Math 427, 527
Homework 5
2026-03-19

1. Suppose A is an abelian group. Recall that an element $a \in A$ is said to be *torsion* if there exists a positive integer n such that $na = 0$. Let $T(A)$ denote the subgroup of torsion elements in A (you may assume these make up a subgroup). Let **Ab** denote the category of all abelian groups and homomorphisms.Prove that $T(A)$ is a naturally defined subgroup of A. Specifically: prove that $T : \mathbf{Ab} \to \mathbf{Ab}$ is a functor (you can skip verifying it preserves compositions and identities), and prove that the inclusion map $T(A) \to A$ forms part of a natural transformation from T to the identity functor. We can view this as saying that T is a *subfunctor* of the identity functor.

We are allowed to assume that $T(A)$ is a group. Let us next prove that the assignment $A \rightsquigarrow T(A)$ is a functor. Specifically, this requires us to show that if $\phi : A \to B$ is a homomorphism, then there exists an induced $T(\phi) : T(A) \to T(B)$. If $a \in A$ is torsion, then there exists some natural number n for which $na = 0$. Then $n\phi(a) = \phi(na) = \phi(0) = 0$, so that $\phi(a) \in B$ is also torsion. Therefore ϕ restricts to give a homomorphism, which we declare to be $T(\phi)$, from $T(A)$ to $T(B)$.

We do not write out proofs of $T(\phi \circ \psi) = T(\phi) \circ T(\psi)$ and $T(\mathrm{id}) = \mathrm{id}_{T(A)}$, but they are not difficult.

A natural transformation $\nu : T \to \mathrm{id}$ is given by the inclusion $T(A) \subset \mathrm{id}(A) = A$ for all A. To verify that this is indeed a natural transformation, we must verify that the following diagram commutes for all homomorphisms $\phi : A \to B$:

$$\begin{tikzcd}
T(A) \arrow[r, "\nu_A"] \arrow[d, "T(\phi)"] & A \arrow[d, "\phi"] \\
T(B) \arrow[r, "\nu_B"] & B.
\end{tikzcd}$$

This commutes because $T(\phi)$ is the restriction of ϕ to the subgroup $T(A)$, and ν_A, ν_B are the inclusions.

□

2. If M is a topological space and $f : M \to M$ is a homeomorphism, then the *mapping torus* T_f of f is the quotient of the space $M \times [0,1]$ obtained by making the identification $(m,0) \sim (f(m),1)$ for all $m \in M$. It can be covered by two open sets: U, V which are the homeomorphic images in T_f of $M \times (0,1)$ and $M \times ([0,1/2) \cup (1/2,1]$ respectively.

(a) Prove the following useful technical result. It holds for any category of R-modules, but this exercise is stated only in the category of abelian groups. Suppose

$$\cdots \longrightarrow A_3 \xrightarrow{d_3} A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} \cdots$$

is an exact sequence, and that $B_2 \subseteq A_2$ and $B_1 \subseteq A_1$ are subgroups with the property that d_2 maps B_2 isomorphically to B_1. Prove there is an induced exact sequence

$$\cdots \longrightarrow A_3 \xrightarrow{d_3'} A_2/B_2 \xrightarrow{d_2'} A_1/B_1 \xrightarrow{d_1'} A_0 \xrightarrow{d_0'} \cdots.$$

(b) By applying the previous technical result to a Mayer–Vietoris sequence, or otherwise, prove that for any mapping torus T_f associated to $f : M \to M$ there is a long exact sequence in homology of this form:

$$\cdots \longrightarrow \mathrm{H}_q(M) \xrightarrow{\mathrm{id}-f_*} \mathrm{H}_q(M) \longrightarrow \mathrm{H}_q(T_f) \longrightarrow \mathrm{H}_{q-1}(M) \longrightarrow \cdots.$$

(c) Let $M = S^1 \times S^1$ and let $f : S^1 \times S^1 \to S^1 \times S^1$ be the homeomorphism that is the identity of the first factor and a reflection on the second. Calculate the integral homology of the mapping torus T_f.

(a) Consider the diagram of chain complexes:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d, "d_3"] & A_3 \arrow[r] \arrow[d, "d_3'"] & 0 \\
0 \arrow[r] & B_2 \arrow[r] \arrow[d, "d_2|_{B_2}"] & A_2 \arrow[r] \arrow[d, "d_2"] & A_2/B_2 \arrow[r] \arrow[d, "d_2'"] & 0 \\
0 \arrow[r] & B_1 \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d, "d_1"] & A_1/B_1 \arrow[r] \arrow[d, "d_1'"] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & A_0 \arrow[r] \arrow[d] & A_0 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

There is an associated long exact sequence of homology groups, but since the left and middle chain complexes are actually exact sequence, their homology vanishes and therefore so too does the homology of the right hand sequence, i.e., it is exact as required.

(b) Let U and V be as above. The open subsets U and V deformation retract onto the homeomorphic images of $M \times \{1/2\}$ and $M \times \{0\}$, respectively, and the intersection $U \cap V$ deformation retracts onto the homeomorphic image of $M \times \{1/4\} \sqcup M \times \{3/4\}$. If $j_1 : U \cap V \to U$ and $j_2 : U \cap V \to V$ are the inclusions, then using the appropriate deformation retractions, j_1 and j_2 are given (up to homotopy equivalence) by

$$\mathrm{id} \sqcup \mathrm{id} : M \sqcup M \to M, \qquad f \sqcup \mathrm{id} : M \sqcup M \to M,$$

respectively. The Mayer-Vietoris sequence

$$\cdots \to \mathrm{H}_{q+1}(T_f) \to \mathrm{H}_q(U \cap V) \xrightarrow{\begin{bmatrix} j_{1*} \\ -j_{2*} \end{bmatrix}} \mathrm{H}_q(U) \oplus \mathrm{H}_q(V) \to \mathrm{H}_q(T_f) \to \cdots$$

thus reads

$$\cdots \to \mathrm{H}_{q+1}(T_f) \to \mathrm{H}_q(M) \oplus \mathrm{H}_q(M) \xrightarrow{F} \mathrm{H}_q(M) \oplus \mathrm{H}_q(M) \to \mathrm{H}_q(T_f) \to \cdots$$

where $F = \begin{bmatrix} \mathrm{id} & \mathrm{id} \\ -f_* & -\mathrm{id} \end{bmatrix}$. Define

$$B_q := \{(0, \alpha) \mid \alpha \in \mathrm{H}_q(M)\} \subseteq \mathrm{H}_q(M) \oplus \mathrm{H}_q(M), \qquad B_q' := \{(\alpha, -\alpha) \mid \alpha \in \mathrm{H}_q(M)\} \subseteq \mathrm{H}_q(M) \oplus \mathrm{H}_q(M).$$

Then F maps B_q isomorphically onto B'_q, and we may apply part (a) to obtain the long exact sequence

$$\cdots \longrightarrow \mathrm{H}_{q+1}(T_f) \longrightarrow \frac{\mathrm{H}_q(M)\oplus\mathrm{H}_q(M)}{B_q} \xrightarrow{F'} \frac{\mathrm{H}_q(M)\oplus\mathrm{H}_q(M)}{B'_q} \longrightarrow \mathrm{H}_q(T_f) \longrightarrow \cdots \tag{1}$$

where F' is induced by F. The definitions are such that B_q is the kernel of the projection to the first factor $\mathrm{proj}_1 : \mathrm{H}_q(M)\oplus\mathrm{H}_q(M) \to \mathrm{H}_q(M)$, and B'_q is the kernel of the addition map $+ : \mathrm{H}_q(M)\oplus\mathrm{H}_q(M) \to \mathrm{H}_q(M)$. Moreover, the square

$$\begin{tikzcd}
\frac{\mathrm{H}_q(M)\oplus\mathrm{H}_q(M)}{B_q} \arrow[r, "F'"] \arrow[d, "\mathrm{proj}_1", "\cong"'] & \frac{\mathrm{H}_q(M)\oplus\mathrm{H}_q(M)}{B'_q} \arrow[d, "+", "\cong"'] \\
\mathrm{H}_q(M) \arrow[r, "1-f_*"] & \mathrm{H}_q(M)
\end{tikzcd} \tag{2}$$

commutes. For each q, we may replace the map F' in the long exact sequence (1) with $1 - f_*$ using (2) to obtain the desired long exact sequence.

(c) Using the Künneth formula, we have natural isomorphisms

$$\mathrm{H}_2(M) \cong \mathrm{H}_1(S^1)\otimes_{\mathbb{Z}}\mathrm{H}_1(S^1), \qquad \mathrm{H}_1(M) \cong (\mathrm{H}_0(S^1)\otimes\mathrm{H}_1(S^1)) \oplus (\mathrm{H}_1(S^1)\otimes_{\mathbb{Z}}\mathrm{H}_0(S^1)). \tag{3}$$

Let $\alpha \in \mathrm{H}_0(S^1)$ be the generator given by the connected component of M, and let $\beta \in \mathrm{H}_1(S^1)$ be a generator. These generators and the Künneth isomorphisms (3) induce isomorphisms

$$\mathrm{H}_2(M) \cong \mathbb{Z}, \qquad \mathrm{H}_1(M) \cong \mathbb{Z}\oplus\mathbb{Z}$$

where $\beta\otimes\beta$ is a generator of $\mathrm{H}_2(M)$, and $(\alpha\otimes\beta, 0), (0, \beta\otimes\alpha)$ are generators of $\mathrm{H}_1(M)$. Let $r : S^1 \to S^1$ be the reflection, so that $f = \mathrm{id}\times r : M \to M$. Since r has degree -1, the map $f_* : \mathrm{H}_2(M) \to \mathrm{H}_2(M)$ is given by

$$\beta\otimes\beta \mapsto \mathrm{id}_*(\beta)\otimes r_*(\beta) = \beta\otimes(-\beta) = -\beta\otimes\beta.$$

Then $\mathrm{id} - f_* : \mathrm{H}_2(M) \to \mathrm{H}_2(M)$ is given by multiplication by 2. Similarly, the map $f_* : \mathrm{H}_1(M) \to \mathrm{H}_1(M)$ is given on generators by

$$\begin{aligned}
(\alpha\otimes\beta, 0) &\mapsto (\mathrm{id}_*(\alpha)\otimes r_*(\beta), 0) = (\alpha\otimes -\beta) = (-\alpha\otimes\beta, 0), \\
(0, \beta\otimes\alpha) &\mapsto (0, \mathrm{id}_*(\beta)\otimes r_*(\alpha)) = (0, \beta\otimes\alpha),
\end{aligned}$$

and we may identify $\mathrm{id} - f_* : \mathrm{H}_1(M) \to \mathrm{H}_1(M)$ with the matrix

$$\mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}} \mathbb{Z}^2.$$

A portion of the long exact sequence of part (b) is then

$$\begin{tikzcd}
 & 0 \arrow[r] & \mathrm{H}_3(T_f) \arrow[dll] \\
\mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z} \arrow[r] & \mathrm{H}_2(T_f) \arrow[dll] \\
\mathbb{Z}^2 \arrow[r, "{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}}"'] & \mathbb{Z}^2 \arrow[r] & \mathrm{H}_1(T_f) \arrow[dll] \\
\mathrm{H}_0(M) \arrow[r, "\mathrm{id}-f_*=0"'] & \mathrm{H}_0(M) &
\end{tikzcd}$$

We have $\mathrm{H}_0(T_f) = \mathbb{Z}$, since T_f is connected, and $\mathrm{H}_n(T_f) = 0$ when $n \geq 4$ since $\mathrm{H}_n(M) = \mathrm{H}_{n-1}(M) = 0$ in this case. The map $\times 2 : \mathbb{Z} \to \mathbb{Z}$ is injective, and we deduce $\mathrm{H}_3(T_f) = 0$. From the long exact sequence, we obtain a short exact sequence

$$0 \to \operatorname{coker}(\times 2 : \mathbb{Z} \to \mathbb{Z}) \to \mathrm{H}_2(T_f) \to \ker \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \to 0$$

where $\operatorname{coker}(\times 2 : \mathbb{Z} \to \mathbb{Z}) \cong \mathbb{Z}/2$ and $\ker \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \cong \mathbb{Z}$ The short exact sequence splits since $\mathbb{Z}$ is free. Similarly, there is a short exact sequence

$$0 \to \operatorname{coker} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \to \mathrm{H}_1(T_f) \to \ker(0 : \mathrm{H}_0(M) \to \mathrm{H}_0(M)) \to 0.$$

Now, $\operatorname{coker} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \cong \mathbb{Z} \oplus \mathbb{Z}/2$ and $\ker(0 : \mathrm{H}_0(M) \to \mathrm{H}_0(M)) \cong \mathbb{Z}$, so this short exact sequence also splits. To summarize, we have shown

$$\mathrm{H}_n(T_f) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}^2 \oplus \mathbb{Z}/2 & \text{if } i = 1 \\ \mathbb{Z} \oplus \mathbb{Z}/2 & \text{if } i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

□

3. Suppose X is a path-connected space with basepoint x_0. This problem is part of a pair that will establish the following result: the Hurewicz map $\eta : \pi_1(X, x_0) \to \mathrm{H}_1(X; \mathbb{Z})$ is the abelianization of $\pi_1(X, x)$: i.e., the map η is surjective and its kernel is the normal subgroup generated by commutators $\alpha\beta\alpha^{-1}\beta^{-1}$.

You may want to use the result of Example 2.1 in the textbook in answering this question.

(a) We write $c_0^0, c_1^0 \in S_0([0,1])$ for the constant simplices with value 0 and 1. and suppose $k \in S_1([0,1])$ is a chain with the property that $d(k) = c_0^0 - c_1^0$. Prove that the class of k generates $\mathrm{H}_1([0,1], \{0,1\})$. Deduce that the image of k under the map $h : [0,1] \to S^1$ given by $h(t) = (\cos 2\pi t, \sin 2\pi t)$ generates $\mathrm{H}_1(S^1)$.

(b) For all $x \in X$, pick a 1-simplex $\tau_x : \Delta^1 \to X$ such that $\tau_x(0) = x_0$ and $\tau_x(1) = x$. Given any 1-simplex $\sigma : \Delta^1 \to X$, define a 1-cycle

$$l(\sigma) = \tau_x + \sigma - \tau_y, \quad \text{where } x = \sigma(0),\ y = \sigma(1).$$

You do not have to prove this is a 1-cycle. Prove that the homology class of $l(\sigma)$ is in the image of the Hurewicz map $\eta : \pi_1(X, x_0) \to \mathrm{H}_1(X, x_0)$.

(c) Prove that $\eta : \pi_1(X, x_0) \to \mathrm{H}_1(X)$ is surjective.

(a) First, we note that $[c_0^0] = [c_0^1]$ in $\mathrm{H}_0([0,1])$, and $\mathrm{H}_0([0,1]) = \mathbb{Z}[c_0^0]$. From the long exact sequence of the pair $([0,1], \{0,1\})$, we have the following short exact sequence

$$0 = \mathrm{H}_1([0,1]) \to \mathrm{H}_1([0,1], \{0,1\}) \xrightarrow{\partial} \mathrm{H}_0(\{0,1\}) = \mathbb{Z}[c_0^0] \oplus \mathbb{Z}[c_0^1] \xrightarrow{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \mathrm{H}_0([0,1]) = \mathbb{Z}[c_0^0] \to 0.$$

Since $\operatorname{im}\partial = \ker \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is generated by $[c_0^0] - [c_0^1]$, it suffices to show $\partial([k]) = [c_0^0] - [c_0^1]$. This follows from the definition of ∂ in the snake lemma:

$$\partial([k]) = [dk] = [c_0^0 - c_0^1] = [c_0^0] - [c_0^1].$$

The map h induces a homeomorphism $\bar{h} : [0,1]/\{0,1\} \to S^1$, and it follows from the corollary to Excision (Corollary 10.3 in the textbook) that $h_* : \mathrm{H}_1([0,1],\{0,1\}) \to \mathrm{H}_1(S^1,*) = \mathrm{H}_1(S^1)$ is an isomorphism, where $* = (1,0) \in S^1$, so that $h_*([k])$ generates $\mathrm{H}_1(S^1)$.

(b) For $i = 1,2,3$, let

$$\alpha_i : \Delta^1 \to [0,1]$$
$$(t, 1-t) \mapsto \frac{t-1+i}{3}$$

and put $\alpha = \alpha_1 + \alpha_2 + \alpha_3 \in S_1([0,1])$. If $c_t^0 \in S_0([0,1])$ denotes the constant simplex with value $t \in [0,1]$, then

$$d(\alpha) = d(\alpha_1) + d(\alpha_2) + d(\alpha_3) = (c_{1/3}^0 - c_0^0) + (c_{2/3}^0 - c_{1/3}^0) + (c_1^0 - c_{2/3}^0) = c_1^0 - c_0^0,$$

so $h_*[\alpha]$ is a generator of $\mathrm{H}_1(S^1,*)$ by part (a). If $\gamma : I \to X$ is the concatenation of paths $\tau_x * \sigma * \overline{\tau}_y$, where $\overline{\tau}_y$ denotes the reverse path of τ_y, then γ is a loop based at x_0 and so induces a based map $\gamma' : (S^1,*) \to (X,x_0)$ with $\gamma' \circ h = \gamma$. The element $h_*[\alpha] \in \mathrm{H}_1(S^1,*)$ may be used to define a Hurewicz homomorphism η where

$$\eta([\gamma']) = \gamma'_*(h_*[\alpha]) = \gamma_*([\alpha]) = [\gamma_*(\alpha_1) + \gamma_*(\alpha_2) + \gamma_*(\alpha_3)] = [\tau_x + \sigma + \overline{\tau}_y] = [\tau_x + \sigma - \tau_y].$$

In the last equality, we have used the fact that $\tau_y + \overline{\tau}_y$ is a boundary.

(c) Let $k = \sum_{i=1}^n m_i \sigma_i \in Z_1(X)$. Let $x_i = \sigma_i(0)$ and $y_i = \sigma_i(1)$. Note that the list of 0-simplices $\{x_i, y_i\}_{i=1}^n$ has multiplicities, since k is a cycle. In defining the 1-cycles $l(\sigma_i)$, we may "recycle" the 1-simplices τ_{x_i}, τ_{y_i}. More precisely, whenever $x_i = x_j$, $x_s = y_t$, or $y_p = y_q$, we may assume $\tau_{x_i} = \tau_{x_j}$, $\tau_{x_s} = \tau_{y_t}$, and $\tau_{y_p} = \tau_{y_q}$. Since k is a cycle,

$$dk = \sum_i m_i(y_i - x_i) = 0$$

Given our choices of τ_{x_i}, τ_{y_i}, this implies

$$\sum_i m_i(\tau_{y_i} - \tau_{x_i}) = 0,$$

so

$$k = k - \sum_i m_i(\tau_{y_i} - \tau_{x_i}) = \sum_i m_i l(\sigma_i).$$

Since each $[l(\sigma_i)]$ is in the image of η, the class $[k]$ is in the image of η as well. Indeed, if $[l(\sigma_i)] = \eta[\gamma_i]$, then

$$\eta(\prod_i \gamma_i^{m_i}) = [\sum_i m_i l(\sigma_i)] = [k],$$

since η is a group homomorphism.

□

4. We say an abelian group A is *uniquely divisible* if the multiplication-by-n map $\times n : A \to A$ is an isomorphism for all positive integers n. You may assume the following easy (but boring to prove) fact: an abelian group A is uniquely divisible if and only if the abelian group structure on A extends to a $\mathbb{Q}$-vector-space structure.

(a) Let A be an abelian group and suppose that for all prime numbers p, we have $\mathbb{Z}/(p) \otimes_{\mathbb{Z}} A = 0$ and $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/(p), A) = 0$. Prove that A is uniquely divisible.

(b) You may assume that the usual multiplication map $\mu : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ is $\mathbb{Z}$-bilinear. Prove using the universal property that $\mathbb{Q}$ is a tensor product of $\mathbb{Q}$ and $\mathbb{Q}$ over $\mathbb{Z}$: in particular, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$.

(c) Suppose A is an abelian group such that $A \otimes_{\mathbb{Z}} k = 0$ and $\mathrm{Tor}_1^{\mathbb{Z}}(A, k) = 0$ for all

$$k \in \{\mathbb{Q}, \mathbb{Z}/(2), \mathbb{Z}/(3), \mathbb{Z}/(5), \dots\}.$$

(These fields are called *prime fields*). Prove that $A = 0$.

(d) **Not to be handed in:** Give an example of a nonzero abelian group A for which $A \otimes_{\mathbb{Z}} k = 0$ for all prime fields k (and therefore, for all fields k).

(a) Let p be a prime number, and consider the map $\times p : A \to A$. Its kernel is the p-torsion subgroup of A, which is isomorphic to $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/(p), A) = 0$, and its cokernel is $A/pA \cong A \otimes_{\mathbb{Z}} \mathbb{Z}/(p) = 0$. Therefor $\times p : A \to A$ is an isomorphism.

If n is any positive integer, then n can be factored as a product of (possibly repeated) primes $p_1 p_2 \dots p_r$, and similarly $\times n = (\times p_1) \circ (\times p_2) \circ \cdots \circ (\times p_r)$.

Since $\times n$ is a composite of isomorphisms, it is an isomorphism.

(b) Suppose $B : \mathbb{Q} \times \mathbb{Q} \to A$ is a $\mathbb{Z}$-bilinear map. We will produce a linear map $f : \mathbb{Q} \to A$ making the diagram below commute, and prove that f is unique with this property:

$$\begin{tikzcd}
\mathbb{Q} \times \mathbb{Q} \arrow[rr, "\mu"] \arrow[dr, "B"'] & & \mathbb{Q} \arrow[dl, dashed, "f"] \\
& A. &
\end{tikzcd}$$

We define $f(p/q) = B(p/q, 1)$. Bilinearity of B implies that f is linear. To verify that the diagram commutes, we must check that $B(p/q, p'/q') = f(pp'/(qq'))$ for all rational numbers p/q, p'/q'. We may write

$$f(pp'/(qq')) = B(pp'/(qq'), 1) = p' B(p/(qq'), q'(1/q')) = q' B(p/(qq'), p'/q') = B(p/q, p'/q')$$

using bilinearity several times. This concludes the check.

Now suppose f' is some function that also makes this diagram commute. Applying commutativity of the diagram to $(p/q, 1) \in \mathbb{Q} \times \mathbb{Q}$, we see that $f'(p/q) = B(p/q, 1) = f(p/q)$. This establishes uniqueness.

(c) Using (a), the group A is uniquely divisible and is therefore a $\mathbb{Q}$-vector space: $A \cong \bigoplus_I \mathbb{Q}$. Then

$$0 = A \otimes_{\mathbb{Z}} \mathbb{Q} \cong \left(\bigoplus_I \mathbb{Q}\right) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_I (\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}) \cong \bigoplus_I \mathbb{Q} \cong A$$

where in the penultimate isomorphism, we have used part (b).

(d) Let $A = \mathbb{Q}/\mathbb{Z}$, where $\mathbb{Z}$ is included in $\mathbb{Q}$ in the obvious way. That is, there is a short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0. \tag{4}$$

Since $\mathbb{Z} \not\cong \mathbb{Q}$, the group $\mathbb{Q}/\mathbb{Z}$ is not 0.

Apply the exact functor $\mathbb{Q} \otimes_{\mathbb{Z}} -$ to (4) to get

$$0 \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z} \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$

We know what almost all the terms in this sequence are. $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$. The induced map between them is the identity, leaving $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) = 0$.

Similarly, applying $\mathbb{Z}/(p) \otimes_{\mathbb{Z}} -$ to (4) gives us

$$\mathbb{Z}/(p) \otimes_{\mathbb{Z}} \mathbb{Z} \longrightarrow \mathbb{Z}/(p) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Z}/(p) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$

and since $\mathbb{Z}/(p) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}/p\mathbb{Q} = 0$, we conclude $\mathbb{Z}/(p) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) = 0$.

Remark. An abelian group A such that $\times p : A \to A$ is surjective is called *p-divisible*.

□

5. Not to be handed in.

Let **C** and **D** be two categories. Suppose $L : \mathbf{C} \to \mathbf{D}$ and $R : \mathbf{D} \to \mathbf{C}$ are functors with the property that for all objects $c \in \mathbf{C}$ and $d \in \mathbf{D}$, there are bijections

$$\mathbf{D}(L(c), d) \leftrightarrow \mathbf{C}(c, R(d))$$

that are natural in both c and d.

(Two functors with this property are said to be *adjoint* functors, with L being the *left adjoint* and R the *right adjoint*.)

(a) If p is a pushout of $z \leftarrow x \to y$ in **C**, prove that $L(p)$ is a pushout of $L(z) \leftarrow L(x) \to L(y)$ in **D**.

(b) Suppose

$$A \xrightarrow{f} B \xrightarrow{q} C \to 0$$

is an exact sequence. Prove that C is a pushout of $0 \leftarrow A \xrightarrow{f} B$.

(c) Let D be another abelian group. You may assume $- \otimes_{\mathbb{Z}} D$ is left adjoint to $\mathrm{Hom}(D, -)$, both functors being $\mathrm{Ab} \to \mathrm{Ab}$. Use adjunction to prove that

$$A \otimes_{\mathbb{Z}} D \xrightarrow{f \otimes \mathrm{id}_D} B \otimes_{\mathbb{Z}} D \xrightarrow{q \otimes \mathrm{id}_D} C \otimes_{\mathbb{Z}} D \longrightarrow 0$$

is an exact sequence.

(a) We wish to show that $L(p)$ satisfies the universal property of pushouts. Suppose $w \in \mathbf{D}$ is an object and that the square

$$\begin{tikzcd} L(x) \arrow[r] \arrow[d] & L(y) \arrow[d] \\ L(z) \arrow[r] & w \end{tikzcd}$$

commutes. Using the natural bijections $\mathbf{D}(L(c), d) \leftrightarrow \mathbf{C}(c, R(d))$, we obtain a corresponding commuting square

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[d] \\ z \arrow[r] & R(w) \end{tikzcd}$$

in **C**. Since p is a pushout of $z \leftarrow x \rightarrow y$, there is a unique map $\phi : p \to R(w)$ such that the two triangles in

$$\begin{tikzcd}
x \arrow[r] \arrow[d] & y \arrow[d] \arrow[ddr, bend left] & \\
z \arrow[r] \arrow[drr, bend right] & p \arrow[dr, "\phi"] & \\
& & R(w)
\end{tikzcd}$$

commute. By adjunction, we obtain a map $\bar{\phi} : L(p) \to w$ which makes the two triangles in

$$\begin{tikzcd}
L(x) \arrow[r] \arrow[d] & L(y) \arrow[d] \arrow[ddr, bend left] & \\
L(z) \arrow[r] \arrow[drr, bend right] & L(p) \arrow[dr, "\bar{\phi}"] & \\
& & w
\end{tikzcd}$$

commute. Since ϕ is uniquely determined and $\mathbf{D}(L(p), w) \leftrightarrow \mathbf{D}(p, R(w))$ is a bijection, the map $\bar{\phi}$ is uniquely determined.

(b) Suppose $g : B \to D$ is a map with $g \circ f = 0$. In other words, suppose

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d, "g"] \\
0 \arrow[r] & D
\end{tikzcd}$$

commutes. We define a map $\phi : C \to D$ by $c \mapsto g(b)$, where $b \in B$ satisfies $q(b) = c$. To see that ϕ is well-defined, suppose $q(b) = q(b') = c$. Then $q(b - b') = 0$, so $b - b' = f(a)$ for some $a \in A$. Then $q(b) - q(b') = q(b - b') = q \circ f(a) = 0$. Also, we observe $\phi \circ q = g$ by construction. To show that ϕ is the unique map with this property, suppose $\phi' : C \to D$ is another map satisfying $\phi' \circ q = g$. If $c \in C$ and $b \in B$ such that $q(b) = c$, then $\phi \circ q(b) = \phi' \circ q(b)$, or $\phi(c) = \phi'(c)$ (in other words, surjections in **Ab** are epimorphisms).

(c) First, we establish the converse of (b): if

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d, "q"] \\
0 \arrow[r] & C
\end{tikzcd}$$

is a pushout square, then $A \xrightarrow{f} B \xrightarrow{q} C \to 0$ is an exact sequence.

The sequence

$$A \xrightarrow{f} B \xrightarrow{q'} B/\operatorname{im}(f) \to 0$$

is exact, where q' is the natural quotient map. Using part (b), the abelian group $B/\operatorname{im}(f)$ is also a pushout of $0 \leftarrow A \xrightarrow{f} B$. There is thus a unique isomorphism $\phi : B/\operatorname{im}(f) \to C$ making

$$\begin{tikzcd}
B \arrow[r, "q'"] \arrow[dr, "q"'] & B/\operatorname{im}(f) \arrow[d, "\phi", "\cong"'] \\
& C
\end{tikzcd}$$

commute. So $A \xrightarrow{f} B \xrightarrow{q} C \to 0$ is exact.

Now we prove (c). By part (b), we know C is a pushout of $0 \leftarrow A \xrightarrow{f} B$, and part (a) implies that $C \otimes_{\mathbb{Z}} D$ is a pushout of

$$0 \longleftarrow A \otimes_{\mathbb{Z}} D \xrightarrow{f \otimes \mathrm{id}_D} B \otimes_{\mathbb{Z}} D.$$

Using the converse of (b), we deduce that

$$A \otimes_{\mathbb{Z}} D \xrightarrow{f \otimes \mathrm{id}_D} B \otimes_{\mathbb{Z}} D \xrightarrow{q \otimes \mathrm{id}_D} C \otimes_{\mathbb{Z}} D \longrightarrow 0$$

is exact.

□