HOMEWORK 5

Due on 21 April 2020

(1) This question is about homotopy pullbacks. Suppose

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

is a diagram of pointed fibrant objects in a simplicial model category. Let P denote the homotopy limit of this diagram.

There exists an inclusion $\partial\Delta[1] \to \Lambda^2[2]$ given by including the two endpoints of the Λ.

(a) Produce a pullback (not a homotopy pullback) square

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Z^{\Lambda^2[2]} \arrow[d] \\ X \times Y \arrow[r, "f{,}g"] & Z \times Z \end{tikzcd}$$

(b) Hence produce a homotopy fibre sequence

$$\Omega Z \to P \to X \times Y.$$

The associated long exact sequence of homotopy groups (when it exists) is called the *homotopy Mayer–Vietoris sequence.*

(2) Suppose $\mathbf{M}$ is a right proper model category. In the last part, it is also assumed to be simplicial. Suppose

$$\begin{tikzcd} X_1 \arrow[rr, "f_1"] \arrow[dr, "g_1"] \arrow[dd, "\phi_X"'] & & Y_1 \arrow[dr, "h_1"] \arrow[dd, "\sim"', "\phi_Y"] & \\ & Z_1 \arrow[rr, two heads, "j_1"] \arrow[dd, "\phi_Z"', "\sim"] & & W_1 \arrow[dd, "\sim"', "\phi_W"] \\ X_2 \arrow[rr, "f_2"] \arrow[dr, "g_2"] & & Y_2 \arrow[dr, "h_2"] & \\ & Z_2 \arrow[rr, two heads, "j_2"] & & W_2 \end{tikzcd}$$

is a commutative diagram where all objects are fibrant, the top and bottom squares are pullback squares, the indicated vertical arrows (ϕ_Y, ϕ_Z and ϕ_W) are weak equivalences and in which j_1 and j_2 are fibrations.

(a) Suppose further that h_1 and h_2 are fibrations and that the square

$$\begin{tikzcd} Z_1 \arrow[r, "j_1"] \arrow[d, "\phi_Z"'] & W_1 \arrow[d, "\phi_W"] \\ Z_2 \arrow[r, "j_2"] & W_2 \end{tikzcd}$$

is a pullback square. Prove that ϕ_X is a weak equivalence.

(b) Now suppose h_1 and h_2 are fibrations. Let $Z' = Z_2 \times_{W_2} W_1$. Do not assume the natural map $Z_1 \to Z'$ is a weak equivalence. By comparing with the diagram

$$\begin{tikzcd} Z' \times_{W_1} Y_1 \arrow[rr, "f'"] \arrow[dr, "g'"] \arrow[dd, "\phi'_X"'] & & Y_1 \arrow[dr, "h_1"] \arrow[dd, "\sim"', "\phi_Y"] & \\ & Z' \arrow[rr, two heads, "j'"] \arrow[dd, "\phi'_Z"', "\sim"] & & W_1 \arrow[dd, "\sim"', "\phi_W"] \\ X_2 \arrow[rr, "f_2"] \arrow[dr, "g_2"'] & & Y_2 \arrow[dr, "h_2"] & \\ & Z_2 \arrow[rr, two heads, "j_2"] & & W_2 \end{tikzcd}$$

prove that ϕ_X in the original diagram is a weak equivalence.

(c) Now do not assume h_1 and h_2 are fibrations. By exploiting functorial factorizations of h_1 and h_2 as trivial cofibrations followed by fibrations, prove that ϕ_X is a weak equivalence.

(d) Suppose

$$\begin{tikzcd} & Y \arrow[d, two heads] \\ Z \arrow[r] & W \end{tikzcd}$$

is a diagram of fibrant objects and the indicated morphism is a fibration. Prove that the canonical map $Y \times_W Z \to Y \times^h_W Z$ is a weak equivalence.

Remark 1. A dual problem to this one, in a left proper model category, applies to the pushout and homotopy pushout.

(3) Let $n \geq 1$. Consider a horn inclusion $\Lambda^i[n] \to \Delta[n]$. Form the normalized chain complex map $j : N\mathbb{Z}\Lambda^i[n] \to N\mathbb{Z}\Delta[n]$ in the category of chain complexes of abelian groups.

(a) Let $Q[n]$ denote the chain complex having $\mathbb{Z} \to \mathbb{Z}$ in the n-th and $n-1$-st positions, an isomorphism between them, and 0s elsewhere. Show that there is a split short exact sequences of chain complexes

$$0 \to N\mathbb{Z}\Lambda^i[n] \xrightarrow{j} N\mathbb{Z}\Delta[n] \to Q[n] \to 0$$

Hint: $N\mathbb{Z}X$ is a model for the cellular chain complex of $|X|$. Use the homology long exact sequence of a pair.

(b) Suppose $f : X_* \to Y_*$ is a map of nonnegatively-graded chain complexes of abelian groups, and that f is surjective in each degree other than possibly 0. Using the previous part and adjunctions, prove that $K(f)$ is a fibration.

(4) For the purposes of this question, a *flasque* sheaf is a Nisnevich sheaf $\mathcal{F}$ such that $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective whenever $U \hookrightarrow X$ is a Zariski open embedding. An *injective* sheaf is an injective object in the abelian category of (Nisnevich) sheaves. You may assume injective sheaves are flasque, and that there are enough injectives, i.e., it is always possible to form an injective resolution for each sheaf of abelian groups.

Fix a Nisnevich sheaf of abelian groups A on $\mathbf{Sm}_k$.

(a) Let

$$0 \to A \to I_0 \stackrel{d_0}{\to} I_1 \stackrel{d_1}{\to} I_2 \to \dots$$

be an injective resolution. Form the *good truncation* $\tau_{\leq n} I_*$ as

$$0 \to I_0 \stackrel{d_0}{\to} I_1 \stackrel{d_1}{\to} I_2 \to \cdots \to I_{n-1} \to \ker(d_n) \to 0.$$

Shift this complex to form $\tau_{\leq n} I_*[-n]$, a chain complex concentrated in degrees 0 to n, with I_0 in degree n. Prove that $K\tau_{\leq n} I_*[-n]$ has the Brown–Gersten proprty. (Hint: use the previous question).

(b) Let $X \in \mathbf{Sm}_k$. Explain why the homology of the truncated complex above, i.e.: $\mathrm{H}_i(\tau_{\leq n} I_*[-n](X))$ is isomorphic to the cohomology $\mathrm{H}^{n-i}_{Nis}(X, \mathcal{A})$. This should just be a matter of restating definitions and very mild homological algebra, but you may have to look up these definitions, depending on your background.

(c) Define the *Eilenberg–Mac Lane (simplicial) presheaf* $K(A, n)$ by applying K to $A[-n]$, the complex having A in degree n and 0 elsewhere. Show that $K(A, n) \to K\tau_{\leq n} I_*[-n]$ is a Nisnevich local equivalence.

(d) Suppose X is an object of $\mathbf{Sm}_k$. Show that there are isomorphisms:

$$\pi_i(R_{Nis} K(A, n)(X)) \cong \mathrm{H}^{n-i}_{Nis}(X, A).$$

Remark 2. The main point of this exercise, that $\pi_i(R_t K(A, n)(X)) \cong \mathrm{H}^{n-i}_t(X, A)$, holds in any local homotopy theory, not just the Nisnevich one. But in general, we do not have a pleasant Brown–Gersten condition to exploit in order to get the result, so the proof is harder.