two path components are necessarily contractible. They are: this is the content of the Schoenflies theorem, which requires a further argument. All our result tells us is that they have the homology of a contractible space, and in particular, are both path-connected.

□