Lecture Notes on $\mathbb{A}^1$ homotopy - v1.4.39

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Chapter 1

Sites

1.1 Sheaves

The original reference for the notion of a sheaf, in this sense, is [AGV72]. The book [MM92] is very readable and an excellent reference as well. The mammoth resource [dJon17]is excellent on this topic as well. For the category theory that we assume, the resource [Mac98] is the definitive reference.

We start with the notion of a *sheaf* on a topological space.

Definition 1.1. Given a space X we let $o(X)$ denote the category of open sets of X under inclusion. A *presheaf* on X is a functor

$$\mathscr{F} : o(X)^{\mathrm{op}} \to \mathbf{Set}$$

One forms a category of presheaves, $\mathbf{Pre}(X)$, by defining the morphisms to be the natural transformations. By convention, the elements of $\mathscr{F}(U)$ are called *sections* of $\mathscr{F}$ on U. This is also denoted $\Gamma(\mathscr{F}, U)$. One may define the notion of a presheaf of groups, rings and so on similarly. The key idea here is that there are *restriction maps*: $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$ when $V \subset U$.

Definition 1.2. We define a *sheaf* on X to be a presheaf $\mathscr{F}$ satisfying the following axiom:

If $\{U_i\}_{i \in I}$ form an open cover of V, then the following diagram is an equalizer

$$\mathscr{F}(V) \longrightarrow \prod_{i \in I} \mathscr{F}(U_i) \rightrightarrows \prod_{i \in I, J} \mathscr{F}(U_i \cap U_j) \tag{1.1}$$

An *equalizer* is a categorical limit. In practice, this means one may identify $\mathscr{F}(V)$ with the subset of elements $(u_i) \in \prod_{i \in I} \mathscr{F}(U_i)$ such that the restrictions on pairwise intersections agree: $\rho_{ij}(u_i) = \rho_{ji}(u_j)$.

One defines a category of sheaves, $\mathbf{Sh}(X)$, by simply restricting the class of objects. A morphism between sheaves is still just a natural transformation.

Example 1.3.
1. The set of continuous ($\mathbb{R}$-valued) functions on X is a sheaf.
2. The set of bounded ($\mathbb{R}$-valued) functions on X may not be a sheaf if X is not compact.

1.2 Grothendieck Topologies

We generalize the above example by replacing the open covers by a generalization.

Definition 1.4. Let $\mathbf{C}$ be a category. The category of (set-valued) presheaves on $\mathbf{C}$ is the category of functors $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$. We write $\mathbf{Pre}(\mathbf{C})$.

Example 1.5. Given an object $c \in \mathbf{C}$, I can define a presheaf y_c on $\mathbf{C}$ by setting $y_c(d) = \mathrm{Mor}(d, c)$, with the evident restriction maps. Given a morphism $c \to c'$, there is a natural transformation $y_c \to y'_c$ (i.e. a morphism of presheaves). This sets up a functor:

$$y : \mathbf{C} \to \mathbf{Pre}(\mathbf{C}).$$

It is left as an exercise to prove that this is an embedding: i.e. that the natural map $\mathrm{Mor}(c, d) \to \mathrm{Mor}(y_c, y_d)$ is a bijection. The embedding is called the *Yoneda embedding*.

It's important for later application that the Yoneda embedding commutes with limits.

Definition 1.6. A *sieve*, S, on $c \in \mathrm{ob}\mathbf{C}$ is a subfunctor of y_c.

This is a confusing definition, so let's look at it in a more elementary way. Given $f : d \to c$, i.e. an element of $y_c(d)$, we can ask whether $f \in S(d)$. If $f \in S(d)$ and we have a composite $f \circ g : d' \to d \to c$, then $f \circ g \in S(d')$.

If you pretend that the category $\mathbf{C}$ has a set of objects and a set of morphisms, then a sieve $S(c)$ is a subset of the morphisms with codomain c that is closed under left composition.

Definition 1.7. Suppose we have a map $h : c' \to c$ and we have a sieve S on c, then we form the pull-back sieve $h^* S$ on c' by the following rule:

$$h^*(S)(d) = \{g : d \to c' : (h \circ g : d \to c) \in S(d).$$

The verification that this is a sieve is left as an exercise.

Definition 1.8. A *Grothendieck topology*, J, on **C** is an assignment to each object c of **C** of a collection $J(c)$ of sieves on c such that:

1. The maximal sieve $y_c \in J(c)$.

2. If $S \in J(c)$ and $h : c' \to c$ is a map, then $h^*(c) \in J(c')$.

3. If $S \in J(c)$ and if R is a sieve on c such that $h^*(R) \in J(d)$ whenever $h \in S(d)$, then $R \in J(d)$.

The sieves $J(c)$ are said to be *covering sieves* for the topology. This looks complicated and abstract, but there are ways to cope. What we will do is ignore this definition and work with bases instead.

Definition 1.9. Suppose **C** has finite limits. A *basis* for a Grothendieck topology is a function K which assigns to each object c a collection $K(c)$ sets of morphisms $\{f_i : c_i \to c\}_{i\in I}$ in c such that

1. All isomorphisms, as singleton sets, are in K

2. If $\{f_i : c_i \to c\}_{i\in I} \in K(c)$, and if $g : c' \to c$ is any morphism, then the pullback family $\{f_i \times_c g : c_i \to c'\}_{i\in I}$ is in $K(c')$.

3. If $\{f_i : c_i \to c\}_{i\in I} \in K(c)$ and if for each c_i, we have a family $\{g_{i,j} : d_{i,j} \to c_i\}_j$ in $K(c_i)$, then $\{f_i \circ g_{i,j} : d_{i,j} \to c\}_{i,j}$ is in $K(c)$.

These morphisms will be called *K–coverings*, or, loosely and incorrectly, *covering*.

Warning: this does not have a lot in common with the notion of a basis of a topology in the point-set sense.

Second warning: people often specify the basis and call it 'the topology'. A basis K generates a topology J as follows.

$$S \in J(c) \Leftrightarrow \exists R \in K(c),\ R \subset S.$$

Third warning: two different bases may generate the same topology (i.e. have the same covering sieves). If **C** has finite limits, then there is a maximal basis for J, and some may call any element of this basis 'covering' for the topology.

Example 1.10. Let $o(X)$ denote the category of open sets of X. Here is a definition of a basis K. For each V, define $K(V)$ to be the set of families of subsets $\{U_i\}_{i\in I}$ of V that cover V. This forms a basis as above.

The associated sieves $J(V)$ are the following. $S \in J(V)$ is a covering sieve if there exists some open cover $\{U_i\}_{i\in I}$ of V such that $(f : W \hookrightarrow V) \in S(V)$ if and only if W is contained in some U_i.

Definition 1.11. In the presence of a basis K, and pullbacks in $\mathbf{C}$, we define a sheaf. For all basic covering families $\{f_i : y_i \to x\}$, the following diagram is an equalizer

$$\mathscr{F}(x) \longrightarrow \prod_{i \in I} \mathscr{F}(y_i) \rightrightarrows \prod_{i,j \in I} \mathscr{F}(y_i \times_x y_j) \tag{1.2}$$

The category of sheaves $\mathbf{Sh}(\mathbf{C})_K$ is the full subcategory of the category of prehseaves $\mathbf{Pre}(\mathbf{C})$ where the objects are sheaves. That is, a morphism between sheaves is just a morphism of presheaves.

Example 1.12. With $o(X)$ as before, we may define a basis by letting $K(V)$ consist of ordinary covering families. With this definition, once we remember that $U_i \times_X U_i = U_i \cap U_j$, we have recovered the 'topological' definition of a sheaf.

Sheaves can be defined directly from the topology, but we will not need this.

Question: if x is an object of $\mathbf{C}$, is the presheaf y_x actually a sheaf? In our examples, generally the answer will be 'yes'. The object y_x is a*representable* presheaf, and a topology for which all representable presheaves are sheaves is called *subcanonical.*

1.3 The associated sheaf functor

Proposition 1.13. *The limit of a diagram of sheaves is again a sheaf.*

Proof. Limits commute with limits, see [Mac98]. □

Corollary 1.14. *A monomorphism of sheaves is a monomorphism of presheaves, which is defined objectwise.*

Proposition 1.15. *The inclusion (forgetful) functor* $\mathbf{Sh}(\mathbf{C}) \to \mathbf{Pre}(\mathbf{C})$ *has a left adjoint, a, called the 'associated sheaf' functor, or the 'sheafification' functor. It commutes with finite limits.*

We will not give the proof of this in class, you can consult [MM92, Chapter IV]

Corollary 1.16. *The category of sheaves has all small limits and all small colimits.*

Chapter 2

Schemes

For the most part in this course we will do our algebraic geometry relative to a base field, k. It can be done more generally, and perhaps we will touch on that.

Since we will want to read [MV99], the best thing to do is to set $S = \operatorname{Spec} k$, but make a mental note that S may be taken to be an arbitrary noetherian scheme without harming the set up of the theory.

Our reference for the algebraic geometry is [Har77] or [Vak15].

2.1 Varieties

The main object of study is finite type, separated, smooth k-schemes. We will let $\mathbf{Sch}_k$ denote the category of all finite type, separated k-schemes, and let $\mathbf{Sm}_k$ denote the category of finite type separated smooth k-schemes. If you like, $\mathbf{Sm}_k$ is similar to, or identical, to the category of smooth varieties over k. The chief difference is that we allow disconnected objects.

The point of this course is to do 'homotopy theory', whatever that is, with the category $\mathbf{Sm}_k$. The first big problem is that homotopy theory makes big categorical demands that $\mathbf{Sm}_k$ cannot meet. For instance, $\mathbf{Sm}_k$ does not have a lot of colimits.

To this end, we construct $\mathbf{Pre}(\mathbf{Sm}_k)$. This category has all limits and all colimits, and there is a Yoneda embedding $\mathbf{Sm}_k \to \mathbf{Pre}(\mathbf{Sm}_k)$ in it.

Embedding schemes into $\mathbf{Pre}(\mathbf{Sm}_k)$ is the basis of the 'functor of points' methodology in algebraic geometry, [EH00]. As a convention, let if X is a scheme and $\operatorname{Spec} R$ is an affine scheme, write

$$X(R) = \operatorname{Mor}_{\mathbf{Sch}}(\operatorname{Spec} R, X) = y_X(\operatorname{Spec} R).$$

Let $\mathbf{Aff}_k$ denote the category of affine schemes in $\mathbf{Sm}_k$. It is well known that $\mathbf{Aff}_k$ is the opposite category of a category of k-algebras, i.e. $\mathrm{Mor}(\operatorname{Spec} R, \operatorname{Spec} S) = \mathrm{Hom}(S, R)$.

We now list some varieties which will be with us throughout the course:

1. $\mathbb{A}^n_k$. This is $\operatorname{Spec} k[x_1, \dots, x_n]$. From the functor of points point of view, $\mathbb{A}^n_k(R) = R^n$.

2. $\mathbb{A}^n_k - \{0\}$. This is not affine, unless $n = 1$. From the functor of points p.o.v.

$$(\mathbb{A}^n_k - \{0\})(R) = U(R^n)$$

a fact which we will leave as an exercise later. In the case $n = 1$, we get $\mathbb{A}^1 - \{0\}(R) = R^\times$.

3. P^n_k. This represents

$$R^{n+1} \to \mathscr{L} \to 0$$

up to action by $R^\times$.

2.2 The Nisnevich Topology

We embed $\mathbf{Sm}_k \to \mathbf{Pre}(\mathbf{Sm}_k)$ because we want to be able to form colimits. It is worthwhile to note that:

Proposition 2.1. *The Yoneda embedding preserves limits.*

Proof. Exercise. □

Example 2.2. The Yoneda embedding does not preserve colimits, even when they are straightforward.

For instance, the following diagram is a pushout of schemes

$$\begin{tikzcd}
\mathbb{G}_m \times \mathbb{G}_m \arrow[r] \arrow[d] & \mathbb{A}^1 \times \mathbb{G}_m \arrow[d] \\
\mathbb{G}_m \times \mathbb{A}^1 \arrow[r] & \mathbb{A}^2 - \{0\}
\end{tikzcd}$$

The scheme $\mathbb{A}^2 - \{0\}$ represents the functor $R \mapsto U_2(R)$ sending R to pairs of elements $r, s \in R$ such that r, s generate the unit ideal. This fact has been advertised previously, and will appear as an exercise later.

The colimit of presheaves represents pairs (r, s) of elements in R where at least one element is a unit in R.

For instance, the morphism $\operatorname{Spec} k[t] \to \mathbb{A}^2 - \{0\}$ given by $(t, t+1)$ appears in $\mathbb{A}^2(k[t])$, but does not arise from either a morphism $\operatorname{Spec} k[t] \to \mathbb{A}^1 \times \mathbb{G}_m$ or a morphism $\operatorname{Spec} k[t] \to \mathbb{G}_m \times \mathbb{A}^1$.

So passing to presheaves has caused us to lose geometric information, and (vaguely) this loss seems to be to do with patching things together. Using a (Grothendieck) topology will help with this.

Definition 2.3. A *standard étale* map is a map isomorphic to one of the form $R \to (R[x]/(f))_g$, where f, g are polynomials, f is monic and f' is invertible in $(R[x]/(f))_g$

Definition 2.4. A map of rings $f : S \to R$ is *finitely presented* if it is isomorphic to a map $S \to S[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Since everything we look at will be noetherian, we can run this together with *finite type*: $S[x_1, \dots, x_n]/I$. A map of schemes $f : X \to Y$ is *locally finitely presented* if, for each affine open $\operatorname{Spec} B$ in Y, $f^{-1}(\operatorname{Spec} B)$ may be written as a union of affine opens $\operatorname{Spec} A_i$ such that the maps $f : B \to A_i$ are finitely presented.

We give this definition because it's what's required for the most general setup. But we will only ever talk about noetherian schemes, in which case this is the same as *locally of finite type*. If we work over a field, then the schemes we are talking about are finite type over a field and all morphisms between them are of *finite type*. So you can ignore this definition because all maps may be assumed to have this property.

Definition 2.5. A map of schemes $f : X \to Y$ is said to be *étale* if it is locally of finite type and satisfies the following unique lifting condition

$$\begin{tikzcd} \operatorname{Spec} A/I \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where I is a square-0 ideal. One may assume A is a local ring.

Remark 2.6. From this definition, the following observations are immediate: the composite, pull-back of étale maps are étale.

Not proved: this is equivalent to other definitions of étale. We refer to [dJon16] in particular for proofs. One may also consult [Gro66]*Exposé 17.

1. (Locally of finite presentation), flat and unramified.

2. (Locally of finite presentation), flat and for every $y \in Y$, the fibre X_y is a disjoint union of finite, separable field extensions of $\kappa(y)$.

Proposition 2.7. *If $f : X \to Y$ is an étale morphism. Then for every $x \in X$ and V an affine open neighbourhood of $f(x)$, there is an affine open $U \ni x$ such that $f(U) \subset V$ and $f|_U : U \to V$ is standard étale.*

Proof. [dJon16, Tag 02GT] □

Conversely, being étale is a local condition, and standard étale maps are étale. Warning: not every étale map between affine schemes is standard étale.

Example 2.8. The following map is standard étale and therefore étale: $f_n : \mathbb{G}_m \to \mathbb{A}^1$ on the level of rings by $k[y] \mapsto k[t, t^{-1}]$ as $y \mapsto t^n$, provided char $k \not| n$. Indeed, we can write $k[t] = k[y,t]/(t^n - y)$, which is monic. The derivative is nt^{n-1}, which is not 0 unless $t = 0$. Then we may localize by making t invertible, so $t \neq 0$.

Example 2.9. An open immersion is étale.

Example 2.10. A product of separable field extensions is étale.

Definition 2.11. A family of maps $\{f_i : X_i \to Y\}$ is an *étale covering* if each map f_i is étale, and the f_i are jointly surjective.

We will lose nothing by assuming this family is finite, so we will add this as a restriction. In the non-noetherian case it may matter.

Definition 2.12. A family of maps $\{f_i : X_i \to Y\}$ is a *Nisnevich covering* if it is an étale cover and, for every $y \in Y$, there exists some point $x \in X_i$ such that $f_i(x) = y$ and the induced map on residue fields is an isomorphism.

This definition is due to [Nis89].

Remark 2.13. Suppose given any étale cover and two points $x \in X_i$ and $y \in Y$ such that $f_i(x) = y$. To study the map on residue fields we may assume the map f_i is standard étale: i.e. the map on residue fields is obtained as the map on residue fields you get by taking residue fields (localizing and modding out) on a ring map such as

$$\mathrm{Spec}(R[t]/(f))_{(g)} \to \mathrm{Spec}\, R.$$

where f must be monic and have invertible derivative. Since we're working locally, we may ignore the (g). A little work shows one arrives at ultimately one arrives at a map of fields $\mathrm{Spec}\, \kappa[t]/(f_0(t)) \to \mathrm{Spec}\, \kappa$ where f_0 is a (monic) irreducible factor of $f \in \kappa[t]$, and where f_0 has no repeated roots in a splitting field. That is, the map is a separable finite field extension.

Example 2.14. Away from the characteristic, the n-fold cover $\mathbb{G}_m \to \mathbb{G}_m$ is an étale cover (just by itself). It is not a Nisnevich cover when $|n| > 1$, in particular because of what happens at the generic point. Adding in the inclusion $\mathbb{G}_m - \{x_1, \dots, x_s\} \to \mathbb{G}_m$ makes it a Nisnevich cover exactly when the points $\{x_i\}$ have at least one n-th root in the ground field.

Remark 2.15. In the situation where we consider only finite families $\{f_i : X_i \to Y\}_{i=1}^n$, we may form the disjoint union $\bigcup_{i=1}^n X_i \to Y$, so there's no restriction in assuming the covering family is a singleton.

There are alternative descriptions of the Nisnevich coverings that are sometimes worthwhile.

Proposition 2.16. *A map $f : X \to Y$ is a Nisnevich covering if and only if it is an étale covering and there exists a stratification of the base $\emptyset = Y_0 \subset Y_1 \subset \cdots \subset Y_N = Y$ such that*

1. *Each Y_{i-1} is closed in Y_i*
2. *Write X_i for $f^{-1}(Y_i)$. There exist morphisms $s_i : Y_i - Y_{i-1} \to X_i - X_{i-1}$ splitting $f|_{X_i - X_{i-1}}$.*

Proof. Suppose we have local sections. Given any point $y \in Y$, it lies in some stratum $Y_i - Y_{i-1}$. Consider the points $\{x_j\} = f^{-1}(y)$. The maps on residue fields correspond to separable finite field extensions $\kappa(y) \to \kappa(x_i)$, and the existence of the section implies one of these is necessarily an isomorphism.

Conversely, suppose we have a Nisnevich map with irreducible target Y. We will find a generically-defined section. Let $\xi \in X$ lie above the generic point $\eta \in Y$ such that $\kappa(y) \cong \kappa(x)$, and find a standard étale map around ξ and η. Such a map is of the form $\operatorname{Spec} R[t]/(f) \to \operatorname{Spec} R$, and we may assume both $R[t]/(f)$ and R are integral domains, and the induced map on fraction fields is an isomorphism. This implies that f is linear and so the map $\operatorname{Spec} R[t]/(f) \to \operatorname{Spec} R$ is an isomorphism, and therefore has a section.

The proof now proceeds by noetherian induction. □

Proposition 2.17. *Both étale and Nisnevich coverings satisfy the requirements to be a basis for a Grothendieck topology on* $\mathbf{Sm}_k$.

Proof. Let's do the étale case first.

In the first place, an isomorphism is an étale covering. This is obvious.

Second, if $f : X \to Y$ is an étale covering, and $g : W \to Y$ is a map of schemes, then, if we denote $g^{-1}(X)$ by U, a diagrammatic argument shows that there is a unique lift in

$$\begin{tikzcd}
\operatorname{Spec} A/I \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} A \arrow[r] \arrow[ur, dashed] \arrow[urr, dashed] & W \arrow[r] & Y
\end{tikzcd}$$

That implies the pullback of an étale map is étale (up to checking finite presentation, which we do not care about).

The pullback of a surjective map of schemes is surjective, this is an exercise in scheme theory: [Vak15, Exercise 9.4.D]. This settles axiom 2 in the étale case.

In our current setup, axiom 3 simply asks that a composite of surjective étale maps be surjective étale. This is not hard.

As for the Nisnevich case, we further have to check the "completely decomposed" part of the coverings are preserved by pullback and composition. Composition is clear.

For pullback, we argue as follows. There is a stratification of Y and there exist sections of $f : X \to Y$ on the locally closed strata. These are preserved by pullback. $\square$

Remark 2.18. We observe in passing that the Zariski topology is coarser (fewer coverings) than the Nisnevich, which is coarser than the étale.

2.3 Interlude on Morphisms of Schemes

Definition 2.19. Suppose X is a topological space and $\mathscr{P}$ is a presheaf on X (i.e., on $o(X)$). Let $x \in X$ be a point. We define the *stalk* of $\mathscr{P}$ at x, denoted $x^*\mathscr{P}$, as the colimit

$$\operatorname*{colim}_{U \ni x} \mathscr{P}(U).$$

This is a special case of a *filtered colimit*, in that the diagram over which we take the limit is *cofilitered*. For any two U, U' in the diagram, there exists a U'' such that $U \to U''$ and $U' \to U''$ appear in the diagram, and for any two arrows $f, g : U \to U'$, there exists some further composite $h : U' \to U''$ so that $h \circ f = h \circ g$. In our case, $h = \mathrm{id}_{U'}$ works, but in a general filtered colimit, you might need nontrivial h.

Remark 2.20. A map of presheaves $\mathscr{P} \to \mathscr{P}'$ induces an isomorphism on associated sheaves if and only if it induces an isomorphism on all stalks. This is an exercise in the topological case, and will be important later in a more general setting.

Exercise 2.21. If $x \in X$ is a point, then taking the stalk x^* preserves finite limits and all colimits of sheaves on X.

An affine scheme consists of a topological sheaf $X = \operatorname{Spec} R$ and a sheaf of rings $\mathscr{O}_X$ on X. The sheaf $\mathscr{O}_X$ is often called R in an abuse of notation. If $U \subset \operatorname{Spec} R$ is the open subset obtained by discarding the closed subscheme defined by $r = 0$, then $\mathscr{O}_X(U) = R_x$ (i.e., invert r).

Exercise 2.22. Let $p \in \operatorname{Spec} R$ be a point (i.e., a prime ideal). Then $p^*(\mathscr{O}_X) = R_p$, the localization of R at p.

A scheme is a topological space X equipped with a sheaf of rings $\mathcal{O}_X$ such that X may be covered by affine schemes $\operatorname{Spec} R$ in such a way that $\mathcal{O}_X$ restricts to R on $\operatorname{Spec} R$.

Definition 2.23. A topological space with a sheaf of rings is called a *ringed space*. A ringed space $(X, \mathcal{O})$ such that the stalks of $\mathcal{O}$ are local rings is called a *locally ringed space*. These are specific examples of a *ringed site* (a site with a sheaf of rings) and a *locally ringed site*—it's slightly harder to formulate the 'local' condition in general. See [Gro68].

Morphisms of sites

Suppose $(\mathbf{C}, J)$ and $(\mathbf{C}', J')$ are categories with Grothendieck topologies (we're specifying bases for these topologies here. Suppose further for simplicity that both $\mathbf{C}$ and $\mathbf{C}'$ have all pullbacks.

Given a functor $f^{-1} : \mathbf{C}' \to \mathbf{C}$, we may define a pushforward functor on presheaves $f_*(P)(U) = P(f^{-1}(U)$. You may verify that this functor preserves limits, and so by the adjoint functor theorem, [Mac71], this functor has a left adjoint, $f^*_{\text{pre}} : \mathbf{Pre}_{\mathbf{C}'} \to \mathbf{Pre}_{\mathbf{C}}$.

Definition 2.24. A *continuous map of sites* $f : (\mathbf{C}, J) \to (\mathbf{C}', J')$ is a functor $f^{-1} : \mathbf{C}' \to \mathbf{C}$ such that f_* preserves sheaves.

In general, f^*_{pre} does not preserve sheaves, so we define f^* to be the composite of f^*_{pre} with the associated sheaf functor: $f^* : \mathbf{Sh}_{J'}(\mathbf{C}') \to \mathbf{Pre}(\mathbf{C}) \to \mathbf{Sh}_J(\mathbf{C})$.

Exercise 2.25. If f^{-1} preserves pull back diagrams of objects and takes covering families in J' to covering families in J, then f^{-1} is continuous.

Example 2.26. If $f : X \to Y$ is a continuous map of topological spaces, then $f^{-1} : o(Y) \to o(X)$ is a functor on categories of open subsets (given by $U \mapsto f^{-1}(U)$ in the point-set sense). Then $f_*(P)(U) = P(f^{-1}(U))$.

As an exercise, verify that f^*_{pre} is given by

$$f^*_{\text{pre}}(Q)(U) = \operatorname*{colim}_{V \supset f(U)} Q(U).$$

(Compare [Vak15, Exercise 2.6.B]).

To prove f is continuous, we check that f^{-1} preserves (finite) intersections and open covers, which it does.

Definition 2.27. A *morphism of sites* is a continuous map of sites such that f^* preserves finite limits.

Exercise 2.28. A continuous function $f : X \to Y$ between topological spaces induces a morphism of sites—to figure out what f^* does to finite limits of sheaves, examine the behaviour on stalks x^*. It was an exercise earlier to prove that taking stalks preserves finite limits.

Back to schemes

A morphism of schemes $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of locally ringed spaces. We haven't defined what this means yet, so here it is.

First of all, there is a continous function $f : X \to Y$ of topological spaces. Then there is a map of ring objects $\phi : f^*\mathcal{O}_Y \to \mathcal{O}_X$ such that on a stalk x^*, the map $x^*(\phi) : x^* f^* \mathcal{O}_Y \to x^*\mathcal{O}_X$ is a map of local rings in that the maximal ideal is mapped to the maximal ideal (the complement of the maximal ideal consists of invertible elements, so the complements are mapped one to the other).

If $f : X \to Y$ is a map of topological spaces, and $x \in X$, and P is a presheaf on Y, then the map on stalks $x^*(f^*P) \cong f(x)^*(P)$.

Proposition 2.29. *Suppose X, Z are schemes and $\{U_i\}_{i\in I}$ is an open cover of X. Suppose morphisms are given $f_i : U_i \to Z$ such that the restrictions $f_i|_{U_j} : U_i \cap U_j \to Z$ and $f_j|_{U_i} : U_i \cap U_j \to Z$ agree. Then there exists a unique morphism of schemes $f : X \to Z$ such that $f|_{U_i} = f_i$.*

In other words, the presheaf $\mathrm{Mor}_{\mathbf{Sch}}(\cdot, Z)$ on $o(X)$ which we might (slightly inaccurately) denote y_Z is a sheaf.

Proof. By easy point-set topology, the continuous functions f_i glue to give a unique continuous function $f : X \to Z$.

There are extra conditions to check. First, that there is a map of structure sheaves $f^*\mathcal{O}_Z \to \mathcal{O}_X$, but this map exists on an open cover $\{U_i\}$ of X, and can be assembled from that data. Second we should verify that this is a map of locally ringed spaces, but this can be tested on stalks, which are calculated locally, so the result follows. □

Proposition 2.30. *Suppose $(\mathbf{C}, J)$ is a site, assumed to have pull-backs, as usual. Suppose further that the Yoneda presheaves y_X are in fact sheaves, i.e. the Yoneda embedding factors through a functor*

$$y : \mathbf{C} \to \mathbf{Sh}_J(\mathbf{C})$$

(also denoted y). Then, if $\{f_i : U_i \to X\}$ is a covering family in J, then the obvious diagram

$$\coprod_{i,i} y_{U_i \times_X U_j} \rightrightarrows \coprod_i y_{U_i} \to y_X$$

is a coequalizer diagram.

In order to prove this, we employ a lemma

Lemma 2.31. *There is a natural isomorphism.*

$$\mathrm{Mor}_{\mathbf{Pre}}(y_U, \mathscr{F}) = \mathscr{F}(U).$$

Proof. This is part of Yoneda's lemma. In outline, the proof proceeds as: Given an element $x \in of \mathscr{F}(U)$, one may produce a map of presheaves

$$\hat{x} : y_U \to \mathscr{F}$$

as follows. For any V, define the map $\hat{x}(V)$ by $\hat{x}(V) : y_U(V) \to \mathscr{F}(U)$ by sending a map $f : V \to U$ to the restriction of x along f, i.e., the image of x under $\mathscr{F}(U) \to \mathscr{F}(V)$. One verifies that $\hat{x}$ is a map of presheaves.

Conversely, given a map of presheaves $f : y_U \to \mathscr{F}$, we may define an element x of $\mathscr{F}(U)$ by $x = f(\mathrm{id}_U)$. One verifies that these two constructions are inverse to one another, and natural in $\mathscr{F}$ and U.

□

Proof of Proposition. Let $\mathscr{F}$ be a sheaf and consider the result of applying the functor $\mathrm{Hom}_{\mathbf{Sh}_J(\mathbf{C})}(\cdot, \mathscr{F})$ to the diagram in the proposition. By use of the lemma, we obtain

$$\prod_{i,j} \mathscr{F}(U_i \times_X U_j) \leftleftarrows \prod_i \mathscr{F}(U_i) \leftarrow \mathscr{F}(X)$$

which is an equalizer diagram since the family $\{U_i \to X\}$ is covering and $\mathscr{F}$ is a sheaf.

Unwinding what this equalizer says: we see that a map $y_X \to \mathscr{F}$ is equivalent to the data of a map

$$f : \coprod_{i \in I} y_{U_i} \to \mathscr{F}$$

such that the two restrictions to maps

$$\coprod_{i,j \in I} y_{U_i \times_X U_j} \to \mathscr{F}$$

agree. But this is precisely the defining property to say y_X is the coequalizer of the two maps

$$\coprod_{i,j \in I} y_{U_i \times U_j} \rightrightarrows \coprod_{i \in I} y_{U_i}.$$

□

Example 2.32. A special case of the above result is the statement that if U, V are open subschemes of X such that $X = U \cup V$, then the diagram

$$\begin{tikzcd} y_{U\cap V} \arrow[r] \arrow[d] & y_U \arrow[d] \\ y_V \arrow[r] & y_X \end{tikzcd}$$

is a colimit of sheaves on the big Zariski—and on the big étale and Nisnevich, see below—site.

2.3.1 Outline of an argument for FPQC descent of morphisms

We continue to cut corners by assuming facts about étale morphisms.

Definition 2.33. A morphism $f : X \to Y$ is *flat* if $\mathcal{O}_x$ is flat over $\mathcal{O}_{f(x)}$ for all $x \in X$.

Exercise 2.34 (Ring theoretic). A map of rings $f : A \to B$ is flat (i.e., B is flat as an A-module) iff the induced map $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat.

Definition 2.35. A morphism $f : X \to Y$ is *faithfully flat* if it is flat and surjective.

Exercise 2.36 (Ring theoretic). A map of rings $f : A \to B$ is faithfully flat (i.e., B is faithfully flat as an A-module) iff the induced map $\operatorname{Spec} B \to \operatorname{Spec} A$ is faithfully flat.

1. étale morphisms are flat.
2. flat morphisms are open.
3. surjective flat morphisms are faithfully flat.

Definition 2.37. A morphism of schemes $f : Y \to X$ is quasicompact if, for any affine open $\operatorname{Spec} A \subset X$, the set $f^{-1}(\operatorname{Spec} A)$ can be covered by finitely many affine open sets.

As with similar notions of finiteness of schemes, this property will not really matter for us, because it's automatically going to be satisfied for the schemes we think about (finite type k-schemes).

Given a ring map $f : A \to B$, we may view B as an A-module via f. We may set up a map

$$d : B \to B \otimes_A B$$

by $b \mapsto b \otimes 1 - 1 \otimes b$.

Lemma 2.38. *If $f : A \to B$ is a faithfully flat map of rings, then the complex*

$$0 \to A \to B \to B \otimes_A B$$

is exact.

Proof. First we prove that $0 \to A \to B$ is exact. We can check this after applying $\cdot \otimes_A B$. Let A' and B' be the rings so obtained. Then $A' \to B'$ has a section μ (multiplication) , and we can write $B' = A' \oplus I$, where I is a B'-module.

Now $B' \otimes_{A'} B' = A' \otimes A' \oplus A' \otimes I \oplus I \oplus A' \oplus I \otimes I$ (all tensors over A'). If $i \in I$ is not 0, then $d(i) = i \otimes 1 - 1 \otimes i$, which is not 0 since it's the difference of nonzero elements in different summands. It's easy to check that $\ker(d) = A'$, and we're done. □

Proposition 2.39. *Suppose $f : A \to B$ is a faithfully flat map of rings, interpreted as the inclusion of a subring, and $h : C \to B$ is a map of rings such that the two composites $C \to B \to B \otimes_A B$ agree. Then there exists a unique lift $h' : C \to A$ such that $f \circ h' = h$.*

Proof. Consider the short exact sequence

$$0 \to A \to B \to B \otimes_A B$$

of A modules. Apply $\mathrm{Hom}_{\mathbb{Z}}(C, \cdot)$ to this exact sequence to deduce that there is a sequence

$$0 \to \mathrm{Hom}(A, C) \to \mathrm{Hom}(B, C) \to \mathrm{Hom}(B \otimes_A B, C)$$

Now h is an element of the middle group that vanishes in the right-hand group. Therefore there exists a map $h' : C \to A$ (as abelian groups) such that h is $f \circ h'$. But this implies that the image of h in B is contained in $A \subset B$, and it follows that h' is a ring map (it's the same on elements). □

The above is the result we want to prove in the case of affine schemes. Now let's set about generalizing it.

Proposition 2.40. *Let $f : Y \to X$ be a faithfully flat and quasi-compact map of schemes, and let Z be a scheme. Suppose $g : Y \to Z$ is a morphism such that the two composites $Y \times Y \to Y \to Z$ agree. Then there exists a unique descent of g to $g' : X \to Z$ such that $g' \circ f = g$.*

Proof.

1. We reduce to the case where Z is affine. Replace Z by $Z' = \operatorname{Spec} C$ and replace Y by $g^{-1}(Z')$, and X by $X' = f(g^{-1}(Z))$. Since f is an open map, this last is an open set in X. If we can construct g' (uniquely) on an open cover of X, such as that provided by these X', then by Zariski gluing, we're finished.

2. We reduce to the case where X' is affine. Let $X'' = \operatorname{Spec} A$ be an affine open. Since it suffices to produce the map g' on an open cover, checking that the maps we've produced agree on the overlaps, this is enough. We write Y'' for $f^{-1}(X')$.

3. We now reduce to the case where Y'' is itself affine. Since f was assumed quasicompact, and X'' is affine, Y'' is covered by a finite set of affine open subsets $\{\operatorname{Spec} B_i\}$. Now write $B = \prod_i B_i$. Now $\operatorname{Spec} B = \coprod_i \operatorname{Spec} B_i$ is an affine open cover. We replace $g : Y'' \to Z'$ by $\tilde{g} : \operatorname{Spec} B \to Z'$. I leave it to you to verify that it's enough to show we can force this $\tilde{g}$ to descend to $X'' = \operatorname{Spec} A$.

4. Now we are wholly in the affine case, and we're done by a previous proposition.

$\square$

Chapter 3

Simplicial Sheaves

The modern reference for simplicial sets is [GJ99]. The older references are [May92], [Cur71]. The expository paper [Fri12] is a good place to understand these objects geometrically.

Definition 3.1. A *simplicial object* in a category **C** is a family of objects $\{X_n\}_{n=0}^{\infty}$ equipped with *face* d_i and *degeneracy* s_i maps, as follows: For each n there are $n+1$ face maps

$$d_i : X_n \to X_{n-1}.$$

There are $n+1$ degeneracy maps

$$s_i : X_n \to X_{n+1}.$$

And these satisfy simplicial identities

$$\begin{aligned} d_i d_j &= d_{j-1} d_j \quad i < j \\ s_i s_j &= s_j s_{i-1} \quad i > j \\ d_i s_j &= \begin{cases} s_{j-1} d_i & i < j \\ \mathrm{id} & i = j, \text{or } i = j+1 \\ s_j d_{i-1} & i > j+1 \end{cases} \end{aligned}$$

One may define a category of simplicial objects in **C**—the maps of simplicial objects have to commute with the simplicial structure maps. We'll denote this category by **sC**. Objects are often denoted $X_\bullet$.

The object X_n consists of the *n-simplices* of $X_\bullet$. If the category **C** has the notion of an "image" lying around, and unions, the union of the images of the degeneracy maps are the *degenerate* simplicies. The others, in so far as this is a meaningful utterance, are the nondegenerate simplicies.

Two categories are really important in this course.

1. Simplicial sets: **sSet**
2. Simplicial presheaves on a category: **sPre**(**C**).

We give an alternative construction of this category. Let Δ denote the category where the objects are the sets $\mathbf{n} = \{0, \ldots, n\}$ and the maps are opposite weakly increasing maps. Call $d^i : \mathbf{n} \to \mathbf{n+1}$, skipping i, the standard i-th coface map (for n), and $s^i : \mathbf{n} \to \mathbf{n-1}$ given by duplicating i is the i-th standard codegeneracy (for n).

Exercise 3.2. A functor $X : \Delta^{\mathrm{op}} \to \mathbf{C}$ is equivalent to a simplicial object in $\mathbf{C}$ as defined above. The map $X(d^i)$ is $d_i : X_n \to X_{n+1}$ and similarly for the degeneracies. Verify he maps that are called for as $X(f)$ where f is a more general map in the category Δ may be written as composites of these standard maps (this is all done in [GJ99]).

Definition 3.3. A functor $X : \Delta \to \mathbf{C}$ is called a *cosimplicial object* in $\mathbf{C}$. It can also be described in terms of relations between maps, as in the case of simplicial objects, but we will not write down those relations here.

Definition 3.4. The standard topological n-simplex, $|\Delta^n|$, is the set of solutions in $[0,1]^{n+1}$ of $\sum_{i=0}^n x_i = 1$.

The standard n-simplices assemble to give a standard cosimplicial object $|\Delta^\bullet|$. The cosimplicial maps are given as follows: coface maps are given by including $d^i : (x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_i, 0, x_{i+1}, \ldots, x_n)$ and codegeneracy maps are given by adding two neighbouring coordinates.

Definition 3.5. The standard *algebraic* n-simplex, Δ^n_{alg} is the variety $k[x_0, \ldots, x_n]/(\sum x_i - 1)$. These also assemble to give a cosimplicial scheme object, with the coface and codegeneracies as above.

Definition 3.6. The *geometric realization* of a simplicial set is a topological space constructed as follows

$$|X| = \coprod_{n=0}^{\infty} X_n \times |\Delta^n| / \sim$$

where $(x, p) \sim (y, q)$ if for some i, $d_i x = y$ and $d^i q = p$ or $s_i x = y$ and $s^i q = p$.

The motivating idea is that a simplicial set is combinatorial data that allow one to reconstruct a topological space assembled out of simplices.

Example 3.7. An ordinary object X of $\mathbf{C}$ may be viewed as a simplicial object by putting $X_0 = X$ and defining the higher X_i to be entirely degenerate.

3.1 Digression on slice categories

In addition to the previously-cited sources on sheaves and Grothendieck topologies ([dJon17], [MM92]), we also recommend [Fan+05, Chapter 2].

Definition 3.8. Suppose $\mathbf{C}$ is a category, and U is an object of $\mathbf{C}$. The *slice* category of $\mathbf{C}$ over U, denoted $\mathbf{C}/U$, is the category whose objects are pairs (V, f) where $f : V \to U$ is a morphism in $\mathbf{C}$. Informally, we say V is an object *over* U—the map f is a necessary part of the data, but is sometimes left implied. The morphisms consist of the obvious commutative triangles.

If $(\mathbf{C}, J)$ is a site (here J is a basis for a topology), assuming $\mathbf{C}$ has pull-backs, then if U is an object of $\mathbf{C}$, the category $\mathbf{C}/U$ also carries a topology, generated by the coverings in J in an obvious way. In fact, if $U \to V$ is a morphism in $\mathbf{C}/U$, we obtain a functor $\mathbf{C}/U \to \mathbf{C}/V$ and, in fact (exercise) morphisms

$$\mathbf{Sh}_J(\mathbf{C}/V) \to \mathbf{Sh}_J(\mathbf{C}/U)$$

of categories of sheaves for the respective J-topologies.

Special case: if $\mathbf{C}$ has a terminal object (an object $*$ such that all objects U are equipped with a unique morphism $U \to *$), then $\mathbf{C}/*$ is equivalent to $\mathbf{C}$ itself. The category $\mathbf{Sm}_k$ has a terminal object, $\operatorname{Spec} k$.

3.2 Homotopy Groups

The definitive references for this material are [Jar87] and [JSS15].

Definition 3.9. If $X_\bullet$ is a simplicial set, the set $\pi_0(X_\bullet)$ is defined as $\pi_0(|X|)$, the set of path components. If $x \in X_0$, and $i \geq 1$, we define $\pi_i(X_\bullet, x)$ as $\pi_i(|X_\bullet|, x)$, the i-th higher homotopy group. If $i \geq 2$, then π_i is abelian.

Definition 3.10. A *weak equivalence* of simplicial sets is a map $f : X_\bullet \to Y_\bullet$ such that for all choices of basepoint $x \in X_0$ and all $i \geq 0$, the map $\pi_i(f) : \pi_i(X_\bullet, x) \to \pi_i(Y, f(x))$ is an isomorphism.

The idea of the homotopy "group" of a simplicial presheaf $\mathscr{X}$ is that we'd like to formalize

$$\pi_i^{\text{pre}} : U \mapsto \pi_i(\mathscr{X}(U), x)$$

to give a presheaf $\pi_i^{\text{pre}} : \mathbf{sC} \to \mathbf{Set}$. The big problem here is where the basepoints lie.

First, the easy case:

Definition 3.11.

$$\pi_0^{\mathrm{pre}}(\mathscr{X}) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$$

is defined by $\pi_0^{\mathrm{pre}}(\mathscr{X})(U) = \pi_0^{\mathrm{pre}}(\mathscr{X}(U))$.

Now the harder case:

Definition 3.12. Let $\mathbf{C}$ be a category, $\mathscr{X}$ a simplicial presheaf on $\mathbf{C}$, U an object of $\mathbf{C}$ and $x \in \mathscr{X}(U)$, and $i \geq 1$. Then $\pi_i^{\mathrm{pre}}(\mathbf{X}, x) : \mathbf{C}/U^{\mathrm{op}} \to \mathbf{Grp}$ is the functor

$$\pi_i^{\mathrm{pre}}(\mathbf{X}, x)(V \xrightarrow{f} U) = \pi_i^{\mathrm{pre}}(\mathbf{X}(V), f^* x).$$

Pay attention to what has happened to the basepoint.

The definitions above of π_*^{pre} are functorial in maps $f : \mathscr{X} \to \mathscr{Y}$. For π_0^{pre}, this is straightforward. The construction π_0^{pre} is a functor from $\mathbf{sPre}(\mathbf{C})$ to $\mathbf{sPre}$. For the higher homotopy groups, the construction π_i^{pre} is a collection of functors, one for each object U of $\mathbf{C}$, from the category of pointed simplicial presheaves on $\mathbf{C}/U$, denoted $\mathbf{sPre}(\mathbf{C}/U)_{\bullet}$ to $\mathbf{Grp}$, the category of groups.

Definition 3.13. A morphism $f : \mathscr{X} \to \mathscr{Y}$ of simplicial presheaves is a *global weak equivalence* if it induces isomorphisms on all homotopy presheaves, for all possible choices of basepoint.

Equivalently, for all U, the map $f : \mathscr{X}(U) \to \mathscr{Y}(U)$ is a weak equivalence of simplicial sets.

We now define the homotopy sheaves

Definition 3.14. With the same data as before, but now suppose J is (a basis for) a topology on $\mathbf{C}$. The homotopy sheaf $\pi_i(\mathscr{X}, x)$ is the sheaf associated to $\pi_i^{\mathrm{pre}}(\mathscr{X}, x)$.

Definition 3.15. A J-local weak equivalence is a map $f : \mathscr{X} \to \mathscr{Y}$ inducing an isomorphism on all homotopy sheaves (for the J topology).

Remark 3.16. Every global weak equivalence is a local weak equivalence.

3.3 Examples and Constructions

The following remark should have come earlier somwhere.

Remark 3.17. If $\mathbf{C}$ is a category with a terminal object, $*$, and if $\mathscr{X}$ is a (simplicial) presheaf on $\mathscr{X}$, then $\Gamma(\mathscr{X}) = \mathscr{X}(*)$ is the set of *global sections*.

All categories over which we take presheaves will have terminal objects henceforth.

Proposition 3.18. *The category of simplicial presheaves has all limits and colimits.*

Proof. It's really a presheaf category on $\mathbf{C} \times \Delta$. □

Definition 3.19. A simplicial presheaf $\mathscr{X}_\bullet$ is a *simplicial sheaf* if every level of $\mathscr{X}$ is a sheaf.

There is a category of simplicial sheaves and an associated sheaf functor from $\mathbf{sPreSm}_k \to \mathbf{sShSm}_k$.

Example 3.20. Let $K_\bullet$ be a simplicial set. Then we can define a constant simplicial presheaf on $\mathbf{C}$, also denoted K, by setting $K(U) = K$. We can sheafify this to make a constant simplicial presheaf.

Notation 3.21. Here are some standard simplicial sets: Δ^n, represented by $[n]$. $\partial\Delta^n$ (the $n-1$ skeleton of Δ^n. We use this as a model for S^n. Λ^n_r, the r-th horn, obtained by omitting the r-th face.

Definition 3.22. Let $\mathbf{K}$ denote the category of compactly generated spaces. These are the topological spaces X such that $U \subset X$ is open if and only if $U \cap C$ is open in C for all compact subsets $C \subset X$.

Definition 3.23. Let Sing denote the singular functor on $\mathbf{K}$. This takes X to the simplicial set whose n-th level consists of $\mathrm{Map}_{\mathbf{K}}(|\Delta^n|, X)$.

Proposition 3.24. *Geometric realization is left adjoint to the Singular functor.*

$$|\cdot| : \mathbf{sSet} \leftrightarrows \mathbf{K} : \mathrm{Sing}$$

[Hov99, Chapter 3.1]

Proposition 3.25. *If X is a topological space, $|\mathrm{Sing}\, X|$ has the same homotopy groups as X.*

Proof. Deferred. □

Proposition 3.26. *Geometric realization preserves finite limits (when considered as a functor with target in* $\mathbf{K}$*). It preserves all colimits, by virtue of being a left adjoint.*

[Hov99, Lemma 3.2.3]

Notation 3.27. Suppose $\mathbf{C}$ is a category with a (chosen) terminal object $*$. A *pointed* object is a pair (X, x) where X is an object of $\mathbf{C}$ and $x : * \to X$ is a map. There is a category of pointed objects in $\mathbf{C}$, the maps are the *based* or *pointed* maps:

$$\begin{tikzcd} & * \arrow[dl] \arrow[dr] & \\ X \arrow[rr] & & X' \end{tikzcd}$$

The category of based objects in $\mathbf{C}$ is denoted $\mathbf{C}_\bullet$.

Example 3.28. Given two pointed simplicial sets, K and J, we may form $K \vee J$ and $K \wedge J$. The realizations of these simplicial sets are the wedge- and smash-products of the realizations of J and K, respectively.

Example 3.29. We may carry out the same constructions for simplicial presheaves.

Example 3.30. We may define the (reduced) suspension of a based simplicial presheaf $\Sigma\mathscr{X}$ as the suspension $S^1 \wedge \mathscr{X}$. Unwinding definitions, $(\Sigma\mathscr{X})(U) = \Sigma(\mathscr{X}(U))$.

A great many constructions one wishes to carry out consist of colimits and finite limits, possibly with standard simplicial objects. These constructions can all be carried out for simplicial presheaves, and the calculation can be seen 'objectwise', as with the suspension.

Example 3.31. Suppose we have a diagram of sheaves

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"] & A \\ B \end{tikzcd} \tag{3.1}$$

such that the colimit as presheaves is not the colimit as sheaves. For instance, $A = \mathbb{A}^1 \times \mathbb{G}_m$, $B = \mathbb{G}_m \times \mathbb{A}^1$ and $C = \mathbb{G}_m \times \mathbb{G}_m$ in the Nisnevich (or Zariski or étale) topology.

Then we may construct a simplicial sheaf via $X_0 = A \coprod B$, $X_1 = A \coprod B \coprod C$—the first two terms being degnerate—, and the higher X_i wholly degenerate. Here the boundary maps from C are f and g respectively. Note that $\pi_0(X_\bullet(U)) = \pi_0^{\mathrm{pre}}(X_\bullet)(U)$ is the colimit of presheaves in (3.1) evaluated at U. Hence $\pi_0^{\mathrm{pre}}(X_\bullet)$ is the colimit of presheaves. This is not a sheaf, even though $X_\bullet$ is a simplicial sheaf.

Example 3.32. Consider e.g. the Nisnevich topology on $\mathbf{Sm}_k$. Let P be a presheaf that is a strict subpresheaf of its sheafification $P \hookrightarrow a(P)$. For instance, the presheaf sending a ring to pairs of elements, at least one of which is a unit is a strict subpresheaf of the sheaf represented by $\mathbb{A}^2 - \{0\}$.

Then $P \to a(P)$ is not a global weak equivalence of discrete objects, but it is a local weak equivalence.

Example 3.33. We can make this more higher homotopic' by replacing P and $a(P)$ by bouquets of spheres indexed by P and $a(P)$.

In the specific example mentioned, note that $P(\operatorname{Spec} k)$ consists of elements in $k^2 - \{(0,0)\}$, as does $a(P)(\operatorname{Spec} k)$. So in this instance, we see it's not enough to consider only choices of basepoint in $P(*)$. We could fashion a simplicial presheaf Q that consists of a bouquet of circles indexed by P, and another as $Q' = Q \cup_* a(P)/P$, adjoining a point to $Q(U)$ for every element of $a(P)(U)$ not in $P(U)$.

Then, as far as any basepoint chosen in $Q(\operatorname{Spec} k)$ can tell, these presheaves are equivalent. But Q' is not connected, while Q is.

3.3.1 Internal Mapping and Hom Objects

Here's another construction that we're going to need, and we should introduce at some point. For the sheaf-theoretic part, we refer to [MM92].

Definition 3.34. A *cartesian closed category* is a category $\mathbf{C}$ such that

1. There is a terminal object
2. Any two objects X, Y have a cartesian product $X \times Y$
3. Any two objects Y, Z have an *exponential* or (internal) *mapping* object Z^Y or $\operatorname{Map}(Y, Z)$

such that there is an adjunction (i.e., a natural isomorphism)

$$\operatorname{Mor}_{\mathbf{C}}(X \times Y, Z) \cong \operatorname{Mor}_{\mathbf{C}}(X, \operatorname{Map}(Y, Z))$$

This is a specific instance of a closed (symmetric) monoidal category with the part of $\otimes$ being played by $\times$.

Exercise 3.35. One specific case of the above is when $X = \operatorname{Map}(Y, Z)$. Then there is the identity on the right, and on the left this corresponds to a canonical map $e : \operatorname{Map}(Y, Z) \times Y \to Z$, the evaluation map. This is the counit of the adjunction.

One may verify that $\operatorname{Map}(Y, Z)$ is the universal object such that there is a map $\operatorname{Map}(Y, Z) \times Y \to Z$, and this characterizes it.

Example 3.36. The category of sets is cartesian closed. The internal mapping object is the set of functions $Y \to Z$.

Example 3.37. Let $\mathbf{C}$ be a category. The category of presheaves of sets on $\mathbf{C}$ is cartesian closed. Given a presheaf Y on $\mathbf{C}$ and any object U of $\mathbf{C}$, there is a presheaf $Y|_{\mathbf{C}/U}$ on $\mathbf{C}/U$

defined by $(V \to U) \mapsto Y(V)$. This is the obvious way of viewing Y as a presheaf on $\mathbf{C}/U$. We mentioned last time that if Y is a sheaf, then so too is this restricted presheaf. We may occasionally denote this by simply Y when we think no confusion can arise.

The internal mapping object $\mathrm{Map}(Y,Z)$ is given as follows: $\mathrm{Map}(Y,Z)(U)$ is the set of maps $Y|_{\mathbf{C}/U} \to Z|_{\mathbf{C}/U}$ as presheaves on $\mathbf{C}/U$. Check that this is a presheaf.

Articulating the adjunction is slightly harder. We can construct an evaluation map $\mathrm{Map}(Y,Z) \times Y \to Z$. If we evaluate the presheaves at any U, since $\mathrm{Map}(Y,Z)(U) \subset \mathrm{Map}(Y(U),Z(U))$, we obtain the required map. The definition of $\mathrm{Map}(Y,Z)$ ensures that the evaluation maps are compatible with one another. It's an exercise to check that it is also universal [MM92, Chapter I, Exercise 8].

Definition 3.38. A category $\mathbf{C}$ is *locally cartesian closed* if, for any object U, the category $\mathbf{C}/U$ is cartesian closed.

Example 3.39. The category of simplicial sets is cartesian closed. This may be seen as a special case of the statement regarding presheaf categories, since this is just presheaves on Δ.

By the adjunction,

the internal mapping object, $\mathrm{Map}(X_\bullet, Y_\bullet)$ is often denoted by $S(X_\bullet, Y_\bullet)$.

By the adjunction,

$$\mathrm{Mor}_{\mathbf{sSet}}(\Delta^n, S(X_\bullet, Y_\bullet)) = \mathrm{Mor}(X \times \Delta^n, Y)$$

so the n-simplices of $S(X_\bullet, Y_\bullet)$ consist of maps $\mathrm{Mor}_{\mathbf{sSet}}(X_\bullet \times \Delta^n, Y_\bullet)$. Since the objects Δ^n assemble to form a cosimplicial object in simplicial sets, $S(X_\bullet, Y_\bullet)$ is, in fact, simplicial.

Since $\mathbf{s}Set$ is really just the category of presheaves of sets on Δ, this implies that we can write every simplicial set as a colimit of 'representable' simplicial sets (the standard simplicial sets Δ^n).

Example 3.40. We may put the last two examples together in order to construct an internal mapping object for simplicial presheaves. We define $\mathrm{Map}(\mathscr{X},\mathscr{Y})$ by declaring the n-th level at U to be $\mathrm{Map}_{\mathbf{sPre}(\mathbf{C}/U)}(\mathscr{X}|_U \times \Delta^n, \mathscr{Y})$. The verification that this works is left to the reader.

Exercise 3.41. In these categories, there are terminal objects, denoted $*$. If you run through the implication of the adjunction $\mathrm{Map}(*, Y) \cong Y$.

Remark 3.42. By the Yoneda lemma, for presheaves, $\mathrm{Map}(X,Z)(U)$ is $\mathrm{Mor}(U, \mathrm{Map}(X,Z)) = \mathrm{Mor}(U \times X, Z)$, and for simplicial presheaves $\mathrm{Map}(\mathscr{X},\mathscr{Y})(U)_n = \mathrm{Mor}_{\mathbf{sPre}}(U \times \mathscr{X} \times \Delta^n, \mathscr{Y})$.

Example 3.43. For pointed simplicial sets or pointed simplicial presheaves (X,x) and (Y,y), the appropriate product is the smash product, $X \wedge Y$. There is a closed symmetric monoidal structure here too, the mapping objects $\mathrm{Map}_\bullet(X,Y)$ are always computed in a basepoint preserving way. Now $\mathrm{Map}_\bullet((X,x),(Y,y))$ is defined as a pullback:

$$\begin{tikzcd} \mathrm{Map}_\bullet((X,x),(Y,y)) \arrow[r] \arrow[d] & \mathrm{Map}(X,Y) \arrow[d] \\ * & \mathrm{Map}(*,Y) \end{tikzcd}$$

As a special case of these structures, we may say that $\mathbf{sPre}(\mathbf{C})$ and $\mathbf{sPre}(\mathbf{C})_\bullet$ have simplicial structures. Specifically, in the unpointed case, if $\mathscr{X}$ is a simplicial presheaf and if K is a simplicial set, then we may define $K \times \mathscr{X}$, and we may define a simplicial mapping space $S(\mathscr{X},\mathscr{Y}) = \Gamma\,\mathrm{Map}(\mathscr{X},\mathscr{Y})$. These constructions are adjoint.

Similar constructions exist in the pointed case.

Proposition 3.44. *Let K be a simplicial set, viewed as a constant simplicial presheaf, and let $\mathscr{X}$ be a simplicial presheaf. Then* $\mathrm{Map}(K,\mathscr{X})(U) = \mathrm{Map}(K,\mathscr{X}(U))$*, and similarly in the pointed case.*

3.4 Stalks

Definition 3.45. Let X be an object in $\mathbf{Sm}_k$ and $x \in X$ a point. A *Nisnevich neighbourhood* of $x \in X$ is an étale morphism $f : V \to X$ of schemes and a point $v \in V$ such that f induces an isomorphism on residue fields $\kappa(x) \to \kappa(v)$.

There is an evident notion of maps of Nisnevich neighbourhoods. The resulting category of Nisnevich neighbourhoods is essentially small, and filtered in the obvious sense. For a given point $x \in X$ and a presheaf $F : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Set}$ we may form

$$p^*F = \operatorname*{colim}_{(V,v)} F(V)$$

over all Nisnevich neighbourhoods of the point $x \in X$. This construction p^*F is called the *stalk* of F at $x \in X$. The formation of the stalk is plainly functorial in F, so we obtain a functor

$$p^* : \mathbf{Pre}(\mathbf{Sm}_k) \to \mathbf{Set}$$

and similarly a functor

$$p^* : \mathbf{sPre}(\mathbf{Sm}_k \to \mathbf{sSet}$$

by applying the first p levelwise.

Proposition 3.46. *The functor p^* as defined above preserves finite limits and all colimits.*

Proof. Filtered colimits commute with colimits and with finite limits. □

Proposition 3.47. *If $F \to a(F)$ is the associated sheaf functor, then $p^* F \to p^* a(F)$ is an isomorphism.*

Remark 3.48. A functor $p^* : \mathbf{Pre}(\mathbf{Sm}_k) \to \mathbf{Set}$ that preserves all colimits must have a right adjoint, [Mac71]. The right adjoint to p^* is the *skyscraper sheaf* functor: $p_* : \mathbf{Set} \to \mathbf{Pre}(\mathbf{Sm}_k)$. It follows from the above that it really is a sheaf.

The following is a useful result in general:

Proposition 3.49. *If $\mathbf{C}$ is a small category, then every object of $\mathbf{Pre}(\mathbf{C})$ may be written as a colimit of representables.*

Proof. For a given presheaf F, we can define a category y/F, the objects of which are maps $y_U \to F$ in $\mathbf{Pre}(\mathbf{C})$. This is also the *category of elements* of F.

There is a forgetful functor, $p : y/F \to \mathbf{Pre}(\mathbf{C})$.

We form the colimit $\operatorname{colim}_{y/F} p(y_U \to F)$ in $\mathbf{C}$. The verification that this colimit is really F is slightly brain-bending. It's not at all hard to see there's a map $\operatorname{colim}_{y/F} p(y_U \to F) \to F$.

To show this is universal, suppose $\operatorname{colim}_{y/F} p(y_U \to F) \to G$ is given, and let V be an object of $\mathbf{C}$. Then we may define a map $F(V) \to G(V)$ as follows: $x \in F(V)$ corresponds to an element (V, x) of the diagram over which we took the colimit. The map $\operatorname{colim}_{y/F} p(y_U \to F) \to G$ induces a map from $(V, x) \to G$, and therefore an element $x' \in G(V)$. Then define the map $F(V) \to G(V)$ by sending $x \mapsto x'$. The verification that this assembles to form a unique universal presheaf map is left as an exercise. □

Corollary 3.50. *Every object of $\mathbf{Pre}(\mathbf{Sm}_k)$ is a colimit of presheaves represented by affine schemes.*

Proposition 3.51. *Let $x \in X$ be a point, and let p^* denote the associated stalk functor. Then there exists a local ring $R = \mathcal{O}^h_{X,x}$ such that for any representable Y, there is an isomorphism $p^* Y \cong Y(R)$. Moreover $\mathcal{O}^h_{X,x}$ is Hensel local.*

Proof. We may assume $Y = \operatorname{Spec} S$ is affine. We may also assume we form $p^* Y$ by means of a filtered colimit of affine schemes (such a filtered colimit is cofinal in the defining colimit).

But now we are calculating a limit of the form

$$\operatorname*{colim}_i Y(\operatorname{Spec} R_i) = \operatorname*{colim}_i \operatorname{Mor}_{k-\mathrm{alg}}(S, R_i) = \operatorname{Mor}_{k-\mathrm{alg}}(S, \lim R_i)$$

It's instructive to see why R is Hensel local. Suppose S is a local ring of a point on a smooth variety, and write κ for $S/\mathfrak{m}$, the residue field. Write $\bar{f}$ etc. for the restriction of f to κ etc. Suppose $f \in S[t]$ is such that

1. f' is a unit in S, i.e., $\bar{f}' \neq 0$.
2. There exists $\alpha \in \kappa$ such that $f(\alpha) = 0$.

Now take some affine neighbourhood Spec A of $x \in X$ such that the following is true.

1. $A \subset S$ and A is an integral domain.
2. $f \in A[t] \subset S[t]$.
3. f' is a unit in A
4. $\alpha \in A/\mathfrak{m} \cap A \subset \kappa$.

Now we can form the standard étale ring extension

$$A \to A[t]/f$$

We claim that this generically (i.e., after inverting some elements) has a section. If we invert everything, we can find a section, since

$$\kappa \to \kappa[t]/f \cong \kappa[t]/(t-\alpha) \oplus \kappa[t]/f_1.$$

Hence we can define a section map $\psi : S[t]/f \to \kappa$, and after inverting $\psi(t)$, this gives a map $(S[t]/f)_{\psi(t)} \to S_{\psi(t)}$,

From this, it's easy to deduce that if the ring $\mathcal{O}^h_{X,x}$ exists as claimed, then it will have the henselian property.

□

Proposition 3.52. *Suppose $\phi : F \to G$ is a map of sheaves on* $\mathbf{Sm}_k$ *that is not an isomorphism (resp. monomorphism, epimorphism). Then there exists some stalk such that* $p^*(F) \to p^*(G)$ *is not an isomorphism (resp. monomorphism, epimorphism).*

Definition 3.53. Let $(R, \mathfrak{m})$ be a local ring with residue field K. We say R is *henselian* or *a Hensel local ring* if it satisfies the following condition, known as Hensel's lemma:

Suppose $f \in R[t]$ is a monic polynomial and $\alpha_0 \in K$ is a simple (non-repeated) root of $\bar{f}$. Then there exists α in R such that $\bar{\alpha} = \alpha_0$ and such that α is a root of f.

If K is also separably closed (so that one can find the roots of $\bar{f}$ without passing to an extension) then R is called *strictly Hensel local*.

Remark 3.54. Since every presheaf is expressible as a colimit of representables, at least in principle we have a method of calculating p^*F for any given presheaf:

1. Express $F = \operatorname{colim} X_i$ where the X_i are representable.
2. Find $\mathcal{O}^h_{X,x}$.
3. Then $p^* F = \operatorname{colim} p^* X_i = \operatorname{colim} X(\mathcal{O}^h_{X,x})$.

Remark 3.55. A *complete local ring* is a local ring R such that $R \cong \lim_n R/\mathfrak{m}^n$. Alternatively, R is complete and Hausdorff in the $\mathfrak{m}$-adic topology. A complete local ring satisfies Hensel's lemma. Two prototypical examples are the case of $R = K[[t]]$, the ring of formal power series over a field, and the ring $\mathbb{Z}_p$ of p-adic integers.

3.5 Group Objects and the Two Sided Bar Construction

Discuss group schemes, group sheaves.

Define $B(A, G, C)$, BG, $K(A, n)$ as sheaves.

Chapter 4

Model Categories

4.1 Definition

The motivating idea for the invention of model categories is that we have a category **C** and some maps in **C** are considered to be "equivalences" which may not be isomorphisms but that we'd like to treat as isomorphisms. If we formally add inverses of these equivalences, even disregarding any set-theoretic problems that might arise, without extra structure on the category **C** we might find ourselves in a difficulty when we tried to calculate morphisms in the new category.

In what follows, we'll define a model category to be a category equipped with extra structure. That extra structure is called a *model structure*. For any given category with a model structure, there usually are other model structures that can be given.

Definition 4.1. A morphism $g : C \to D$ is a *retract* of a morphism $f : A \to B$ if there exists a commutative diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & C \arrow[d] \\ D \arrow[r] & B \arrow[r] & D \end{tikzcd}$$

such that the composites along the top and along the bottom are the appropriate identity maps.

This generalizes the notion of a retract of an objects—C is a retract of A if and only if id_C is a retract of id_A. Moreover, any isomorphism $g : C \to D$ is a retract of $\mathrm{id}_C : C \to C$.

Definition 4.2. A *model category* is a category having all small limits and colimits and equipped with the following *model structure*:

Three subcategories **W**, **Cof** and **Fib**, each having the same objects as **C**—but presumably fewer morphisms—called the weak equivalences, cofibrations and fibrations of the structure.

- The weak equivalences will be denoted $A \xrightarrow{\sim} B$
- The cofibrations will be denoted $A \rightarrowtail B$
- The fibrations will be denoted $A \twoheadrightarrow B$.

These subcategories satisfy the following axioms:

1. **W** satisfies the two-out-of-three rule: if f and g are composable arrows of **C** and two of f, g and $g \circ f$ are in **W**, then so is the third.
2. **W**, **Cof** and **Fib** are closed under retracts.
3. There are *functorial factorizations*: any morphism $f : A \to B$ of **C** has a functorial decomposition as $A \to A' \to B$ where $A \to A'$ is a cofibration and a weak equivalence and $A' \to B$ is a fibration, and a functorial decomposition $A \to B' \to B$ where $A \to B$ is a cofibration and $B' \to B$ is a fibration and a weak equivalence.
4. The lifting axiom. Given a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{4.1}$$

   in which $A \to B$ is a cofibration and $X \to Y$ is a fibration, and at least one of these two maps is a weak equivalence, then there exists a lift $B \to X$ as illustrated.

Notation 4.3. A morphism that is both a weak equivalence and a cofibration is a *trivial cofibration*. Similarly, a *trivial fibration* is a fibration and a weak equivalence.

Remark 4.4. If we know the weak equivalences and the cofibrations, then we can determine the fibrations. They are exactly those maps $X \to Y$ such that in any square like (4.2) where $A \to B$ is a trivial cofibration, a lift exists. This is proved in [Hov99, Section 1.1]

Remark 4.5. The axioms for a model category are self-dual. If **C** has a model structure, then so too does $\mathbf{C}^{\mathrm{op}}$; the cofibrations of one are the opposites of the fibrations of the other, and vice versa.

Definition 4.6. If $\mathbf{C}$ is a model category (with $\mathbf{W}$, $\mathbf{Cof}$ and $\mathbf{Fib}$ and the functorial factorizations implied), then the *homotopy category* is another category, $\operatorname{Ho}\mathbf{C}$ and a functor

$$\Phi : \mathbf{C} \to \operatorname{Ho}\mathbf{C}$$

such that

- $\mathbf{C}$ and $\operatorname{Ho}\mathbf{C}$ have the same objects.
- Φ is the identity on objects.
- If $w : X \to Y$ is an arrow in $\mathbf{W}$, then $\phi(w)$ is an isomorphism.
- $\Phi : \mathbf{C} \to \operatorname{Ho}\mathbf{C}$ is initial among functors taking arrows in $\mathbf{W}$ to isomorphisms.

Notation 4.7. If X and Y are isomorphic in the homotopy category, then we say they are *weakly equivalent* and write $X \simeq Y$. Note that there may be no arrow in $\mathbf{C}$ realizing this weak equivalence, there might be a zigzag of weak equivalences in $\mathbf{C}$ from X to Y.

Proposition 4.8. *Given a model category* $\mathbf{C}$*, its homotopy category* $\operatorname{Ho}\mathbf{C}$ *exists.*

The proof is not at all difficult. Nor does this require the full strength of the model structure. The proof is to be found in [Hov99, Section 1.2].

The technically hard part of all this, for which model categories are useful, is computing sets of morphisms in $\operatorname{Ho}\mathbf{C}$.

4.2 Some examples

Example 4.9. The most important example of a model category is **sSet**. The standard model structure on this category is defined as follows:

1. Weak equivalences are those maps $f : X_\bullet \to Y_\bullet$ that induce isomorphisms on π_0 and on all higher homotopy groups (for all choices of basepoint in X_0).
2. Cofibrations are monomorphisms, i.e., levelwise inclusions.
3. Fibrations are more mysterious. They may be defined by the lifting condition, as in (4.2). They are called *Kan fibrations* in this instance.

The proof that this is indeed a model structure is involved. It can be found in [Hov99]Chapter 3 or [GJ99], among other places.

Often the fibrations are not defined independently of the cofibrations, but given instead by a lifting property. It is also the case that often, it is sufficient to test against a restricted class of trivial cofibrations, rather than all of them. For instance, one may define a Kan fibration as a map $X \to Y$ admitting lifts

$$\begin{tikzcd} \Lambda^n_r \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{4.2}$$

for horn-inclusions.

Remark 4.10. A model category **C** has an initial object, $\emptyset$, and a final object, $*$—by virtue of having all limits and colimits. An object such that $\emptyset \to X$ is a cofibration is *cofibrant* and an object such that $X \to *$ is a fibration is *fibrant*.

All simplicial sets are cofibrant, but not all are fibrant. Those that are are called *Kan complexes*.

Remark 4.11. If X is an object of a model category, then QX is cofibrant and weakly equivalent to X. Furthermore the map $\emptyset \to QX \to RQX$ is a composite of cofibrations, and so RQX is also cofibrant and weakly equivalent to X. It is fibrant by construction. In particular, every object is equivalent to one that is both fibrant and cofibrant.

Example 4.12. There are several model structures on categories of topological spaces. One of the most commonly-used is the following, the *Serre model structure*

1. Weak equivalences are the maps $f : X \to Y$ inducing isomorphisms on π_0 and higher homotopy groups (again, for all choices of basepoint in X).

2. The fibrations are the *Serre fibrations*, they have the lifting property for the inclusions of the n-disk D^n as caps of the cylinders $D^n \times I$.

3. The cofibrations include cellular inclusions of CW complexes—but there are others.

Example 4.13. The *local model structures* will be central to what follows. Here is the first, and most important, one. Let **C** be a category and J a (basis for a) Grothendieck topology on **C**. Then there is a model structure on **Pre**(**C**) where

- The weak equivalences are the J-local weak equivalences: i.e., the maps inducing an isomorphisms on homotopy sheaves.

- The cofibrations are the monomorphisms: i.e., the maps $f : \mathscr{X} \to \mathscr{Y}$ inducing cofibrations $\mathscr{X}(U) \to \mathscr{Y}(U)$ for all objects U in $\mathbf{C}$.
- The fibrations are mysterious, and defined by the lifting condition.

The existence of this model structure is the main result of the first half of the paper [Jar87]. This structure is called the *injective local* model structure:

- *injective* because the cofibrations are those inherited from **sSet**.
- *local* because the weak equivalences are the local equivalences.

We remark that all objects are cofibrant in this structure.

There are other model structures. We will encounter at least one of them.

4.3 The calculation of sets of morphisms

Throughout $\mathbf{C}$ will denote a model category.

Definition 4.14. Any object in $\mathbf{C}$ is equipped with unique maps $\emptyset \to X$ and $X \to *$. We can factor these maps (functorially) to give

$$\emptyset \rightarrowtail RX \overset{\sim}{\twoheadrightarrow} X$$

and

$$X \overset{\sim}{\rightarrowtail} QX \twoheadrightarrow *.$$

Here RX is a *cofibrant replacement* for X and QX is a *fibrant replacement*.

When all objects are already cofibrant—as will be the case for us—, the cofibrant replacement is not very useful. On the other hand, the fibrant replacement is a very powerful and, at times, mysterious, construction.

Definition 4.15. Given an object X of $\mathbf{C}$, a *cylinder object* for X is a factorization of the fold map $X \coprod X \to X$ as

$$X \coprod X \rightarrowtail X' \xrightarrow{\sim} X.$$

A *path object* is a factorization of the diagonal map $X \to X \times X$

$$X \xrightarrow{\sim} X'' \twoheadrightarrow X \times X.$$

Functorial choices of cylinder objects and path objects exist—in which case the weak equivalences are also (co)fibrations as well—, but we do not want to restrict the definition to these.

One highly-illuminating example of a cylinder object is the following: If X is a topological set, then we can include $X \coprod X$ in $X \times I$ as the two endpoints. This is a cofibration in the Serre model structure. Then $X \times I \to X$, projection on the first factor, is a fibration. A similar example can be be written down for simplicial sets: $X \coprod X \to X \times I \to X$, where I is any model for the usual interval, e.g.,Δ^1.

Similarly, but less easily verifiably until we discuss (co)fibrancy and mapping objects, if I is a fibrant interval, then $X \to \mathrm{Map}(I, X) \to X \times X$ (in each of the two cases above) gives a path object. Here $X \to \mathrm{Map}(I, X)$ is given by inclusion as constant maps, and $\mathrm{Map}(I, X) \to X \times X$ is given by projection onto either endpoint of the path.

Definition 4.16. If $f_1, f_2 : X \to Y$ are two maps, then a *left homotopy* from f_1 to f_2 is a map $H : X' \to Y$ from a cylinder object on X to Y such that the two composites $X \to X \coprod X \to X' \to Y$ agree with f_1 and f_2 respectively.

A *right homotopy* from f_1 to f_2 is a map $H : X \to Y''$ from X to a path object on Y such that the two composites $X \to Y'' \to Y \times Y \to Y$ agree with f_1 and f_2 respectively.

Notation 4.17. There are two obvious inclusions $X \to X \coprod X$, and we will write i_0 and i_1 for the composites of these inclusions with the cofibration $X \coprod X \to X'$.

Lemma 4.18. *If X is cofibrant, then the composites $i_0, i_1 : X \to X \coprod X \to X'$ are trivial cofibrations.*

Proof. First, either inclusion $X \to X \coprod X$ is a cofibration if X is cofibrant; this is may be checked directly by verifying the lifting property for trivial fibrations. Consequently, $i_0 : X \to X'$ is a cofibration, as is i_1. Finally for either i_0 or i_1, the composite $X \to X' \to X$ is the identity, which is a weak equivalence, and therefore by two-out-of-three, $X \to X'$ is a weak equivalence. □

As always, there is a dual statement for fibrations.

Proposition 4.19. *If X is cofibrant and Y is fibrant and $f_1, f_2 : X \to Y$ are two maps, then f_1 and f_2 are left homotopic if and only if they are right homotopic. Moreover, if there is a left homotopy from f_1 to f_2 using one cylinder object, then there is a left homotopy using any cylinder object, and similarly for right homotopy.*

We refer to [Hov99, Proposition 1.2.5] for the proof of this. It is not particularly difficult, and it is similar in the style of argument to the proofs of some subsequent results that we will prove in the notes.

We will call such maps *homotopic*. In general, homotopy theory in **C** works best if the sources of maps are cofibrant and the targets are fibrant.

Exercise 4.20. The pushout of a (trivial) cofibration is a (trivial) cofibration.

Proposition 4.21. *If X is cofibrant and Y is fibrant, then homotopy is an equivalence relation on maps* $\mathrm{Mor}(X, Y)$.

Proof. Reflexivity and transitivity are easy, (and require no particular hypotheses on X and Y for either left- and right-homotopy).

The sticking point is transitivity. We'll prove the result for left homotopy. Suppose f_1 is left homotopic to f_2 using the cylinder object $X' \to X$ and the map $H_1 : X' \to Y$ and f_2 is left homotopic to f_3 using the cylinder object X'' and the map $H_2 : X'' \to Y$.

First we try to glue the two homotpies together in the naive way. This is, we form the pushout

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[d, "i_0"] & X' \arrow[d] \\ X'' \arrow[r] & Z \end{tikzcd}$$

In this case $X \to Z$ is a trivial cofibration. The idea is that Z is the gluing-together of two cylinder objects. The two maps $X' \to X$ and $X'' \to X$ glue to give a map $s : Z \to X$. The composite $X \to Z \to X$ is the identity, and the first map $X \to Z$ is a trivial cofibration, so it follows that $s : Z \to X$ is also a weak equivalence.

There are two maps $X \xrightarrow{i_0} X' \to Z$ and $X \xrightarrow{i_1} X'' \to Z$, which we'll call j_0 and j_1. Furthermore the maps H_1 and H_2 assemble to give a map $H : Z \to Y$, which resembles a homotopy from f_1 to f_3, in that $H \circ j_0 = f_1$ and $H \circ j_1 = f_3$.

The two maps j_0 and j_1 extend to a map $X \coprod X \xrightarrow{j_0 \coprod j_1} Z$.

So far so good, but Z is not necessarily a cylinder object for X. The map $X \coprod X \to Z$ need not be a cofibration. But we can use the model structure axioms to factor it, giving $X \coprod X \rightarrowtail Z' \overset{\sim}{\twoheadrightarrow} Z$, and now the composite $Z' \to Z \xrightarrow{J} Y$ gives a homotopy from f_1 to f_3. □

Proposition 4.22. *If X is cofibrant and Y and Z are fibrant, and if f_1, f_2 are homotopic maps $X \to Y$ and g is a map $Y \to Z$, then $g \circ f_1$ is homotopic to $g \circ f_2$. A similar result holds for a cofibrant W and a map $h : W \to X$.*

Proof. The proof is straightforward. □

Proposition 4.23. *Suppose X is cofibrant and $g : Y_1 \to Y_2$ is a weak equivalence of fibrant objects. Then the induced map* $\mathrm{Mor}_{\mathbf{C}}(X, Y_1)/\sim \to \mathrm{Mor}_{\mathbf{C}}(X, Y_2)/\sim$ *is a bijection.*

The dual result also holds, of course: If $g : X_1 \to X_2$ is a weak equivalence of cofibrant objects and Y is fibrant, then the induced map on homotopy classes of maps $X_1 \to Y$ and $X_2 \to Y$ is a bijection.

Proof. First we assume that $Y_1 \to Y_2$ is a trivial fibration. Given a map $f : X \to Y_2$, we can lift it to a map $\tilde{f} : X \to Y_1$, so the map $\mathrm{Mor}_{\mathbf{C}}(X, Y_1)/\sim \to \mathrm{Mor}_{\mathbf{C}}(X, Y_2)/\sim$ is surjective even before taking equivalence classes. We want to show it is injective. Suppose $g \circ f_1 \sim h \circ f_2$ for some maps $f_1, f_2 : X \to Y_1$. Choose a cylinder object and a homotopy $H : X' \to Y_2$. Then we can find a lift in

$$\begin{tikzcd} X \coprod X \arrow[r, "f_1 \coprod f_2"] \arrow[d] & Y_1 \arrow[d] \\ X' \arrow[r, "H"] \arrow[ur, dashed, "K"] & Y_2, \end{tikzcd}$$

since the map on the left is a cofibration and the one on the right is a trivial fibration. Then the lift, K, is a homotopy from f_1 to f_2.

To handle the case where $g : Y_1 \to Y_2$ is a weak equivalence between fibrant objects, not necessarily a trivial fibration, we appeal to the following lemma.

Lemma 4.24 (Ken Brown's Lemma)**.** *Suppose* $\mathbf{C}$ *is a model category and* $\mathbf{D}$ *is a category (maybe a model category) with a subcategory of equivalences satisfying the two-out-of-three property. Suppose* $F : \mathbf{C} \to \mathbf{D}$ *is a functor that takes trivial fibrations between fibrant objects to equivalences. Then* F *takes weak equivalences between fibrant objects to equivalences.*

The dual statement for cofibrations also holds.

Proof of Ken Brown's Lemma. Suppose $f : A \to B$ is a weak equivalence of fibrant objects. Factor $(\mathrm{id}, f) : A \to A \times B$ into a trivial cofibration $A \to C$ followed by a fibration $C \to A \times B$. The projection maps $A \times B \to A$ and $A \times B \to B$ are fibrations (this can be checked by testing lifting properties). Now $C \to A \times B \to A$ is a fibration, and $A \to C \to A \times B \to A$ is the identity, so by two-out-of-three $C \to A$ is a trivial fibration. Additionally $B \to C \to A \times B \to A$ is f, a weak equivalence, so by two-out-of-three again $C \to B$ is also a trivial fibration. Hence $F(C \to A)$ and $F(C \to B)$ are equivalences. Now we play a cancellation game. The map $F(\mathrm{id}_A : A \to A)$ is an equivalence, and so by two-out-of-three $F(A \to C)$ must also be an equivalence. But now $F(A \to C \to B) = F(f)$ is an equivalence, as required. □

Applying Ken Brown's lemma to the functor $\mathrm{Mor}_{\mathbf{C}}(X, \cdot)/\sim$ (using left homotopy, strictly speaking, since the input need not be fibrant) from $\mathbf{C}$ to the category of sets with bijections as the set of equivalences, completes the proof. □

Ken Brown's lemma is surprisingly useful. In particular, it can often be applied when one has a model category where all objects are cofibrant, in which case in order to prove a functor between model categories preserves weak equivalences it's sufficient to prove it preserves trivial cofibrations, a smaller collection of maps.

Notation 4.25. The subcategory of $\mathbf{C}$ consisting of fibrant-cofibrant objects is denoted $\mathbf{C}_{cf}$. As a consequence of the previous results, we can define a category $\mathbf{C}_{cf}/\sim$ where the objects are those of $\mathbf{C}_{cf}$ but the morphisms are homotopy classes of maps (rather than maps themselves).

The isomorphisms in $\mathbf{C}_{cf}/\sim$ are the *homotopy equivalences*. Explicitly, these are (homotopy classes of) maps $f : A \to B$ and $g : B \to A$ such that $f \circ g \sim \mathrm{id}_B$ and $g \circ f \sim \mathrm{id}_A$.

Notation 4.26. We can also define a category $\mathrm{Ho}\,\mathbf{C}_{cf}$, defined by the universal property that weak equivalences in $\mathbf{C}_{cf}$ map to isomorphisms in $\mathrm{Ho}\,\mathbf{C}_{cf}$. It is an exercise [Hov99, Proposition 1.2.3] to show that the obvious functor $\mathrm{Ho}\,\mathbf{C}_{cf} \to \mathrm{Ho}\,\mathbf{C}$ is an equivalence of categories; the inverse functor may be given by fibrant-cofibrant replacement: $X \mapsto RQX$.

Proposition 4.27. *A map between fibrant–cofibrant objects $f : X \to Y$ is a weak equivalence if and only if it is a homotopy equivalence.*

This proposition is key to the whole endeavor. The proof is cribbed shamelessly from [Hov99, Proposition 1.2.8].

Proof. Suppose first that $f : A \to B$ is a weak equivalence of fibrant-cofibrant objects. Let X be a fibrant-cofibrant object. Then the functor $\mathrm{Mor}_{\mathbf{C}_{cf}/\sim}(X, \cdot)$ sends trivial fibrations to bijections, so by Ken Brown's lemma, sends weak equivalences (between fibrant objects) to bijections. Applying this functor with $X = B$ to the map $f : A \to B$, and considering $\mathrm{id}_B \in \mathrm{Mor}_{\mathbf{C}_{cf}/\sim}(B, B)$, we deduce there must exist a corresponding map $g : B \to A$, or strictly speaking, a homotopy class of maps with some representative $g : B \to A$, such that $f \circ g \sim \mathrm{id}_B$.

Postcomposing with f, we see that $f \circ g \circ f \sim f = \mathrm{id}_A \circ f$. Since postcomposition by f induces a bijection

$$\mathrm{Mor}_{\mathbf{C}_{cf}/\sim}(A, A) \to \mathrm{Mor}_{\mathbf{C}_{cf}/\sim}(A, B)$$

on homotopy classes, it follows that $f \circ g \sim \mathrm{id}_A$.

This proves that a weak equivalence between fibrant-cofibrant objects is a homotopy equivalence.

The converse direction is somewhat technical. First of all, we observe that a map homotopic to a weak equivalence is a weak equivalence—the verification of this is left as an exercise.

Second, we consider the easier case of $f : A \to B$ with homotopy inverse $g : B \to A$ such that $f \circ g = \mathrm{id}_B$. In this case

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & A \arrow[r] \arrow[d, "gf"] & A \arrow[d, "fgf"] \\ B \arrow[r, "g"] & A \arrow[r, "f"] & B \end{tikzcd}$$

(where unmarked arrows are identity maps) exhibits $f : A \to B$ as a retract of the weak equivalence gf, and so f is a weak equivalence.

Finally, we reduce to this case. Suppose we are given a map $f : A \to B$ with homotopy inverse $g : B \to A$ and a homotopy $H : B' \to B$ from fg to id_B. Factor f into a trivial cofibration $f_1 : A \to C$ followed by a fibration $f_2 : C \to B$. It will suffice to prove that f_2 is a weak equivalence. Note that f_1 is a weak equivalence between fibrant-cofibrant objects, and therefore a homotopy equivalence. Form

$$\begin{tikzcd} B \arrow[r, "f_1 g"] \arrow[d, "i_0"'] & C \arrow[d, "f_2"] \\ B' \arrow[r, "H"] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

Since $C \to B$ is a fibration and i_0 is a trivial cofibration, there is a lift in this square. Denote such a lift by H'. Let $q = H' \circ i_1 : B \to C$. Then $f_2 \circ q = \mathrm{id}_B$, and H' is a homotopy from $f_1 g$ to q. Letg_1 be a homotopy inverse for f_1. Then $f_2 \sim f_2 f_1 g_1 \sim f g_1$, and so $q f_2 \sim (f_1 g)(f g_1) \sim f_1 g_1 \sim \mathrm{id}_C$.

This rather tortured sequence of homotopies shows that f_2 has q as a homotopy inverse, and $f_2 \circ q = \mathrm{id}_B$, so we have reduced to the special case already handled. □

Proposition 4.28. *Suppose $\Psi : \mathbf{C}_{cf} \to D$ is a functor that takes weak equivalences to isomorphisms, then Ψ identifies homotopic maps.*

Proof. Let $f, g : A \to B$ be homotopic maps in $\mathbf{C}_{cf}$, let $A \coprod A \to A' \xrightarrow{s} A$ be a cylinder object for A and let $H : A' \to A$ be a homotopy from f to g. Then $si_0 = si_1$ are both the identity on A and s is a weak equivalence, so $\Psi(i_0) = \Psi(i_1)$. But then it follows that $\Psi(f) = \Psi(Hi_0) = \Psi(Hi_0) = \Psi(g)$, as required. □

Corollary 4.29. *The category $\mathbf{C}_{cf}/\sim$ satisfies the universal property of* $\mathrm{Ho}\,\mathbf{C}_{cf}$*. In particular, it follows that $\mathbf{C}_{cf}/\sim$ is equivalent to* $\mathrm{Ho}\,\mathbf{C}$.

These results give us a way of calculating maps $\mathrm{Mor}_{\mathrm{Ho}\,\mathbf{C}}(X, Y)$. Namely, take fibrant, cofibrant replacements of X and Y, denoted QRX and QRY, then calculate homotopy classes of maps: $\mathrm{Mor}_{\mathbf{C}}(QRX, QRY)/\sim$. In fact, we can do better as the following proposition shows:

Proposition 4.30. *Suppose X is cofibrant and Y is fibrant. Then* $\mathrm{Mor}_{\mathbf{C}}(X,Y)/\sim = \mathrm{Mor}_{\mathrm{Ho}\,\mathbf{C}}(X,Y)$.

Proof. Since X is cofibrant, the map $X \to RX$ induces a bijection $\mathrm{Mor}_{\mathbf{C}}(RX,Y)/\sim \to \mathrm{Mor}_{\mathbf{C}}(X,Y)/\sim$, and dually for Y. Therefore $\mathrm{Mor}_{\mathbf{C}}(X,Y)/\sim = \mathrm{Mor}_{\mathbf{C}}(RX,QY)$, where both source and target are fibrant-cofibrant. So $\mathrm{Mor}_{\mathbf{C}}(RX,QY) = \mathrm{Mor}_{\mathrm{Ho}\,\mathbf{C}}(RX,QY) = \mathrm{Mor}_{\mathrm{Ho}\,\mathbf{C}}(X,Y)$ as asserted. □

Remark 4.31. In the sequel we'll generally work in model categories in which all objects are cofibrant. In these categories, $\mathrm{Mor}_{\mathrm{Ho}\,\mathbf{C}}(X,Y)$ is calculated by taking a fibrant replacement RY of Y, and then calculating homotopy classes $\mathrm{Mor}_{\mathbf{C}}(X,RY)/\sim$.

4.4 Quillen Adjunctions

Definition 4.32. Let $\mathbf{C}$ and $\mathbf{D}$ be model categories. A *Quillen adjunction* $F : \mathbf{C} \leftrightarrows \mathbf{D} : U$ is an adjoint pair of functors satisfying either of the following equivalent conditions:

1. F preserves cofibrations and trivial cofibrations
2. U preserves fibrations and trivial fibrations.

Remark 4.33. It's an exericise to verify that these conditions are equivalent. We remark that the left adjoint F preserves weak equivalences between cofibrant objects, by Ken Brown's lemma. This is noteworthy because, again, for us in general all objects will be cofibrant.

Remark 4.34. Often only one of the two functors will be specified (generally the left adjoint F).

Definition 4.35. If $F : \mathbf{C} \to \mathbf{D}$ is a left Quillen functor between model categories, then we define the *total (left) derived functor* as follows

$$\mathrm{Ho}\,\mathbf{C} \xrightarrow{\mathrm{Ho}\,F\circ Q} \mathrm{Ho}\,\mathbf{D}.$$

That is, we take a cofibrant replacement, then apply F in the homotopy categories. This is well defined because F preserves weak equivalences between fibrant objects.

The *total right derived functor* of a right Quillen functor is defined similarly. The left- and right-derived functors behave well in composites, but we do not go into this story here. We refer to [Hov99, Section 1.3].

Remark 4.36. The left derived functor may be denoted LF, but this is going to be troublesome, because L also stands for "localization". As a matter of practice, a functor is seldom left-and-right Quillen, so it generally will have at most one meaningful derived functor, which we may denote, in an abuse of notation by the same symbol. So if $F : \mathbf{C} \to \mathbf{D}$ is a left Quillen functor, then both $LF : \mathrm{Ho}\,\mathbf{C} \to \mathrm{Ho}\,\mathbf{D}$ and $F : \mathrm{Ho}\,\mathbf{C} \to \mathrm{Ho}\,\mathbf{D}$ will also denote the total left derived functor.

Proposition 4.37. *If $F : \mathbf{C} \to \mathbf{D}$ and $U : \mathbf{D} \to \mathbf{C}$ are Quillen adjoint, then the left- and right-derived functors are adjoint.*

The proof is left as an exercise.

Example 4.38. In some cases, the functors in the Quillen adjunction are identity functors $\mathrm{id} : \mathbf{C} \leftrightarrows \mathbf{C} : \mathrm{id}$, but the model structures on the left and right are different. For instance, if we have two model structures, denoted v_1 and v_2 for the time being on $\mathbf{C}$ having the same cofibrations, but different weak equivalences $\mathbf{W}_1$ and $\mathbf{W}_2$, such that every weak equivalence in $\mathbf{W}_1$ is a weak equvialence in $\mathbf{W}_2$ (but not necessarily vice versa), then the identity functor $\mathrm{id} : (\mathbf{C}, v_1) \to (\mathbf{C}, v_2)$ is left Quillen (the identity functor the other way is right Quillen, being the right adjoint).

This situation actually has arisen already for us, although we have not drawn attention to it. There are two model structures one can impose on $\mathbf{sPre}(\mathbf{Sm}_k)$: the injective local structure and the *injective global* structure, where cofibrations are the same, but the weak equivalences are the global weak equivalences (see [Isa05] for a good account of this and other model structures). Then

$$\mathrm{id} : \mathbf{sPre}(\mathbf{Sm}_k),\ \text{injective global} \to \mathbf{sPre}(\mathbf{Sm}_k),\ \text{injective local}$$

is left Quillen.

4.5 Pointed Model Categories

Remark 4.39. If $\mathbf{C}$ is a model category, with $\mathbf{W}$, $\mathbf{C}of$ and $\mathbf{F}ib$ as categories of weak equivalences, cofibrations and fibrations, then $\mathbf{C}_\bullet$, the category of pointed objects in $\mathbf{C}$, inherits a model category structure. The weak equivalences, cofibrations and fibrations in $\mathbf{C}_\bullet$ are those based maps that have the required property after forgetting the pointed structures.

Notation 4.40. The homotopy category $\mathrm{Ho}\,\mathbf{sPreSm}_k$ for the injective local model structure will come up repeatedly. It is often denoted $\mathscr{H}_s(k)$. The pointed version is denoted $\mathscr{H}_{s,\bullet}(k)$.

4.6 Closed Monoidal and Simplicial Structures

We refer to [Hov99, Chapter 4] in particular for this. For what we need, all the monoidal structures will be symmetric, which simplifies the exposition somewhat.

A symmetric monoidal category is a category $\mathbf{C}$ equipped with an operation $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, along with a unit element and associator isomorphisms. We will not stress these here. It is *symmetric* if there is also (as part of the data) a natural isomorphism $A \otimes B \to B \otimes A$, satisfying some further coherence axioms. The two examples to keep in mind here are $\times$ (the cartesian product) or $\wedge$ (the smash product). The symmetric monoidal categories are *closed* if there is an 'internal hom' functor

$$\mathrm{Mor}(A \otimes B, C) \cong \mathrm{Mor}(A, \mathrm{Map}(B, C)).$$

Definition 4.41. Suppose the category $\mathbf{C}$ is a model category and a closed symmetric monoidal category. Then we say $\mathbf{C}$ is a *symmetric monoidal model category* if the following holds: if $f : U \to V$ and $g : W \to X$ are cofibrations in $\mathbf{C}$, then

$$f \square g : (V \otimes W) \coprod_{U \otimes W} (U \otimes X) \to V \otimes X$$

is a cofibration, and is a trivial cofibration if either f or g is a trivial cofibration.

The construction $f \square g$ is the *pushout product* of f and g.

Exercise 4.42.

1. If C is cofibrant, then the above makes $\otimes C$ a left Quillen functor. A similar statement holds in the other variable.

2. Given a cofibration $g : W \to X$ and a fibration $p : Y \to Z$, the induced map

   $$\mathrm{Map}(X, Y) \to \mathrm{Map}(X, Z) \times_{\mathrm{Map}(W,Z)} \mathrm{Map}(W, Y)$$

   is a fibration which is a weak equivalence if either g or p is.

Example 4.43. Setting various objects equal to specific special objects in the above yield interesting special cases. For instance, if $\emptyset \to V$ is a cofibration (V is cofibrant), and $W \to X$ is a cofibration, then

$$(\emptyset \to V) \square (W \to X) : V \otimes W \coprod_{\emptyset \otimes W} (\emptyset \otimes V) \to V \otimes X$$

but $\otimes W$ is a left adjoint, therefore $\emptyset \otimes W = \emptyset$ and so this reduces to saying

$$V \otimes W \to V \otimes X$$

is a cofibration, and is a trivial cofibration if $W \to X$ is. If, moreover, all objects are cofibrant, then Ken Brown's lemma says that $V \otimes \cdot$ preserves all weak equivalences.

Example 4.44. If $Y \to \mathbf{pt}$ is a fibration (i.e., Y is fibrant) and $W \to X$ is a cofibration, then dually $\mathrm{Map}(X, Y) \to \mathrm{Map}(X, \mathbf{pt}) \times_{\mathrm{Map}(W,\mathbf{pt})} \mathrm{Map}(W, Y)$ is a fibration, and is a weak equivalence if $W \to X$ is. Again, if all objects are cofibrant, then Ken Brown's lemma says $\mathrm{Map}(\mathbf{pt}, Y)$ preserves weak equivalences (assuming, as we are here, that Y is fibrant).

Definition 4.45. A model category $\mathbf{C}$ is *left proper* if weak equivalences are preserved by pushouts along cofibrations. It is called *right proper* if weak equivalences are preserved by pullbacks along fibrations. It is *proper* if it is both left- and right-proper.

Proposition 4.46. *A model category in which every object is cofibrant is left proper.*

Proof. Trivial cofibrations are closed under pushout. This is a lifting exercise. Then a version of Ken Brown's lemma applies to say that pushing out preserves weak equivalences between cofibrant objects. The details are not given here. □

Proposition 4.47. *The injective local model category on* $\mathbf{sPre}(\mathbf{Sm}_k)$ *is a symmetric monoidal model category (for the cartesian product).*

Proof. It's easy to show that if f, g are cofibrations (i.e., monomorphisms) then $f \square g$ is a cofibration as well. This is done at the level of elements. It is slightly harder to prove that $f \square G$ IsZ a weak equivalence if either f or g is. Suppose $f : U \to V$ is a trivial cofibration and $g : W \to X$ is a cofibration.

We use the properness of this model category. One verifies directly that $(\cdot \times Y)$ preserves local weak equivalences, since one has $\pi_i(X \times Y) = \pi_i(X) \times \pi_i(Y)$. Then the properness implies $W \times U \to W \times V$ is a weak equivalence, and properness says that the pushout of this along the cofibration $W \times U \to W \times X$ is a weak equivalence. But then $X \times U \to X \times V$ is a weak equivalence and $X \times U \to X \times U \coprod_{W \times U} W \times V$ is a weak equivalence, so, by 2-out-of-3, so too is the map $X \times U \coprod_{W \times U} W \times V \to X \times V$. □

Corollary 4.48. *The injective local model category on* $\mathbf{sPre}(\mathbf{Sm}_k)_\bullet$ *is a symmetric monoidal model category with the smash product.*

The argument here is formal, provided $*$ is cofibrant.

Example 4.49. Simpler examples: $\mathbf{sSet}$ and $\mathbf{sSet}_\bullet$. These are monoidal model categories with $\times$ and $\wedge$.

Definition 4.50. A *simplicial* model category is a model category $\mathbf{C}$ equipped with an action $\otimes$ of $\mathbf{sSet}$ and simplicial mapping objects $S(\mathscr{X}, \mathscr{Y})$ and mapping objects $\mathrm{Map}(K, \mathscr{Y})$ in $\mathbf{C}$ and adjunction isomorphisms

$$\mathrm{Mor}_{\mathbf{sSet}}(K, S(\mathscr{X}, \mathscr{Y})) \cong \mathrm{Mor}_{\mathbf{C}}(K \otimes \mathscr{X}, \mathscr{Y}) \cong \mathrm{Mor}_{\mathbf{C}}(\mathscr{X}, \mathrm{Map}(K, \mathscr{Y}))$$

such that if $f : K \to L$ is a cofibration of $\mathbf{sSet}$ and $g : W \to X$ is a cofibration in $\mathbf{C}$, then $f \square g$ is a cofibration, that is trivial if either f or g is.

Example 4.51. The injective local model category on $\mathbf{sPre}(\mathbf{Sm}_k)$ is simplicial. All that's really missing is the simplicial mapping object $S(\mathscr{X},\mathscr{Y})$. This is given by global sections of $\mathrm{Map}(\mathscr{X},\mathscr{Y})$, as we discussed earlier.

There is an analogous notion of pointed simplicial model categories.

One may verify that $\mathbf{pt} \otimes B \cong B$, but we'll assume this without proof. It's obviously the case in the injective local model category.

Example 4.52. In a simplicial model category in which every object is cofibrant, we can define a cylinder object in a direct sort of way. Recall a cylinder object for B is an object B' such that the fold map $B \coprod B \to B$ factors $B \coprod B \rightarrowtail B' \to B$ where the first map is a cofibration and the second is a weak equivalence. That is, we can take $S^0 \otimes B \cong B \coprod B$ and $S^0 \rightarrowtail I \to \mathbf{pt}$ as a basic model. Then $S^0 \otimes B \to I \otimes B \to \mathbf{pt} \otimes B$ gives us $I \otimes B$ as a cylinder object for B.

In such a category, one can therefore define a (left) homotopy between $f : B \to X$ and $g : B \to X$ as a map $H : I \otimes B \to X$ such that $H(0,b) = f(b)$ and $H(1,b) = g(b)$. If X is fibrant, then homotopy classes of maps, defined in this way, agree with the set of maps $\mathrm{Mor}_{\mathrm{Ho}\,\mathbf{C}}(B,X)$.

4.7 (Left) Bousfield Localization

Exercise 4.53. Suppose $\mathscr{X}$ is a Nisnevich sheaf on $\mathbf{Sm}_k$, viewed as an object of $\mathbf{sPre}(\mathbf{Sm}_k)$ by placing it in degree 0, and $\mathscr{Y}$ is a simplicial presheaf. Then any map $\mathscr{Y} \to \mathscr{X}$ factors through $\mathscr{Y} \to a\pi_0(\mathscr{Y})$.

Exercise 4.54. Suppose $\mathscr{X}$ is a Nisnevich sheaf on $\mathbf{Sm}_k$. Then $\mathscr{X}$ is fibrant.

Exercise 4.55. If $\mathscr{X}$ and $\mathscr{Y}$ are Nisnevich sheaves (e.g. if they are schemes) then the maps in the homotopy category $\mathrm{Mor}_{\mathscr{H}_s}(\mathscr{X},\mathscr{Y}) = \mathrm{Mor}_{\mathbf{Pre}(\mathbf{Sm}_k)}(\mathscr{X},\mathscr{Y})$.

The reference for this material is the first part of [Hir03], especially [Hir03, Chapters 3, 4]. The following definition is given:

Definition 4.56. Let $\mathbf{C}$ be a model category and let A be a class of maps in $\mathbf{C}$. A *left localization* of $\mathbf{C}$ with respect to A is a model category $L_A\mathbf{C}$ and a left Quillen functor $j : \mathbf{C} \to L_A\mathbf{C}$ such that the total left derived functor of $j : \mathrm{Ho}\,\mathbf{C} \to \mathrm{Ho}\,L_A\mathbf{C}$ takes the morphisms in A (strictly speaking, their images in $\mathrm{Ho}\,\mathbf{C}$) to isomorphisms in $\mathrm{Ho}\,L_A\mathbf{C}$, and such that moreover $L_A\mathbf{C}$ is universal in the following sense:

If $\mathbf{D}$ is any other model category and $\phi : \mathbf{C} \to \mathbf{D}$ is a left Quillen functor taking S to isomorphisms in $\mathrm{Ho}\,\mathbf{D}$, then there is a unique left Quillen functor $\delta : L_A\mathbf{C} \to \mathbf{D}$ making the obvious triangle commute.

Such things, if they exist, are unique up to unique isomorphism.

Definition 4.57. Let **C** be a simplicial model category in which every object is cofibrant and A a class of maps in **C**. An object W of **C** is *A-fibrant* if it is fibrant and for every $f : X \to Y$ in A, the induced map on simplicial mapping spaces $S(Y, W) \to S(X, W)$ is a weak equivalence.

In heuristic terms, as far as W knows, the map f is a weak equivalence.

Remark 4.58.

1. In the reference ([Hir03]) it is not assumed that **C** is a simplicial model category, nor is it assumed that every object is cofibrant. The reason for this is that it's possible to produce, without further hypotheses, a "homotopy mapping complex" which is a simplicial set $M(X, W)$. It has the right homotopy type even if X is not cofibrant and if W is not fibrant, but since we have the simplicial structure already, we choose to use $S(X, W)$. If X is cofibrant and W fibrant, then $S(X, W)$ has the same homotopy type as $M(X, W)$.

2. In the homotopy theory references, what we call A-fibrant is called simply *A-local*. Historically, A-local has a weaker meaning in $\mathbb{A}^1$ homotopy theory. So, in this course, but not in the homotopy-theory references: an object W is A-local if a fibrant replacement (and therefore all fibrant replacements) for W is A-fibrant.

Definition 4.59. With the same assumptions on **C**, a map $g : X \to Y$ in **C** is an *A-local equivalence* or *A-equivalence* if for all A-fibrant W, the map $S(Y, W) \to S(X, W)$ is a weak equivalence.

In heuristic terms, as far as W knows, the map g is a weak equivalence. That is, the spaces that think that maps in A are weak equivalences also think g is a weak equivalence.

Definition 4.60. An *A-localization* of an object X of **C** is a map $X \to \tilde{X}$ such that $\tilde{X}$ is A-local and $X \to \tilde{X}$ is an A-equivalence.

Our terminology here too differs from [Hir03], since we do not require $\tilde{X}$ to be fibrant.

Definition 4.61. Let **C** be a model category and A a class of maps in **C**. The *left Bousfield localization* of **C** at A is a model category structure $L_A\mathbf{C}$ on **C** (i.e., the underlying categories are the same)

1. The weak equivalences of $L_A\mathbf{C}$ are the A-equivalences.

2. The cofibrations of $L_A\mathbf{C}$ are the cofibrations of **C**.

3. The fibrations of $L_A\mathbf{C}$ are the maps having the right lifting property with respect to the cofibrations that are A-equivalences (the trivial cofibrations of $L_A\mathbf{C}$).

We are not asserting that the left Bousfield localization exists in general.

Exercise 4.62. The class of A-equivalences is closed under retracts and has the 2-out-of-3 property.

In order to say the left Bousfield localization exists, the main obstacle is showing that there are functorial factorizations of maps into cofibrations followed by fibrations, either one of which is trivial.

See [Hir03, Chatper 4] for the following.

Theorem 4.63. *If* $\mathbf{C}$ *is a left proper cellular model category, and* A *is a set of maps in* $\mathbf{C}$*. Then the left Bousfield localization of* $\mathbf{C}$ *at* A *exists.*

The "cellular" condition above is technical. It means there is a set of generating cofibrations, I, a set of generating trivial cofibrations J, such that the domains and codomains of the elements of I are compact (in a category theoretic sense), the domains of the elements of J are small relative to I and the cofibrations are effective monomorphisms. This will not be discussed further, but applies to the injective local model structure.

The proof of this theorem will not be given in class.

Theorem 4.64. *A left Bousfield localization is a localization.*

Proof. First of all, the Quillen adjunction between $\mathbf{C}$ and $L_A\mathbf{C}$ is given by the identity functors in each direction.

Now suppose $F : \mathbf{C} \leftrightarrows \mathbf{D} : U$ is a Quillen pair such that the maps in A become isomorphisms in $\operatorname{Ho}\mathbf{D}$. Then there is a unique extension of (F, U) to an adjoint pair of functors between $L_A\mathbf{C}$ and $\mathbf{D}$, namely (F, U) again, since the categories are the same. It remains to prove that $F : L_A\mathbf{C} \to \mathbf{D}$ is left Quillen. The category of cofibrations hasen't changed, so there's no problem there. As for trivial cofibrations, F takes them to cofibrations (by the previous observation) and to weak equivalences (since A-equivalences are mapped to weak equivalences in $L_A\mathbf{C}$ by hypothesis). □

Our aim here is to reduce, as far as possible, working in the Bousefield local category to working in the original category.

Proposition 4.65. *The class of trivial fibrations in* $\mathbf{C}$ *and* $L_A\mathbf{C}$ *agree.*

Proof. They are defined by the same lifting condition. □

Lemma 4.66. *If $g : X \to Y$ is a map of fibrant-cofibrant objects in a simplicial model category, then g is a weak equivalence if and only if the induced maps $S(Y,X) \to S(X,X)$ and $S(Y,Y) \to S(X,Y)$ induced by g are weak equivalences.*

Proof. The forward direction is a consequence of the hypotheses of simplicial model structures.

For the reverse, we observe that the n-simplices of $S(X,Y)$ are $\mathrm{Mor}_{\mathbf{sSet}}(\Delta^n, S(X,Y)) = \mathrm{Mor}_{\mathbf{C}}(\Delta^n \otimes X, Y)$. From this, it's easy to deduce that $\pi_0(S(X,Y)) = \mathrm{Mor}_{\mathbf{C}}(X,Y)/\sim = \mathrm{Mor}_{\mathrm{Ho}\,\mathbf{C}}(X,Y)$. □

The following lemma is very helpful:

Lemma 4.67. *A map between A-fibrant, cofibrant objects is a weak equivalence in* **C** *if and only if it is an A-equivalence.*

This can be done without the 'cofibrant' hypothesis, but we will not need it, so we don't bother.

Proof. One direction is immediate. Weak equivalences in **C** are A-equivalences.

For the other direction, we use the previous lemma. Let $g : X \to Y$ be an A-equivalence between A-fibrant, cofibrant objects. Then the induced map

$$S(Y,X) \to S(X,X)$$

is induced by testing an A-equivalence against an A-fibrant object, and similarly in the other case. It follows that these maps are weak equivalences. □

Chapter 5

$\mathbb{A}^1$ Homotopy Theory

Fix a field k.

Definition 5.1. The *injective* $\mathbb{A}^1$ model structure on $\mathbf{sPre}(\mathbf{Sm}_k)$ is the left Bousfield localization of the injective local model structure at the set of projections

$$\mathbb{A}^1 \times U \to U$$

as U ranges over all objects of $\mathbf{Sm}_k$.

Remark 5.2. The cofibrations are monomorphisms. All objects are cofibrant and the structure is left proper. The same argument as for the injective local structure implies that this is a closed symmetric monoidal model category and a simplicial model category.

Definition 5.3. The $\mathbb{A}^1$ *homotopy category* is the homotopy category of the injective $\mathbb{A}^1$ model structure. It is denoted $\mathscr{H}(k)$.

Remark 5.4. There are pointed analogues of each of the above, and the pointed $\mathbb{A}^1$ homotopy category is denoted $\mathscr{H}_\bullet(k)$.

Example 5.5. Since $\mathbb{A}^1 \times \cdots \times \mathbb{A}^1 = \mathbb{A}^n_k$, it follows that all affine spaces $\mathbb{A}^n$ are contractible.

Example 5.6. Although the cofibrant objects are easily understood, and the local (or simplicial) weak equivalences are relatively easy to understand, the process of localization has made the already-mysterious fibrations even worse, and we also do not fully understand the $\mathbb{A}^1$ weak equivalences (it is difficult in general to detect if a map is an isomorphism in $\mathscr{H}(k)$).

One example where we can say something is if X is a scheme such that any map $\mathbb{A}^1 \times U \to X$ factors through the projection $\mathbb{A}^1 \times U \to U$. In this case, we already know that (the simplicial presheaf represented by) X is injective local fibrant.Therefore, we deduce that (levelwise) the following two simplicial sets are isomorphic

$$S(U,X) \to S(\mathbb{A}^1 \times U, X)$$

and so X is also $\mathbb{A}^1$-fibrant.

One important object that meets this condition is $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$. Let us calculate $\mathbb{A}^1 \times U \to \mathbb{G}_m$ where $U = \operatorname{Spec} R$ and R is an integral k-algebra. It is $\mathbb{G}_m(R[t])$, which is the set of units in $R[t]$, so the map $\mathbb{G}_m(R) \to \mathbb{G}_m(R[t])$ induced by inclusion of constants is a bijection. The result follows for general U, not just affine, by Zariski descent.

Definition 5.7. The scheme $\mathbb{G}_m$ is often known as the *Tate circle*. It may be viewed as a pointed scheme, with basepoint $\mathbf{pt} \to \mathbb{G}_m$ given by 1.

5.1 Digression on homotopy (co)limits

We observe that the colimit of a diagram is not a homotopy invariant. The easiest example is the pushout of two points from S^0:

$$* \leftarrow S^0 \to *$$

which gives a point, whereas the equivalent

$$I \leftarrow S^0 \to I$$

gives a space homotopy equivalent to S^1.

If you wanted to avoid this behaviour, you might try to define the colimit in the homotopy category. But homotopy categories do not generally have limits and colimits.

The homotopy colimit gets us out of this difficulty. To define it, we will need some technicalities:

Definition 5.8. A $\mathbf{D}$-shaped diagram in a model category $\mathbf{C}$ is a functor $\Phi : \mathbf{D} \to \mathbf{C}$ from a category $\mathbf{D}$ to a category $\mathbf{C}$. Given such a diagram, we form a *Bar Construction* as a simplicial object in $\mathbf{C}$ as follows:

$$B(\mathbf{pt}, \mathbf{D}, \Phi)_n = \coprod_{i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n} \Phi(i_n)$$

where the coproduct ranges over all composable strings of n arrows between $n+1$ objects in D. This is given a simplicial structure; the face map d_m is given by composing $i_{m+1} \to i_m \to i_{m-1}$ or by omitting i_0 (when $m = 0$) or by omitting i_n when $n = m$, in which case one also uses $i_n \to i_{n-1}$ to map $\Phi(i_n) \to \Phi(i_{n-1})$. The degeneracies are given by introducing identity maps.

Remark 5.9. Suppose $B_\bullet$ is a simplicial object in a simplicial model category $\mathbf{C}$, i.e., a functor $B : \Delta^{\mathrm{op}} \to \mathbf{C}$. Then the *realization* of $B_\bullet$ may be constructed in $\mathbf{C}$ in the same way the realization of a simplicial set was constructed in **Top**.

Definition 5.10. If $\mathbf{C}$ is a simplicial model category and $\Phi : \mathbf{D} \to \mathbf{C}$ is a diagram such that every object in Φ is cofibrant, then the *homotopy colimit* of Φ is the realization in $\mathbf{C}$ of $B(\mathbf{pt}, \mathbf{D}, \Phi)$.

Example 5.11. A very common example is given by the homotopy pushout. Given $A \xleftarrow{f} B \xrightarrow{g} C$ in $\mathbf{C}$, the homotopy pushout is the homotopy colimit of this diagram. The bar construction involved has three 0-objects, A, B and C, and two nondegenerate 1-objects, $I \times A$ corresponding to f and $I \times A$ corresponding to g. The rest is degenerate. The realization consists of an object obtained by gluing $I \times A$ to $I \times A$ along a common A, and then gluing B and C to the free ends.

A common example of this construction again arises when C is a point. In this case, one obtains the *mapping cone* of g.

Proposition 5.12. *Suppose Φ and Φ' are two diagrams of the same shape, where the objects in each are cofibrant, and suppose we have a map $\Phi \to \Phi'$ inducing weak equivalences on objects. Then* $\operatorname{hocolim} \Phi \simeq \operatorname{hocolim} \Phi'$.

This is not proved here. We refer to Dugger's notes.

There is also the Fubini theorem. Suppose $\mathbf{I}$ and $\mathbf{I}'$ are two small categories and $X : \mathbf{I} \times \mathbf{I}' \to \mathbf{C}$ is a diagram. Then one can define X_i for each object $i \in I$, it is a diagram $X_i : \mathbf{I}' \to \mathbf{C}$. Moreover, these diagrams assemble to give a diagram of diagrams $(X_i) : \mathbf{I} \to \mathbf{C}^{\mathbf{I}}$.

Proposition 5.13. $\operatorname{hocolim}_{i \in \mathbf{I}} \operatorname{hocolim}_{I'} X_i \simeq \operatorname{hocolim}_{\mathbf{I} \times \mathbf{I}'} X$.

Proposition 5.14. *Suppose $X : A \leftarrow B \to C$ and $X' : A' \leftarrow B' \to C'$ are two diagrams in which all objects are cofibrant and in which the maps $B \to C$ and $B' \to C'$ are cofibrations. Suppose further that there is a map of diagrams $X \to X'$ that is a weak equivalence on objects. Then* $\operatorname{colim} X \to \operatorname{colim} X'$ *is a weak equivalence.*

Proof. The only proof I know in this generality is slightly involved, it's given in [Hov99, Sections 5.1, 5.2]. In outline, it's like this:

One imposes a model structure on the category $\mathbf{C}^{\Lambda}$ of diagrams of this shape $\bullet \leftarrow \bullet \to \bullet$ in $\mathbf{C}$. The weak equvialences are defined objectwise, the cofibrations are the maps from $A \leftarrow B \to C$ to $A' \leftarrow B' \to C'$ such that $A \to A'$ is a cofibration, $B \to B'$ is a cofibration and $C \coprod_B B' \to C'$ is a cofibration. A fibration is a map where $B \to B'$, $C \to C'$ and $A \to B \times_A A'$ is a fibration in $\mathbf{C}$.

Then one proves that $\operatorname{colim} : \mathbf{C}^\Lambda$ is left Quillen adjoint to the functor that includes X in $\mathbf{C}^\Lambda$ as the constant diagram. Then colim is homotopy invariant on cofibrant objects, which is the result. □

Remark 5.15. A similar argument proves the same result for directed colimits, i.e., colimits over $\bullet \to \bullet \to \bullet \to \dots$, provided all objects in the diagram are cofibrant and all maps are cofibrations.

Proposition 5.16. *Suppose $X : B \leftarrow A \to C$ is a diagram in $\mathbf{C}$ in which each object is cofibrant and in which $A \to C$ is a cofibration. Then* $\operatorname{colim} X \simeq \operatorname{hocolim} X$.

Proposition 5.17. *The same holds for direct limits where all objects are cofibrant and all maps fibrations.*

Definition 5.18. Suppose $f : X \to Y$ is a map in simplicial model category $\mathbf{C}$. One method of replacing f by a cofibration is to form $X \to \Delta^1 \otimes X$, which is a cofibration, then forming $(X \coprod Y) \coprod_{X \coprod X} \Delta^1 \otimes X$. It's an exercise to verify that the obvious map from Y is a trivial cofibration and the map from X is a fibration and equivalent to f. Then C_f, the *cone on f* is the colimit of $\mathbf{pt} \leftarrow X \to \operatorname{Cyl}(f)$.

The cone on f has the homotopy type of the pushout $\mathbf{pt} \leftarrow X \xrightarrow{f} Y$.

Example 5.19. If $f : A \to B$ is a map such that A is contractible, then $B \to B/A$ is a weak equivalence.

Example 5.20. The cone on the map $X \to \mathbf{pt}$ is $I \otimes X/(X \coprod X) = SX$, the (unreduced) suspension of X. If X has a basepoint, $SX \simeq S^1 \wedge X$ (this works for any choice of basepoint).

5.2 Back to $\mathbb{A}^1$

The space $\mathbb{A}^n - \{0\}$ is canonically based at $(1, 0, \dots, 0)$.

Proposition 5.21. $\mathbb{A}^n - \{0\} \simeq_{\mathbb{A}^1} (S^1)^{n-1} \wedge \mathbb{G}_m^{\wedge n} \simeq_{\mathbb{A}^1} S^{n-1+n\alpha} = S^{2n-1,n}$.

Proof. The proof is by induction. When $n = 1$, there is nothing to be done.

We can write $\mathbb{A}^n - \{0\}$ as a colimit of the Zariski cover $\mathbb{A}^{n-1} - 0 \times \mathbb{A}^1$ and $\mathbb{A}^{n-1} \times (\mathbb{A}^1 - \{0\})$. Now consider the diagram

$$\begin{tikzcd}
* & * \arrow[l] \arrow[r] & * \\
\mathbb{A}^{n-1} - \{0\} \arrow[u] \arrow[d] & \mathbb{A}^{n-1} - \{0\} \vee \mathbb{A}^1 - \{0\} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \mathbb{A}^1 - \{0\} \arrow[u] \arrow[d] \\
\mathbb{A}^{n-1} - \{0\} & \mathbb{A}^{n-1} - \{0\} \times \mathbb{A}^1 - \{0\} \arrow[l] \arrow[r] & \mathbb{A}^1 - \{0\}
\end{tikzcd}$$

Here the homotopy pushouts of the top and middle rows are contractible. The top row is easy. To see this for the middle row, we do the following.

1. First, we claim that the crushing map factors $\mathbb{A}^{n-1} - \{0\} \vee \mathbb{A}^1 - \{0\} \to \mathbb{A}^{n-1} \vee \mathbb{A}^1 - \{0\} \to \mathbb{A}^1 - \{0\}$, that the first of these two is a cofibration, and the second is an $\mathbb{A}^1$ weak equivalence.

   The first of these statements is easily checked, as is the second. To see that the second map $\mathbb{A}^{n-1} \vee \mathbb{A}^1 - \{0\} \to \mathbb{A}^1 - \{0\}$ is an $\mathbb{A}^1$ equivalence, recall that we can present $X \wedge Y$ as

$$\begin{tikzcd} * \arrow[r, "x_0"] \arrow[d, "y_0"] & X \\ Y & \end{tikzcd}$$

   Since the maps in this diagram are necessarily cofibrations, an $\mathbb{A}^1$ weak equivalence $X \to X'$ induces an $\mathbb{A}^1$ weak equivalence $X \wedge Y \to X' \wedge Y$. Applying this in the case at hand to the weak equivalence $\mathbb{A}^n \to \mathbf{pt}$ gives the claim.

2. Now we claim that the (homotopy) pushout of

$$\begin{tikzcd} \mathbb{A}^{n-1} - \{0\} \wedge \mathbb{A}^1 - \{0\} \arrow[r] \arrow[d] & \mathbb{A}^{n-1} \wedge \mathbb{A}^1 - \{0\} \\ \mathbb{A}^{n-1} - \{0\} \wedge \mathbb{A}^1 & \end{tikzcd}$$

   is contractible—note that the homotopy pushout is equivalent to the ordinary pushout because at least one (in fact both) of the maps is a cofibration. This pushout is the functor sending a ring R to $\mathbb{A}^{n-1} \wedge \mathbb{A}^1 \simeq_{\mathbb{A}^1} \mathbf{pt}$.

The homotopy pushout of the bottom row is simply the pushout, because the horizontal maps are cofibrations. By Zariski descent, it is $\mathbb{A}^n - \{0\}$.

On the other hand, we can compute the homotopy pushout bv going down columns. For the left column and right column, the answer is **pt** (or at least, $\mathbb{A}^1$ equivalent to it). In the middle row, one has $(\mathbb{A}^{n-1} - \{0\}) \vee (\mathbb{A}^1 - \{0\})$.

But now we compute that the homotopy colimit of the whole diagram is the homotopy colimit of $\mathbf{pt} \leftarrow (\mathbb{A}^{n-1} - \{0\}) \vee (\mathbb{A}^1 - \{0\}) \to \mathbf{pt}$, which is equivalent to $S^1 \vee \$\mathbb{A}^{n-1} \vee \mathbb{A}^1 - \{0\} \vee S^1 \simeq S^{n-2+(n-1)\alpha} \vee S^1 \vee S^\alpha \simeq S^{(n-1)+n\alpha}$, by induction. □

Proposition 5.22. $\mathbb{P}^1 \simeq_{\mathbb{A}^1} S^1 \wedge \mathbb{G}_m$

Proof. We can write $\mathbb{P}^1$ as the pushout of $\mathbb{A}^1 \leftarrow \mathbb{G}_m \to \mathbb{A}^1$. □

In fact, the argument generalizes to

Proposition 5.23. $\mathbb{P}^n/\mathbb{P}^{n-1} \simeq_{\mathbb{A}^1} S^1 \wedge \mathbb{A}^n - \{0\}$

Proof. We cover $\mathbb{P}^n$ by the open cover $\mathbb{P}^{n-1} \times \mathbb{A}^1$ (where the first n projective coordinates are not all 0) and $\mathbb{A}^n$ (where the last projective coordinate is not 0). Then a diagram chase proves the result. □

Proposition 5.24. *Suppose $\pi : X \to Y$ is a map of schemes such that there exists a finite Zariski open cover $\{U_i\}$ of Y such that each $\pi^{-1}(U_i) \to U_i$ is an $\mathbb{A}^1$ equivalence. Then π is an $\mathbb{A}^1$ equivalence.*

Proof. It's sufficient to handle the case when there are two open sets, U_1, U_2. But then Y is the homotopy pushout of $U_1 \leftarrow U_1 \cap U_2 \to U_2$ while X is the homotopy pushout of $\pi^{-1}(U_1) \leftarrow \pi^{-1}(U_1) \cap \pi^{-1}(U_2) \to \pi^{-1}(U_2)$. Since π induces objectwise $\mathbb{A}^1$ equivalences between these two diagrams, the result follows. □

I introduced a spurious hypothesis here in class. The above version of the proposition works.

5.2.1 Jouanolou's device

Definition 5.25. Let R_n denote the coordinate ring of $n \times n$ matrices with rank ≤ 1 and trace 1. That is

$$R_n = \frac{k[x_{11}, \ldots, x_{nn}]}{\left((x_{ij}x_{k\ell} - x_{i\ell}x_{kj})_{i,j,k,\ell}, \sum_{i=1}^n x_{ii}\right)}$$

here i, j, k, ℓ range over all possible quadruples. Note that the rank of a matrix is 0 if and only if the matrix is the 0 matrix, so this is equally the coordinate ring of $n \times n$ matrices with rank 1 and trace 1.

Observe that a map $A \to R_n$ is exactly equivalent to the data of an $n \times n$ matrix of entries in A such that all the 2×2 minors vanish and such that the trace is 1.

Let J_n denote $\operatorname{Spec} R_n$.

Suppose R is a reduced ring (no nilpotents). We say a matrix N over R has rank s if, for all maps $m : R \to F$ where F is a field, the image $m(N)$, a matrix over F, has rank s.

Recall that $\mathbb{P}^{n-1}$, viewed as a functor on affine schemes, represents the functor sending $\operatorname{Spec} R$ to the set of equivalence classes of epimorphisms

$$R^n \to \mathscr{L} \to 0$$

where L is projective of rank 1, considered up to action by $R^\times$.

Proposition 5.26. *Given a reduced k-algebra R, the following data are equivalent*

1. *An $n \times n$ matrix N over R of rank 1 and trace 1.*

2. *An $n \times n$ matrix N such that $\operatorname{im} N$ is a rank 1 projective module and such that $N^2 = N$.*

3. *A short exact sequence $0 \to K \to R^n \to L \xrightarrow{\phi} 0$ of R modules where L is rank 1 projective, along with a splitting map $\psi : L \to R^n$, considered up to the equivalence relation $(\phi, \psi) \sim (r\phi, r^{-1}\psi)$ for $r \in R^\times$.*

Proof. 1. Suppose N is of rank 1 and of trace 1. Let $R \to F$ be a map from R to a field and let $\bar{N}$ denote the image of N over F. Then $\bar{N}$ has the eigenvalue 0 with (geometric) multiplicity $n-1$ and 1 with multiplicity 1, the first for rank reasons, the second because the trace is 1. Observe $\bar{N}$ is diagonalizable, and on a basis of eigenvectors it is easy to verify that $\bar{N}^2 = \bar{N}$. Then for the original matrix N, the matrix $N^2 - N$ must vanish under all maps $R \to F$. Since R is reduced, this implies that $N^2 - N = 0$.

Observe that $\ker N = \operatorname{im}(I_n - N)$ and $\ker(I_n - N) = \operatorname{im} N$. Then $\vec{v} \mapsto N\vec{v} \oplus (I_n - N)\vec{v}$ gives a direct sum decomposition of $R^n \cong \operatorname{im} N \oplus \ker N$. This proves that $\operatorname{im} N$ is projective (and it is obviously rank 1 over R since its image over any field is rank 1).

2. Conversely, if N has a projective module of rank 1 as its image, and satisfies $N^2 - N$, then over any field, the eigenvalues of $\bar{N}$ will be 0 with (geom) multiplicity $n-1$ and 1 with multiplicity 1. Therefore the trace is 1.

3. Suppose given N such that $\operatorname{im} N$ is rank 1 projective and such that $N^2 = N$. Then define $L = \operatorname{im} N$, define $\phi : R^n \to L$ to be multiplication by N and define ψ to be inclusion.

4. Given ϕ and ψ as stated, we consider the self map $R^n \to R^n$ given by the composite $\psi \circ \phi$. This is linear, has image projective of rank 1, and is idempotent. Therefore it is represented by a matrix N such that $N^2 = N$ and such that $\operatorname{im} N$ has rank 1. Observe that the construction of N doesn't change if we replace (ϕ, ψ) by $(r\phi, r^{-1}\psi)$.

□

Proposition 5.27. *There is a forgetful morphism $j : J_n \to \mathbb{P}^{n-1}$; on the level of represented functors, this takes the $n \times n$ matrix N to the class of the map $R^n \to \operatorname{im} N \to 0$. This map j is an $\mathbb{A}^1$ equivalence.*

Proof. The fact that such a map exists is a corollary of the previous proposition (and the Yoneda lemma).

To see that it's an $\mathbb{A}^1$ equivalence, we use a standard open cover of $\mathbb{P}^{n-1}$, that where the open sets U_i are the sets where the i-th coordinate doesn't vanish. For instance, let us consider U_1, where the first coordinate doesn't vanish. As a functor, this corresponds to those maps $\phi : R^n \to L \to 0$ where the first basis element $e_1 \in R^n$ maps to a nonvanishing element of L. This implies $\phi(e_1)$ is not 0 even after mapping to any field from R. It follows that for these maps $\phi(e_1)$ is a generator of $L \cong R$. After scaling, we may assume $\phi(e_1) = 1$. Then the images of the other e_i are an $n-1$-tuple $(r_2, \dots, r_n) \in R^{n-1}$. So this open set is isomorphic to $\mathbb{A}^{n-1}$, as we might have expected from our geometry course. Now to consider the set of all possible splittings $R\phi(e_1) \to R^n$. All that is necessary is that $\phi(e_1)$ be taken to something of the form $s_1e_1 + s_2e_2 + \cdots + s_ne_n$ where $s_1 + r_2s_2 + r_3s_3 + \cdots + r_ns_n = 0$. But this is an affine space, given by setting $s_1 = -r_2s_2 + r_3s_3 + \cdots + r_ns_n$. So we have shown that $j^{-1}(U_1) \cong U_1 \times \mathbb{A}^{n-1}$, and in particular, $j : j^{-1}(U_1) \to U_1$ is an $\mathbb{A}^1$ equivalence. $\square$

Corollary 5.28. *Let $X \subset \mathbb{P}^{n-1}$ be a closed subvariety, then $j|_{j^{-1}(X)} : j^{-1}(X) \to X$ is an $\mathbb{A}^1$ equivalence and $j^{-1}(X)$ is an affine variety.*

In fact, with a little more thought, this extends to all quasiprojective varieties.

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