Math 427, 527
Homework 6
2026-04-09

1. Suppose A is an abelian group. Recall that an element $a \in A$ is said to be *torsion* if there exists a positive integer n such that $na = 0$, and *nontorsion* otherwise. Let $T(A)$ denote the subgroup of torsion elements in A. Let **Ab** denote the category of all abelian groups and homomorphisms, and **fgAb** denote the full subcategory of finitely generated abelian groups.

Prove that there does not exist a "nontorsion subgroup" functor, even for finitely generated abelian groups. That is, prove there is no functor $F : \mathbf{fgAb} \to \mathbf{fgAb}$ with the following two properties:

(a) There exists a natural transformation $\mu : F \to \mathrm{id}$ that yields an injection $\mu_A : F(A) \to A$ for all $A \in \mathbf{fgAb}$.

(b) For all $A \in \mathbf{fgAb}$, there is an isomorphism $\mathrm{im}(\mu_A) \oplus T(A) \cong A$.

Suppose for the sake of contradiction that such a functor exists. Let $A = \mathbb{Z} \oplus \mathbb{Z}/(2)$.

The subgroup $T(A)$ consists of $\{(0,0),(0,1)\}$. Since $F(A) \oplus \mathbb{Z}/(2) \cong \mathbb{Z} \oplus \mathbb{Z}/(2)$, the classification of finitely generated modules over abelian groups, specifically, the uniqueness part of this classification, implies that $F(A)$ is an infinite cyclic group. It is embedded by μ_A as a subgroup of A, and we write B for the image $\mathrm{im}(\mu_A)$. The group B is generated by some element $(a,b) \in \mathbb{Z} \oplus \mathbb{Z}/(2)$, where $a \neq 0$ (since the group is of infinite order). Without loss of generality, we may suppose $a \geq 0$. In order for $(1,0)$ to be expressible as a sum of elements of B and $T(A)$, the equation

$$x(a,b) + y(0,1) = (1,0)$$

must have a solution in integers x, y. We deduce that $a = 1$.

There is an isomorphism $\phi : A \to A$ given by the formula $\phi(x,y) = (x, \bar{x} + y)$, where $\bar{x}$ denotes the reduction of x modulo 2. Since the inclusion $F(A) \to A$ is natural, the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\mu_A} & A \\ \downarrow{\scriptstyle F(\phi)} & & \downarrow{\scriptstyle \phi} \\ F(A) & \xrightarrow{\mu_A} & A \end{array}$$

commutes. In particular, $\phi(B) \subset B$. Taking the difference between two elements of B gives us $(0,1) = (1,b) - (1,1+b) \in B$. Since $(0,1)$ is 2-torsion, but $B \cong \mathbb{Z}$, this is a contradiction. □

2. On p55 of the textbook, a method for producing *Moore spaces* $M(A,k)$ is given. Here A is an abelian group and k is a positive integer. These spaces satisfy

$$\tilde{\mathrm{H}}_q(M(A,k);\mathbb{Z}) \cong \begin{cases} A & \text{if } k = q; \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the following graded abelian groups:

(a) $\mathrm{H}_*(M(\mathbb{Z}/(3),2) \times M(\mathbb{Z}/(9),4);\mathbb{Z})$;

(b) $\mathrm{H}_*((\mathbb{R}\mathrm{P}^2)^{\times 3}; F)$ where F is a field (the answer should be given in terms of F and will depend on the characteristic of F).

(c) $\mathrm{H}^*((\mathbb{R}\mathrm{P}^2)^{\times 3};\mathbb{Z})$.

(a) Using the Künneth formula (all tensor products taken over $\mathbb{Z}$):

$$H_q(M(\mathbb{Z}/(3),2) \times M(\mathbb{Z}/(9),4);\mathbb{Z}) \cong \begin{cases} \mathbb{Z} \otimes \mathbb{Z} & \text{if } q = 0; \\ \mathbb{Z}/(3) \otimes \mathbb{Z} & \text{if } q = 2; \\ \mathbb{Z} \otimes \mathbb{Z}/(9) & \text{if } q = 4; \\ \mathbb{Z}/(3) \otimes \mathbb{Z}/(9) & \text{if } q = 6; \\ \mathrm{Tor}(\mathbb{Z}/(3), \mathbb{Z}/(9)) & \text{if } q = 7; \\ 0 & \text{otherwise.} \end{cases}$$

These admit simplifications. Notably, $\mathbb{Z}/(3) \otimes \mathbb{Z}/(9) \cong \mathbb{Z}/(\gcd(3,9)) = \mathbb{Z}/(3)$, and $\mathrm{Tor}(\mathbb{Z}/(3), \mathbb{Z}/(9)) \cong \mathbb{Z}/(3)$ (since $\mathbb{Z}/(3)$ is the subgroup of 9-torsion elements in $\mathbb{Z}/(3)$, for instance).

This gives us the answer

$$H_q(M(\mathbb{Z}/(3),2) \times M(\mathbb{Z}/(9),4);\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } q = 0; \\ \mathbb{Z}/(3) & \text{if } q = 2,6,7; \\ \mathbb{Z}/(9) & \text{if } q = 4; \\ 0 & \text{otherwise.} \end{cases}$$

(b) We know $H_*(\mathbb{R}P^2; F) = F$, concentrated in degree 0, if F is of characteristic different from 2. We also know that if F is of characteristic 2, then $H_q(\mathbb{R}P^2; F) = F$ if $q \in \{0,1,2\}$ and is 0 otherwise. These calculations were derived from the cellular chain complex in lecture.

Over a field F, all modules are free, so that $\mathrm{Tor}_1^F(A,B) = 0$. The Künneth formula over a field therefore says

$$H_n(X \times Y; F) \cong \bigoplus_{p+q=n} H_p(X;F) \otimes_F H_q(Y;F).$$

In this question, all homology groups are F-vector spaces, so they are determined up to isomorphism by their dimension, which happens to be a nonnegative integer.

If F is a field of characteristic different from 2, then $H_*(\mathbb{R}P^2 : F)$ is the homology of a point. Since the homology of a product depends only on the homologies of the terms involved (up to isomorphism), we see that $H_*((\mathbb{R}P^2)^3; F)$ is the homology of a point, i.e., pt × pt × pt as well.

From now on, we assume that the characteristic of F is 2. An easy way to record Künneth calculations for $H_*(X \times Y; F)$ is to make a table whose p,q-th entry is $\dim_F(H_p(X;F)) \times \dim_F(H_p(X;F))$. A diagonal line of slope -1 in this table corresponds to all pairs (p,q) with some constant sum n. Adding terms on these diagonals yields the dimension of $H_n(X \times Y; F)$. Here is the table for $\mathbb{R}P^2 \times \mathbb{R}P^2$ and a field of characteristic 2:

q \ p	0	1	2
2	1	1	1
1	1	1	1
0	1	1	1

Entries not represented in the table are 0. Summing up along the lines of slope -1 gives

$$H_q(\mathbb{R}P^2 \times \mathbb{R}P^2; F) \cong \begin{cases} F & \text{if } q = 0,4; \\ F^2 & \text{if } q = 1,3; \\ F^3 & \text{if } q = 2; \\ 0 & \text{otherwise.} \end{cases}$$

for a field F of characteristic 2.

We repeat the process for $(\mathbb{R}\mathrm{P}^2)^2 \times \mathbb{R}\mathrm{P}^2$, using the newly acquired values. We obtain

q \ p	0	1	2	3	4
2	1	2	3	2	1
1	1	2	3	2	1
0	1	2	3	2	1

Summing up along the lines of slope -1 gives

$$\mathrm{H}_q((\mathbb{R}\mathrm{P}^2)^3; F) \cong \begin{cases} F & \text{if } q = 0, 6; \\ F^3 & \text{if } q = 1, 5; \\ F^6 & \text{if } q = 2, 4; \\ F^7 & \text{if } q = 3; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

for a field F of characteristic 2.

(c) One approach is to apply the Künneth formula twice to calculate $\mathrm{H}_*((\mathbb{R}\mathrm{P}^2)^3; \mathbb{Z})$ and then universal coefficients to pass to cohomology. It may be helpful to know that

$$\mathrm{H}_q(\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } q = 0; \\ \mathbb{Z}/(2)^2 & \text{if } q = 1; \\ \mathbb{Z}/(2) & \text{if } q = 2, 3; \\ 0 & \text{otherwise.} \end{cases}$$

We adopt a different approach, however. This is based on the observation that all positive-degree (co)homology groups of $(\mathbb{R}\mathrm{P}^2)^n$, where n is a positive integer, are finitely generated 2-torsion abelian groups. This is proved by an easy induction: the property holds for $\mathbb{R}\mathrm{P}^2$ and is preserved by the Künneth formula.

Therefore, for all $q > 0$, the group $\mathrm{H}_q((\mathbb{R}\mathrm{P}^2)^3; \mathbb{Z})$ is isomorphic to $\mathbb{Z}/(2)^{r_q}$ for some r. Also, using the universal coefficients theorem for homology, we see that $\mathrm{H}_1((\mathbb{R}\mathrm{P}^2)^3; \mathbb{Z}/(2)) \cong \mathbb{Z}/(2)^{r_1}$ and $\mathrm{H}_q((\mathbb{R}\mathrm{P}^2)^3; \mathbb{Z}/(2)) \cong \mathbb{Z}/(2)^{r_q + r_{q-1}}$ if $q \geq 2$ (the special treatment of $q = 1$ is required because $\mathrm{H}_0((\mathbb{R}\mathrm{P}^2)^3; \mathbb{Z}) = \mathbb{Z}$ is not 2-torsion).

We can calculate the values of r_q iteratively from equation (1). We obtain $r_1 = 3$, $r_2 = 3$, $r_3 = 4$, $r_4 = 2$, $r_5 = 1$ and $r_q = 0$ for $q \geq 6$.

Now we use the universal coefficients theorem to calculate $\mathrm{H}^*((\mathbb{R}\mathrm{P}^2)^3; \mathbb{Z})$ from $\mathrm{H}_*((\mathbb{R}\mathrm{P}^2)^3; \mathbb{Z})$. With the exception of degree-0, all groups appearing are 2-torsion, so the Hom-terms in this theorem contribute 0, and all cohomology arises from Ext-calculations.

Since $\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}/(2)^r, \mathbb{Z}) \cong \mathbb{Z}/(2)^r$, we obtain the following answer:

$$\mathrm{H}^q((\mathbb{R}\mathrm{P}^2)^3; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } q = 0; \\ \mathbb{Z}/(2)^3 & \text{if } q = 2, 3; \\ \mathbb{Z}/(2)^4 & \text{if } q = 4; \\ \mathbb{Z}/(2)^2 & \text{if } q = 5; \\ \mathbb{Z}/(2) & \text{if } q = 6; \\ 0 & \text{otherwise.} \end{cases}$$

□

3. This question is not to be handed in. Suppose X is a path-connected space with basepoint x_0. This problem is the second part of a pair that establishes the following result: the Hurewicz map $\eta : \pi_1(X, x_0) \to \mathrm{H}_1(X; \mathbb{Z})$ is the abelianization of $\pi_1(X, x_0)$. Specifically, let $\pi_1^{\mathrm{ab}}(X, x_0)$ denote the quotient of $\pi_1(X, x_0)$ by the smallest normal subgroup containing all commutators $xyx^{-1}y^{-1}$ for all $x, y \in \pi_1(X, x_0)$. It is an abelian group, and the quotient map $\pi_1(X, x_0) \to \pi_1^{\mathrm{ab}}(X, x_0)$ is surjective. Note that elements of $\pi_1^{\mathrm{ab}}(X, x_0)$ are equivalence classes of loops in X, starting and ending at x_0.

In this question, we will routinely identify the unit interval $[0,1]$ with Δ^1 by means of the obvious linear map sending 0 to $(1,0)$ and 1 to $(0,1)$. This allows us to identify a path $[0,1] \to X$ with the 1-simplex $\Delta^1 \to X$.

For all $x \in X$, pick a 1-simplex $\tau_x : \Delta^1 \to X$. If $\sigma : \Delta^1 \to X$ is a 1-simplex with $\sigma(0) = x$ and $\sigma(1) = y$, we may define a loop $\omega(\sigma)$ as the concatenation of τ_x, σ and the reverse of τ_y, viewing these as paths. By extending linearly, we construct a homomorphism $\omega : S_1(X) \to \pi_1^{\mathrm{ab}}(X, x_0)$.

(a) Suppose ψ is a 2-simplex. Prove that $\omega(d\psi) \in \pi_1^{\mathrm{ab}}(X, x_1)$ is trivial. Deduce that there exists a map $\bar{\omega}$ making the following diagram commute

$$\begin{array}{ccc} Z_1(X) & \xrightarrow{\text{inc}} & S_1(X) \\ \downarrow & & \downarrow \omega \\ \mathrm{H}_1(X) & \xrightarrow{\bar{\omega}} & \pi_1^{\mathrm{ab}}(X, x_0) \end{array}$$

where $Z_1(X) \to \mathrm{H}_1(X)$ is the usual quotient map.

(b) If σ is a 1-simplex, then observe that $\eta(\omega(\sigma))$ is the homology class of $l(\sigma)$ from a previous homework question. Verify that $\eta \circ \bar{\omega} = \mathrm{id} : \mathrm{H}_1(X) \to \mathrm{H}_1(X)$.

(c) If $\gamma : S^1 \to X$ is a loop in X based at x_0, then check that $\bar{\omega}(\eta([\gamma])) = [\gamma] \in \pi_1^{\mathrm{ab}}(X, x_0)$. This establishes that $\bar{\omega}$ is inverse to η, which is therefore an isomorphism.

Solution not yet written. □

4. Suppose V is a finite-dimensional vector space over a field k and $f : V \to V$ is a linear endomorphism. Pick a basis $v_1, \ldots, v_m$ for V, and represent f by a matrix A with respect to this basis. The *trace* of f, denoted $\mathrm{Tr}(f)$, is the sum of the diagonal entries of A. You do not have to prove that this is independent of the choice of basis.

(a) Suppose $W \subseteq V$ is a k-linear subspace and that $f : V \to V$ restricts to give a homomorphism $f|_W : W \to W$. Define $\bar{f} : V/W \to V/W$ by the formula $\bar{f}(v + W) = f(v) + W$. This is well defined and k-linear. Prove that

$$\mathrm{Tr}(f) = \mathrm{Tr}(f|_W) + \mathrm{Tr}(\bar{f}).$$

(b) Suppose C_* is a chain complex of finite-dimensional k-vector spaces for which only finitely many C_n are nonzero. Suppose $f : C_* \to C_*$ is a chain map, which induces maps $f_* : \mathrm{H}_n(C_*) \to \mathrm{H}_n(C_*)$. Prove the identity

$$\sum_{n=-\infty}^{\infty} (-1)^n \mathrm{Tr}(f : C_n \to C_n) = \sum_{n=-\infty}^{\infty} (-1)^n \mathrm{Tr}(f_* : \mathrm{H}_n(C_*) \to \mathrm{H}_n(C_*)).$$

(c) Suppose X is a finite CW complex and $f : X \to X$ is a self map that is cellular (i.e., sends cells to cells) and has the property that f does not map any cell to itself. Let k be a field. Prove that $\mathrm{H}_*(X; k) \not\cong \mathrm{H}_*(\mathrm{pt}; k)$.

(a) Let $\{v_1,\ldots,v_r\}$ be an ordered basis of W and extend this to $\{v_1,\ldots,v_r,v_{r+1},\ldots,v_m\}$, an ordered basis of V. With respect to these ordered bases, the matrix of f takes the form

$$A_f = \begin{bmatrix} A_{f|W} & B \\ 0 & C \end{bmatrix}$$

where the block of 0s is implied by the fact that $f(W)\subset W$. Here A is an $r\times r$ matrix.

The image of $\{v_{r+1},\ldots v_m\}$ constitutes a spanning set for the $m-r$-dimensional space V/W, and therefore an (ordered) basis. With respect to this basis, the matrix of $\bar{f}$ is C.

Simple addition now implies that $\mathrm{Tr}(f|W)+\mathrm{Tr}(\bar{f})=\mathrm{Tr}(f)$.

(b) If C_* is a chain complex of k-vector spaces with only finitely many nonzero terms, and $f : C_* \to C_*$ is a chain map, then we define $\mathrm{Tr}(f)=\sum_{n=-\infty}^{\infty}(-1)^n\,\mathrm{Tr}(f : C_n \to C_n)$.

Suppose

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \xrightarrow{i} & C_* & \xrightarrow{q} & B_* & \longrightarrow & 0 \\ & & \big\downarrow f|_A & & \big\downarrow f & & \big\downarrow \bar{f} & & \\ 0 & \longrightarrow & A_* & \xrightarrow{i} & C_* & \xrightarrow{q} & B_* & \longrightarrow & 0 \end{array}$$

is a commutative diagram of such chain complexes in which the rows are exact sequences. The previous part of this question implies that in each degree, we have $\mathrm{Tr}(f : C_n \to C_n) = \mathrm{Tr}(f|_A : A_n \to A_n) + \mathrm{Tr}(\bar{f} : B_n \to B_n)$, so that taking the alternating sum over all n gives

$$\mathrm{Tr}(f) = \mathrm{Tr}(f|_A) + \mathrm{Tr}(\bar{f}). \tag{2}$$

Now consider the two graded k-vector subspaces $Z_* \subset C_*$, consisting of cycles, and $B_* \subset C_*$ consisting of boundaries. These are both promoted to the level of complexes by giving them the trivial differential 0. The inclusions $B_* \to Z_* \to C_*$ are chain maps, as is the quotient $C_* \to B_*$.

Furthermore, if $f : C_* \to C_*$ is a chain map, and if $x \in Z_n$, then $df(x) = fd(x) = 0$, so $f(x) \in Z$. Similarly, if $x \in B_n$, then $x = dy$ for some y. Then $f(x) = fd(y) = df(y) \in B_*$. If we write $\bar{f}$ for the restriction of f to B_*, then there is a diagram of short exact sequences of chain complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \longrightarrow & C_* & \xrightarrow{d} & B_{*-1} & \longrightarrow & 0 \\ & & \big\downarrow f|_A & & \big\downarrow f & & \big\downarrow \bar{f}_{*-1} & & \\ 0 & \longrightarrow & A_* & \longrightarrow & C_* & \xrightarrow{d} & B_{*-1} & \longrightarrow & 0. \end{array}$$

Note the shift of 1 degree in $\bar{f}$. Apply identity (2) to this diagram to see that

$$\mathrm{Tr}(f) = \mathrm{Tr}(f|_A) - \mathrm{Tr}(\bar{f}). \tag{3}$$

As with Z_* and B_*, consider $\mathrm{H}_*(C_*)$ as a chain complex with trivial differential. Apply identity (2) to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_* & \longrightarrow & Z_* & \longrightarrow & \mathrm{H}_*(C_*) & \longrightarrow & 0 \\ & & \big\downarrow \bar{f} & & \big\downarrow f|_A & & \big\downarrow f_* & & \\ 0 & \longrightarrow & B_* & \longrightarrow & Z_* & \longrightarrow & \mathrm{H}_*(C_*) & \longrightarrow & 0 \end{array}$$

to deduce, using (3), that

$$\mathrm{Tr}\big(f_* : \mathrm{H}_*(C_*) \to \mathrm{H}_*(C_*)\big) = \mathrm{Tr}(f|_A) - \mathrm{Tr}(\bar{f}) = \mathrm{Tr}(f : C_* \to C_*),$$

as required.

(c) Fix a field k. Since f is cellular, it induces a chain map $f_* : C_*(X;k) \to C_*(X;k)$. The n-cells constitute a basis for the finite vector space $C_n(X;k)$. Place this basis in some order and represent each f_n as a matrix. Since f does not map any cell to itself, every diagonal entry of this matrix is 0. In particular, $\mathrm{Tr}(f_* : C_*(X;k) \to C_*(X;k)) = 0$, being an alternating sum of 0s.

Suppose for the sake of contradiction that $\mathrm{H}_*(X;K) \cong \mathrm{H}_*(\mathrm{pt};k)$. This means that $f_* : \mathrm{H}_0(X;k) \to \mathrm{H}_0(X;k)$ is the identity map (it sends the class of a point to the class of a point) and $f_* = 0$ in all other degrees. Since $\mathrm{Tr}(\mathrm{id} : k \to k) = 1$, we calculate $\mathrm{Tr}(f_* : \mathrm{H}_*(X;k) \to \mathrm{H}_*(X;k)) = 1$, a contradiction.

□

5. Let n be a positive integer.

This problem uses some covering-space theory, which we now explain. If $x = (x_0,\dots,x_n)$ is a point in S^n, then we write $-x = (-x_0,\dots,-x_n)$. The quotient $S^n/(x \sim -x)$ is $\mathbb{RP}^n$, and the quotient map $q : S^n \to \mathbb{RP}^n$ is a covering space map. It is known that $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/(2)$ except when $n = 1$, when $\mathbb{RP}^n \approx S^1$. Pick a basepoint $s_0 \in S^n$ and let $q(s_0) = x_0$. The map $q_* : \pi_1(S^n, s_0) \to \pi_1(\mathbb{RP}^n, x_0)$ is injective and its image consists of elements of $\pi_1(\mathbb{RP}^n, x_0)$ that are of the form γ^2.

Throughout the rest of the question, we fix a continuous function $f : S^n \to S^n$ that is *odd* in that $f(-x) = -f(x)$ for all $x \in S^n$. There is an induced function $\bar{f} : \mathbb{RP}^n \to \mathbb{RP}^n$, since $\mathbb{RP}^n = S^n/(x \sim -x)$. Let γ be a based loop in $\mathbb{RP}^n$ that represents a generator of $\pi_1(\mathbb{RP}^n, x_0)$. As a path, γ lifts to a path $\tilde{\gamma}$ in S^n starting at s_0 and ending at $-s_0$ (this follows from covering space theory, and does not require proof here).

If we compose $\bar{f}$ and γ we obtain a loop, based at $\bar{f}(x_0)$, whose lift to S^n starting at $f(s_0)$ is $f(\tilde{\gamma})$. It starts at $f(s_0)$ and ends at $f(-s_0) = -f(s_0)$, so that it is not a loop. Therefore the image of the map $f_* : \pi_1(\mathbb{RP}^n, x_0) \to \pi_1(\mathbb{RP}^n, f(x_0))$ contains at least one element, the class of $f(\gamma)$, that is not a square in $\pi_1(\mathbb{RP}^n, f(x_0))$.

Let C_q denote the mapping cone of the quotient map. One checks from the cell structure that

$$C_q \approx \mathbb{RP}^{n+1}.$$

By construction, there is a natural map $\mathbb{RP}^n \to C_q$, which we do not name. By the Van Kampen theorem, the functorial map $\pi_1(\mathbb{RP}^n) \to \pi_1(C_q)$ is surjective—it is an isomorphism except when $n = 1$, in which case it is isomorphic to the reduction-modulo-2 map $\mathbb{Z} \to \mathbb{Z}/(2)$.

Since the left square in the diagram below commutes, there is an induced continuous map $\hat{f}$ making the whole diagram commute:

$$\begin{array}{ccccc} S^n & \xrightarrow{q} & \mathbb{RP}^n & \longrightarrow & C_q \\ \big\downarrow f & & \big\downarrow \bar{f} & & \big\downarrow \hat{f} \\ S^n & \xrightarrow{q} & \mathbb{RP}^n & \longrightarrow & C_q. \end{array}$$

Using functoriality of π_1, we deduce that the map $\hat{f}_* : \pi_1(C_q) \to \pi_1(C_q)$ is an isomorphism.

(a) Prove that $\hat{f}^* : \mathrm{H}^*(C_q;\mathbb{F}_2) \to \mathrm{H}^*(C_q;\mathbb{F}_2)$ is an isomorphism. Hint: use the Hurewicz map and the cup product. You may assume that $\mathrm{H}^*(\mathbb{RP}^m;\mathbb{F}_2) \cong \mathbb{F}_2[x_1]/(x_1^{m+1})$.

(b) Prove that $f_* : \mathrm{H}_n(S^n;\mathbb{Z}) \to \mathrm{H}_n(S^n;\mathbb{Z})$ is multiplication by an odd integer, i.e., $\deg(f)$ is odd.

(c) Prove there does not exist an odd (i.e., satisfying $g(-x) = -g(x)$ for all x) continuous function $h : S^n \to S^{n-1}$.

(d) Suppose $h : S^n \to \mathbb{R}^n$ is a continuous function. By considering the expression $\frac{h(s)-h(-s)}{\|h(s)-h(-s)\|}$, prove that there is some pair of points $s, -s \in S^n$ with the property that $h(s) = h(-s)$.

(a) Since the cohomology $\mathrm{H}^*(C_q;\mathbb{F}_2)$ has the elements $\{x_1^r\}_{r=0}^{n+1}$ as a basis, it suffices to prove that $\hat{f}^*(x_1^r) = x_1^r$ for all r. The case of $r = 0$ is trivial.

We are told that $\hat{f}_* : \pi_1(C_q) \to \pi_1(C_q)$ is an isomorphism. We also know that the Hurewicz map is natural and in this case $\eta : \pi_1(C_q) \to \mathrm{H}_1(C_q;\mathbb{Z}) \cong \mathrm{H}_1(\mathbb{RP}^{n+1};\mathbb{Z}) = \mathbb{Z}/(2)$ is an isomorphism. Therefore $\hat{f}_* : \mathrm{H}_1(C_q;\mathbb{Z}) \to \mathrm{H}_1(C_q;\mathbb{Z})$ is an isomorphism. Using universal coefficients for cohomology, we see the natural map $\mathrm{Hom}(\mathrm{H}_1(C_q;\mathbb{Z}),\mathbb{F}_2) \to \mathrm{H}^1(C_q;\mathbb{F}_2)$ is an isomorphism. Using naturality, we deduce $\hat{f}^* : \mathrm{H}^1(C_q;\mathbb{F}_2) \to \mathrm{H}^1(C_q;\mathbb{F}_2)$ is an isomorphism, i.e., $\hat{f}^*(x_1) = x_1$.

Using naturality of the cup product, we deduce $\hat{f}^*(x_1^r) = x_1^r$ for all r, which is what we needed to show.

(b) Using the universal coefficient theorem for cohomology, and in particular its naturality properties, we deduce that $f^* : \mathrm{H}^n(S^n;\mathbb{F}_2) \to \mathrm{H}^n(S^n;\mathbb{F}_2)$ is multiplication by the class of $\deg(f)$ in $\mathbb{Z}/(2)$. To prove that $\deg(f)$ is odd, it suffices to prove that this homomorphism f^* is an isomorphism.

Now consider the commutative ladder diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \mathrm{H}^n(\mathbb{RP}^n;\mathbb{F}_2) & \longrightarrow & \mathrm{H}^n(S^n;\mathbb{F}_2) & \xrightarrow{\delta} & \mathrm{H}^{n+1}(C_q;\mathbb{F}_2) & \longrightarrow & \mathrm{H}^{n+1}(\mathbb{RP}^n;\mathbb{F}_2) & \longrightarrow & \cdots \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle f^*} & & \big\downarrow{\scriptstyle \hat{f}^*} & & \big\downarrow & & \\
\cdots & \longrightarrow & \mathrm{H}^n(\mathbb{RP}^n;\mathbb{F}_2) & \longrightarrow & \mathrm{H}^n(S^n;\mathbb{F}_2) & \xrightarrow{\delta} & \mathrm{H}^{n+1}(C_q;\mathbb{F}_2) & \longrightarrow & \mathrm{H}^{n+1}(\mathbb{RP}^n;\mathbb{F}_2) & \longrightarrow & \cdots .
\end{array}$$

Since $\mathrm{H}^{n+1}(\mathbb{RP}^n;\mathbb{F}_2) = 0$, the coboundary maps δ are surjective, and since their sources and targets are all $\mathbb{F}_2$, we obtain a commutative square

$$\begin{array}{ccc}
\mathrm{H}^n(S^n;\mathbb{F}_2) & \xrightarrow{\delta} & \mathrm{H}^{n+1}(C_q;\mathbb{F}_2) \\
\big\downarrow{\scriptstyle f^*} & & \big\downarrow{\scriptstyle \hat{f}^*} \\
\mathrm{H}^n(S^n;\mathbb{F}_2) & \xrightarrow{\delta} & \mathrm{H}^{n+1}(C_q;\mathbb{F}_2),
\end{array}$$

in which all arrows except f^* are already known to be isomorphisms. It follows f^* is an isomorphism, i.e., $\deg(f)$ is odd.

(c) There exists an odd function $i : S^{n-1} \to S^n$: the standard inclusion of S^{n-1} as an equator in S^n will work. Suppose $h : S^n \to S^{n-1}$ is a continuous function. Then $i \circ h$ is not surjective, so has degree 0, and therefore is not odd. The composition of two odd functions is odd, so it follows that h is not odd.

(d) Suppose for the sake of contradiction that $h(s) \neq h(-s)$ for all $s \in S^n$. Let $h(s) = \dfrac{h(s) - h(-s)}{\|h(s) - h(-s)\|}$; which is a continuous function $h : S^n \to S^{n-1}$. It is odd by direct inspection, contradicting the result of the previous part of this question.

Remark. The result in this question is known as the *Borsuk–Ulam theorem*. In the case of $n = 1$, it is a straightforward corollary of the intermediate value theorem, which can also be deduced directly from it. We view the Borsuk–Ulam theorem as a higher dimensional generalization of the intermediate value theorem.

□