HOMEWORK 3

Due on 10 March 2020.

(1) If X is a set, the *constant presheaf* with value X is the presheaf $\mathcal{F}(U) = X$ for all U (all restrictions being identity maps). A *constant sheaf* is a sheaf isomorphic to $a\mathcal{F}$ where $\mathcal{F}$ is constant (one generally speaks of "the" constant sheaf).

  (a) Let X be a set with more than one element. Let $\mathcal{X}$ be the constant sheaf with value X on the topological space S^1. Let $U \subseteq S^1$ be an open subset. Describe $\mathcal{X}(U)$.

  (b) Identify S^1 with the unit complex numbers (with the usual topology). Consider the sheaf $\mathcal{Q}$ on S^1 that assigns to an open subset $U \subset S^1$ the set of continuous functions $\phi : U \to \mathbb{C}^\times$ such that for all $u \in U$, the identity $\phi(u)^2 = u$ holds. Prove this is a sheaf. Consider also the constant sheaf $*$ with value $\{*\}$. There is a unique map $\mathcal{Q} \to *$. Prove this is a sheaf epimorphism but $\mathcal{Q}(S^1) \to *(S^1)$ is not surjective.

(2) Prove that $\mathbb{A}^n_k \setminus \{0\}$ represents the functor sending R to unimodular rows of length n in R^n.

(3) Suppose

$$\begin{tikzcd}
U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"] \\
U \arrow[r, "i"] & X
\end{tikzcd}$$

is an elementary Nisnevich square of varieties (so i is an open embedding, p is an étale map and p restricted to $p^{-1}(X \setminus U)$ is an isomorphism).

  (a) Prove that the diagonal map $V \xrightarrow{\Delta} V \times_X V$ is (formally) étale.

  (b) Prove that $\{\Delta : V \to V \times_X V, i : U \times_X V \times_X V \to V \times_X V\}$ is a Nisnevich covering of $V \times_X V$.

  (c) Prove that for any Nisnevich sheaf $\mathcal{F}$, the equalizer of

$$\mathcal{F}(U) \times \mathcal{F}(V) \rightrightarrows \mathcal{F}(U \times_X V) \times \mathcal{F}(V \times_X V)$$

  (maps being the obvious ones—ask me if you don't think they're obvious) is actually the equalizer of

$$\mathcal{F}(U) \times \mathcal{F}(V) \rightrightarrows \mathcal{F}(U \times_X V).$$

(4) If $\mathcal{G}$ is a sheaf of groups and $\mathcal{X}$ is a sheaf, then it is possible to define a *left action* of $\mathcal{G}$ on $\mathcal{X}$ as a map $\alpha : \mathcal{G} \times \mathcal{X} \to \mathcal{X}$ making certain diagrams commute.

  (a) Explicitly set out what those diagrams are.

  (b) Define $\mathcal{X}/\mathcal{G}$ using a coequalizer diagram in the category of sheaves.

  (c) Consider the constant sheaf of groups C_2, cyclic of order 2, on the big étale site of $\mathbf{Sm}_\mathbb{C}$. If X is a connected variety, then $C_2(X) = \{1, g\}$. This group acts on $\mathbb{G}_m$: if $\operatorname{Spec} R$ is a connected affine variety, then g acts on $\mathbb{G}_m(R) = R^\times$ by $g \cdot r = -r$. Prove that the map $\mathbb{G}_m \to \mathbb{G}_m$ given by $r \mapsto r^2$ identifies $\mathbb{G}_m/C_2$ with $\mathbb{G}_m$ in the big étale site.

  (d) Prove that in the big Nisnevich site, $\mathbb{G}_m/C_2$ (with the same action) is not isomorphic to $\mathbb{G}_m$.