HOMEWORK 2

Due on 24 February 2020.

(1) Let $\mathbf{C}$ be a category with a set of objects and a set of morphisms. For a given integer $m \geq 0$, let $N\mathbf{C}_m$ be the set of sequences

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \xrightarrow{f_3} \dots \xrightarrow{f_m} c_m$$

of composable morhpisms in $\mathbf{C}$. The set $N\mathbf{C}_0$ consists of the objects of $\mathbf{C}$. Define face maps $d_i : N\mathbf{C}_m \to N\mathbf{C}_{m-1}$ by omitting c_i. In the case of $i = 0$ or $i = m$, we omit f_1 or f_m as well. In the other cases, replace the maps $c_{i-1} \xrightarrow{f_i} c_i \xrightarrow{f_{i+1}} c_{i+1}$ by the composite $c_{i-1} \xrightarrow{f_{i+1} \circ f_i} c_{i+1}$. Define degeneracy maps $s_i : N\mathbf{C}_m \to N\mathbf{C}_{m+1}$ by adding an identity map id_{c_i} in the sequence in the obvious place.

(a) Verify that $N\mathbf{C}_\bullet$ forms a simplicial set.

(b) Show that $N\mathbf{C}_\bullet$ is a quasicategory.

(c) Prove that if every morphism in $\mathbf{C}$ is an isomorphism ($\mathbf{C}$ is a groupoid), then $N\mathbf{C}_\bullet$ is a Kan complex.

(2) Suppose $X_\bullet$ is a Kan complex and $x \in X_2$ is a nondegenerate 2-simplex. Prove X contains a nondegenerate 3 simplex.

(3) You may assume that for any k-space, the natural map $|\operatorname{Sing} X| \to X$ is a weak equivalence.

(a) Give an example of a simplicial set A and a weak equivalence of simplicial sets $B \to B'$ such that $\operatorname{Map}(A, B) \to \operatorname{Map}(A, B')$ is not a weak equivalence.

(b) Show that if B and B' are also assumed to be Kan complexes, then $\operatorname{Map}(A, B) \to \operatorname{Map}(A, B')$ is a weak equivalence. Deduce that $\operatorname{Map}(A, B)$ is weakly equivalent to $\operatorname{Map}(A, \operatorname{Sing}|B|)$—this whole question should just be an exercise in Quillen adjunctions.

(c) Show that if B is a Kan complex, then $\operatorname{Map}(|A|, |B|)$ is weakly equivalent to $|\operatorname{Map}(A, B)|$ (i.e., there is an isomorphism in the homotopy category between these two objects).

(4) Identify S^1 with the set of unit complex numbers. Let $L : \mathbf{K} \to \mathbf{K}$ denote the free loop space functor, that is, $LX = \operatorname{Map}(S^1, X)$. Show that $LS^1 \simeq \mathbb{Z} \times S^1$. Hint A: maps $S^1 \to S^1$ admit a pointwise multiplication $fg(x) := f(x)g(x)$, which satisfies $\deg(fg) = \deg(f) + \deg(g)$—you can use singular homology to prove this. This allows you to show that LS^1 decomposes into homeomorphic subspaces, each corresponding to a different degree. Hint B: you can find a homotopy fibre sequence relating LS^1, ΩS^1 and S^1.