USING $\mathbb{A}^1$ HOMOTOPY THEORY TO ADDRESS PROBLEMS IN ALGEBRA

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Part 1. First talk

Work over a field k. Let Sm_k denote the category of smooth, finite-type, separated k-schemes. We embed this category in one with all small limits and colimits.

$$\begin{tikzcd} \eta\colon \mathsf{Sm}_k \arrow[r, hook] & \mathsf{PreSm}_k \arrow[r, hook] \arrow[d, equal] & \mathsf{sPreSm}_k \arrow[d, equal] \\ & \mathsf{Fun}(\mathsf{Sm}_k^{\mathrm{op}}, \mathsf{Set}) & \mathsf{Fun}(\mathsf{Sm}_k^{\mathrm{op}}, \mathsf{sSet}) \end{tikzcd}$$

Here η is the Yoneda embedding: $\eta(X)(U) = \mathrm{Mor}_{\mathsf{Sm}_k}(U, X)$. We'll usually omit η.

The functor $X(-)$ is determined by its value on affine schemes $X(\operatorname{Spec} R)$ — write this as $X(R)$. This is the functor-of-points formalism for schemes. Eisenbud and Harris 1999, Chapter VI

Example.

$$\mathbb{A}^1_k(R) = \mathrm{Mor}_{\mathsf{Sm}_k}\big(\operatorname{Spec} R,\ \operatorname{Spec} k[t]\big) = \mathrm{Mor}_{k\text{-alg}}\big(k[t],\ R\big) = R.$$

$$\mathbb{A}^n_k(R) = R^n,$$
$$\big(\mathbb{A}^n_k \setminus \{0\}\big)(R) = \big\{(x_1, \dots, x_n) \in R^n \ \mid\ \langle x_1, \dots, x_n\rangle = \langle 1\rangle\big\}.$$

$$\mathrm{SL}_n(R) = \big\{M \in \mathrm{Mat}_{n\times n}(R) = R^{n^2} \mid \det(M) = 1\big\}.$$

$$V_t(\mathbb{A}^n)(R) := \big\{(B, C) \in \mathrm{Mat}_{n\times t}(R)^2 \ \mid\ B^T C = I_t\big\}$$

I will call this a *Stiefel variety*. Basepoint $\big(I_t \mid 0\big)^2$.

Provided $t < n$, $\mathrm{SL}_n(R)$ acts on $V_t(\mathbb{A}^n)(R)$ transitively, with stabilizer $\mathrm{SL}_{n-t}(R)$.

$$\mathrm{SL}_n / \mathrm{SL}_{n-t} \ \cong\ V_t(\mathbb{A}^n).$$

The varieties $V_t(\mathbb{A}^n)$ parametrize pairs (B, C) such that $B^T C = I_t$. You can instead define

$$V'_t(\mathbb{A}^n)(R) = \big\{B \in \mathrm{Mat}_{n\times t} \ \mid\ \exists C \text{ for which } B^T C = I_t\big\}.$$

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This happens to be a scheme. There is a morphism

$$f : V_t(\mathbb{A}^n) \longrightarrow V_t'(\mathbb{A}^n) \qquad \text{(forget } C\text{)}.$$

Claim 0.1. There is a (Zariski) open cover $\{U_i\}$ of $V_t'(\mathbb{A}^n)$ with the property that

$$\begin{tikzcd} f^{-1}(U_i) \arrow[r] \arrow[d, "\wr" description, phantom] & U_i \arrow[d, "\wr" description, phantom] \\ U_i \times \mathbb{A}_k^{nt-t^2} \arrow[r, "\text{proj}"'] & U_i \end{tikzcd}$$

(Pick a different row for each column of B — require the element there to be a unit.)

Corollary 0.2. *f is an $\mathbb{A}^1$-equivalence: the base is a colimit of a diagram of the U's and their intersections. Write $U_\bullet$ for this diagram.*

$$\begin{tikzcd} \operatorname{colim} U_\bullet \times \mathbb{A}^{nt-t^2} \arrow[r, "\sim"] \arrow[d, "\cong"'] & V_t(\mathbb{A}^n) \arrow[d, "\sim"', "f"] \\ \operatorname{colim} U_\bullet \arrow[r, "\cong"'] & V_t'(\mathbb{A}^n) \end{tikzcd}$$

$$V_1(\mathbb{A}^n) \xrightarrow{\sim} V_1'(\mathbb{A}^n) = \mathbb{A}^n \setminus \{0\} \simeq S^{2n-1,n} \qquad \text{“}S^{n-1+(n)}\text{”}$$

Stating a problem

An element of $V_s(\mathbb{A}^n)(R)$ gives us a split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \ker B^T \arrow[r] & R^n \arrow[r, "B^T"] & R^s \arrow[l, bend left, "C"] \arrow[r] & 0 \end{tikzcd}$$

Let me write $P = \ker(B^T)$.

$$(\star) \qquad P \oplus R^s \xrightarrow[\cong]{\text{incl}\oplus c} R^n \quad \text{is an isomorphism.}$$

P is said to be *stably free of type* $(n, n-s)$ (notation taken from Raynaud 1968). (P determined by B; $(\star)$ determines B, C.)

If $s' < s$, you can forget columns in B, C and get

$$\begin{tikzcd} \rho : V_s(\mathbb{A}^n) \arrow[r] & V_{s'}(\mathbb{A}^n) \arrow[l, dashed, bend left, "\tau"] \end{tikzcd}$$

The effect on P is to add an $R^{s-s'}$-summand. That is $P \rightsquigarrow P \oplus R^{s-s'}$

Question: for what values of s, n, s' does ρ admit a right inverse τ?

Equivalent question: for what values of n, s does every stably free R-module (over every commutative k-algebra) of type $(n, n-s)$ have a free summand of rank $s - s'$?

This problem is seminal in the history of applying homotopy theory to algebra: Raynaud 1968.

Special case: if $s' = 1$, $s = 2$, n even, then one can construct

$$\tau : \left(\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \right) \rightsquigarrow \left(\begin{bmatrix} b_1 & -c_2 \\ b_2 & c_1 \\ \vdots & \vdots \\ b_{n-1} & -c_n \\ b_n & c_{n-1} \end{bmatrix}, \begin{bmatrix} c_1 & -b_2 \\ c_2 & b_1 \\ \vdots & \vdots \\ c_{n-1} & -b_n \\ c_n & b_{n-1} \end{bmatrix} \right)$$

(Bass, pre-1968.)

Take as read that we have an unstable $\mathbb{A}^1$-homotopy theory of pointed objects

$$(X, x_0), \qquad x_0 : \operatorname{Spec} k \longrightarrow X, \qquad x_0 \text{ often dropped,}$$

and for all R, $X(R)$ is a simplicial set.

Constructions are possible: $X \times Y$, $X \vee Y$,

$$X \wedge Y = (X \times Y)/(X \vee Y) \quad \text{— adjoint to the internal mapping object.}$$

There is the superstructure of homotopy theory — ∞-categorical or model theoretic.

There is the *homotopy category* $\mathscr{H}(k)_*$ in which all $\mathbb{A}^1$-equivalences are declared to be isomorphisms. This forces homotopic maps to be equal, among other things.

We will write $[X, Y]_{\mathbb{A}^1}$ for the set of maps $X \to Y$ in the homotopy category.

We will be talking about the spheres: S^n is a simplicial sphere. By contrast, $S^{(1)}$ means $\mathbb{A}^1 \setminus \{0\}$ with basepoint 1. Putting these together, $S^{a+(b)}$ means $S^a \wedge S^{(1)} \wedge \cdots \wedge S^{(1)}$ (b copies). Other notation is $S^{a+b,b}$.

Realization. Heuristic: if you started with smooth manifolds instead of Sm_k and did the analogous construction for $\mathbb{A}^1$-homotopy, you would recover ordinary homotopy.

Suppose $k \hookrightarrow \mathbb{C}$; then GAGA (Serre 1956) gives us a functor

$$\mathsf{Sm}_k \longrightarrow \mathsf{Top}, \qquad X \longmapsto |X|_{\mathbb{C}}.$$

This propagates through the homotopy theory (this is treated already in Morel and Voevodsky 1999 at the homotopy category level, but a treatment in model categories, and therefore ∞-categories, is given in Dugger and Isaksen 2004)

$$\mathscr{H}(k)_\bullet \longrightarrow \mathscr{H}_\bullet.$$

This sends finite products to finite products and is in the way of being a left adjoint—it preserves colimit-type constructions.

$$S^{p+(q)} \longrightarrow |S^p \wedge \mathbb{G}_m^{\wedge q}|_{\mathbb{C}} = S^p \wedge |\mathbb{C}^\times|^{\wedge q} = S^{p+q}.$$

$$V_t(\mathbb{A}^n) \longrightarrow |\mathrm{SL}_n(\mathbb{C})/\mathrm{SL}_{n-t}(\mathbb{C})| \simeq U_n/U_{n-t} = W_t(\mathbb{C}^n)$$

(see James 1976 for more about the topology of Stiefel manifolds)

If $k \hookrightarrow \mathbb{R}$, then there is a version of this where we take $X(\mathbb{C})$ with $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action ("$C_2$-action"). Taking fixed points gives us

$$|\cdot|_{\mathbb{R}} : \mathscr{H}(k)_\bullet \longrightarrow \mathscr{H}_\bullet \qquad \text{(real realization).}$$

It may be useful to know this is mediated through C_2-equivariant homotopy. This may also be found in Dugger and Isaksen 2004.

Étale realization. This is a difficult topic, and cannot be treated fully in the time available. (Isaksen 2004 is the source for the realization material. A more recent treatment is also in Hoyois 2023. For the étale homotopy type, Friedlander 1982 is the source for what the realization gives. This builds on Artin and Mazur 1969.)

In arbitrary characteristic, one can produce a pro-object

$$X \longmapsto |X|_{\text{ét}} \in \mathsf{pro\text{-}Top},$$

built out of the category of étale covers of X. This gives a functor

$$\mathscr{H}(k)_\bullet \longrightarrow \text{pro-}\mathscr{H}_\bullet \qquad \text{(is this left Quillen).}$$

There are cases where we know almost everything about $|X|_{\text{ét}}$.

I should say a word or two about completion. If ℓ is a prime number, then a topological space Y has an ℓ-completion

$$Y \longrightarrow Y_\ell^\wedge.$$

(Loosely) $Y_\ell^\wedge$ depends only on the aspects of Y that can be recovered from invariants valued in $\mathbb{Z}/(\ell)$-vector spaces. (A reference for completion *per se* is Bousfield and Kan 1972, building on Sullivan 1974).

In particular, if $Y \to Z$ induces an equivalence $Y_\ell^\wedge \xrightarrow{\sim} Z_\ell^\wedge$, then

$$H_*\big(Y;\mathbb{Z}/(\ell)\big) \cong H_*\big(Z;\mathbb{Z}/(\ell)\big).$$

In fact, this is close to a characterizing property for the ℓ-completion Bousfield and Kan 1972, Ch VI, Proposition 5.4.

This allows me to state a comparison result. Assume k is algebraically closed[1]. Let G be a Chevalley group over $\mathbb{Z}$ — e.g. GL_n, SL_n, &c. Let ℓ be a prime invertible in k.

$$\lim\Big(|G_k|_{\text{ét}}{}_\ell^\wedge\Big) \simeq |G_{\mathbb{C}}|_\ell^\wedge$$

(the left side is the characteristic-p construction, the right side the topological one).

This is hard work, but we get something. Assume $k = \bar{k}$. Building from here (Friedlander 1982, Chapter 8 and Friedlander 1973) we can say

$$\lim\Big(|V_t(\mathbb{A}^n)|_{\text{ét}}{}_\ell^\wedge\Big) \simeq W_t(\mathbb{C}^n)_\ell^\wedge, \qquad \ell \text{ invertible in } k.$$

Theorem 0.3 (Raynaud, special case — different proof)**.** *$\rho : V_2(\mathbb{A}^n) \to V_1(\mathbb{A}^n)$ has no right inverse if n is odd ($n \geq 3$) and* $\mathrm{char}\, k \neq 2$.

[1]Thanks to Kirsten Wickelgren for reminding me of this.

Proof. Suppose for the sake of contradiction that it did, via τ. Apply étale realization and 2-completion. We get a section of $|\tau|_{\text{ét}}$:

$$\begin{tikzcd} W_2(\mathbb{C}^n)^\wedge_2 \arrow[r] & W_1(\mathbb{C}^n)^\wedge_2 \arrow[l, bend left, "|\tau|_{\text{ét}}"] \end{tikzcd}$$

But

$$\begin{tikzcd} \pi_{2n-1} W_2(\mathbb{C}^n)^\wedge_2 \arrow[r, "\cong"] \arrow[d] & \mathbb{Z}^\wedge_2 \arrow[d, "\times 2"] \\ \pi_{2n-1} W_1(\mathbb{C}^n)^\wedge_2 \arrow[r, "\cong"'] & \mathbb{Z}^\wedge_2 \end{tikzcd} \qquad \text{(Gilmore 1967)}$$

with $W_1(\mathbb{C}^n) \simeq S^{2n-1}$, so there is no section. $\square$

Primozic 2022 extended this result to characteristic 2.

James numbers

I define a sequence of *James numbers* $b_2, b_3, \dots$ by means of their p-adic valuations:

$$v_p(b_q) = \max\left\{ s + v_p(s) \mid 1 \le s \le \frac{q-1}{p-1} \right\},$$

with the understanding that the maximum of the empty set is 0.

Theorem (Adams and Walker 1965)**.** *The map $\rho : W_s(\mathbb{C}^n) \to W_1(\mathbb{C}^n)$ of manifolds has a right inverse if and only if $b_s \mid n$.*

Part 2. Second Talk

$\mathbb{A}^1_k = \operatorname{Spec} k[t]$ has two points

$$i_0 : \operatorname{Spec} k \xrightarrow{0} \mathbb{A}^1_k, \qquad i_1 : \operatorname{Spec} k \xrightarrow{1} \mathbb{A}^1_k.$$

Given $F : X \times \mathbb{A}^1 \longrightarrow Y$, one can form

$$F_0 = F \circ i_0, \qquad F_1 = F \circ i_1 \ : \ X \longrightarrow Y.$$

We say these are *naively homotopic*. The relation is not an equivalence relation (not transitive), so we define $\simeq_N$ to be the transitive closure.

$$[X_+, Y]_N \ = \ \operatorname{Mor}_{\mathsf{Sm}_k}(X, Y) \big/ {\simeq_N} \, .$$

$$(\star) \qquad [X_+, Y]_N \xrightarrow{\text{natural}} [X_+, Y]_{\mathbb{A}^1}$$

There is a condition of being $\mathbb{A}^1$*-naive*: Asok, Hoyois, and Wendt 2018.

If Y is $\mathbb{A}^1$-naive, then

$$[U_+, Y]_N \xrightarrow{\cong} [U_+, Y]_{\mathbb{A}^1}$$

for affine U.

Most things are not $\mathbb{A}^1$-naive as far as I know; examples are relatively hard to prove Balwe, Hogadi, and Sawant 2015, Section 4.1 (note that the failure of $\mathbb{A}^1$-locality may be witnessed by an affine scheme). Ref

Remark 0.4.

$$[\widetilde{\mathbb{P}}^1_+, -]_N \neq [\mathbb{P}^1_+, -]_N \quad \text{usually.}$$

For instance, if the target is SL_2, then the right hand side is necessarily $*$, whereas the left contains $\pi^{\mathbb{A}^1}_{1+(1)}(\mathrm{SL}_2)$, which is $\mathbf{K}^{\mathsf{MW}}_1(k)$ (Morel 2012, Theorem 1.27, over a perfect k).

Theorem 0.5. *Asok, Hoyois, and Wendt 2018, Theorem 2.4.2 $V_t(\mathbb{A}^n)$ is $\mathbb{A}^1$-naive.*

Theorem 0.6 (Gant and Williams 2025a)**.** *TFAE:*

1. $\rho : V_s(\mathbb{A}^n) \to V_1(\mathbb{A}^n)$ *has a right inverse in* Sm_k*;*
2. $\rho : V_s(\mathbb{A}^n) \to V_1(\mathbb{A}^n)$ *has a right inverse in* $\mathscr{H}(k)_\bullet$*;*
3. *for every commutative k-algebra, every stably free module P of type* $(n, n-1)$ *admits a free summand of rank* $s-1$.

Proof. If we ignore basepoints (the argument that says we can do this is not difficult, and not very interesting), then we see that 1 implies 2 and 1 implies 3. Assume 2.

A major result of Lindel 1981 says that for a regular ring R finite type over a field k, every projective module over $R[t]$ is extended from a module over R. In particular, this implies that if $\operatorname{Spec} R \times \mathbb{A}^1_k \to V_1(\mathbb{A}^n)$ is a naive homotopy, then the stably free modules it represents at times $t = 0, 1$ are isomorphic (note, the B, C matrices are potentially different, but $\ker B^T$ is isomorphic across the family).

Since $V_s(\mathbb{A}^n)$ is affine, smooth over k, and $\mathbb{A}^1$-naive, property 2 implies there is a map $f : V_1(\mathbb{A}^n) \to V_s(\mathbb{A}^n)$ that is naively homotopy inverse to ρ. This means that given P, stably free of type $(n, n-1)$, we can find some Q with the property that $Q \oplus R^{s-1} \cong P$, using Lindel's theorem.

The argument that this allows one to construct a precise right inverse for ρ in Sm_k is given in Raynaud 1968. □

HOMOTOPY GROUPS AND THEIR CALCULATION

We could mean several things by "homotopy group". For (X, x_0):

(1) $\pi_n^{\mathbb{A}^1}(X) = [S^n, X]_{\mathbb{A}^1}$
(2) $\pi_{n+(m)}^{\mathbb{A}^1}(X) = [S^{n+(m)}, X]_{\mathbb{A}^1}$, where $S^{n+(m)} = S^n \wedge (\mathbb{A}^1 \setminus \{0\})^{\wedge m}$

(global sections, since $\operatorname{Spec} k$ is a stalk)

(1) $\pi_{n+(m)}^{\mathbb{A}^1,\mathrm{pre}}(X) : U \longmapsto [S^{n+(m)} \wedge U_+, X]$ — a functor $\mathsf{Sm}_k \to \mathsf{Set}/\mathsf{Ab}$
(2) $\boldsymbol{\pi}_{n+(m)}(X)$ — Nisnevich sheafification of the above.

Homotopy groups are the most widely studied examples of spaces / sets of maps in ($\mathbb{A}^1$-)homotopy theory.

For present purposes,

$$V_1(\mathbb{A}^n) \simeq S^{n-1+(n)}, \qquad \left[V_1'(\mathbb{A}^n) = \mathbb{A}^n \setminus \{0\}\right].$$

Therefore we can consider ourselves interested in

$$\pi_{n-1+(n)}^{\mathbb{A}^1} V_t(\mathbb{A}^n) = \left[V_1(\mathbb{A}^n),\ V_t(\mathbb{A}^n)\right]$$

and its relation to

$$\pi_{n-1+(n)}^{\mathbb{A}^1} V_1(\mathbb{A}^n) \qquad \longleftarrow \text{a homotopy group of spheres.}$$

There is a stable homotopy category, $\mathscr{SH}(k)$:

$$\Sigma_{\mathbb{P}^1}^{\infty} : \mathscr{H}(k)_\bullet \rightleftarrows \mathscr{SH}(k) : \Omega_{\mathbb{P}^1}^{\infty}$$

forcing

$$S^{1+(1)} \wedge (-) = S^1 \wedge S^{(1)} \wedge (-)$$

to be an equivalence of categories.

Important and relatively computable functors

$$\mathsf{Sm}_{k,*} \longrightarrow \mathscr{SH}(k) \longrightarrow \mathsf{GrAb}$$

such as $H_{\mathrm{Mot}}^{*,*}(-; A)$ or $K_*(-)$ factor through $\mathscr{SH}(k)$.

$$\left[S^{a+(b)},\ X\right]_{\mathscr{SH}(k)} =: \pi_{a+(b)}^{s,\mathbb{A}^1}(X).$$

The groups $\pi_{a+(b)}^{s,\mathbb{A}^1}(\mathbb{1})$ are thoroughly studied. There are two big approaches.

(1) **Slice spectral sequence** (see Röndigs, Spitzweck, and Østvær 2024 for the calculations used here). Calculates homotopy sheaves, over a general base (requires inverting p in characteristic $p > 0$). $0 \le a \le 2 + \varepsilon$
(2) **Adams-style spectral sequence**: Isaksen and coauthors (Dugger and Isaksen 2010, Isaksen, Wang, and Xu 2023, Ormsby and Østvær 2014, mostly over $\mathbb{C}$; calculates groups; topologically facing; a very large computation indeed: $\pi_{a+(b)}(\mathbb{1})^\wedge_\ell$, ℓ-complete. (Inputs are $\mathbb{Z}/(\ell)$-valued cohomology theories.) See Gheorghe and Isaksen 2017 for a map.

Observation. It is known from elsewhere that $\pi_{a+(b)}(\mathbb{1})$ is torsion (even torsion of bounded order) when $a > 0$. Even absent finite-generation assumptions, this is enough to reconstruct $\pi_{a+(b)}(\mathbb{1})$ from the ℓ-completions (for all ℓ).

FIBRE AND COFIBRE SEQUENCES

We have the notion of a fibre bundle

$$\left.\begin{tikzcd} U \times F \simeq E \times_B U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ U \arrow[r] & B \end{tikzcd}\right\}\text{quotient map}$$

where $U \to B$ admits a cover by Zariski / Nisnevich / étale / fppf. Well-behaved quotients by free actions give examples.

In classical topology, there is a lifting property that leads to long exact sequences of homotopy groups: $F \to E \to B$ is a *homotopy fibre sequence* $\rightsquigarrow$

$$\cdots \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \longrightarrow \cdots$$

You don't get this in $\mathbb{A}^1$-homotopy—the lifting fails in general. You still get the fibre sequence under $\mathbb{A}^1$-invariance conditions on the base. See for instance Morel 2012, Theorem 6.50 or Wendt 2011, Section 6.

$$\mathrm{SL}_{n-t} \longrightarrow \mathrm{SL}_n \longrightarrow V_t(\mathbb{A}^n) \qquad (t \ge 1)$$

is an $\mathbb{A}^1$-homotopy fibre sequence. Here, a homotopy version of the 3×3 lemma applies.

$$\begin{tikzcd} \mathrm{SL}_{n-t} \arrow[r] \arrow[d] & \mathrm{SL}_{n-t} \arrow[r] \arrow[d] & * \arrow[d] \\ \mathrm{SL}_{n-1} \arrow[r] \arrow[d] & \mathrm{SL}_n \arrow[r] \arrow[d] & V_1\mathbb{A}^n \arrow[d] \\ V_{t-1}\mathbb{A}^{n-1} \arrow[r] & V_t\mathbb{A}^n \arrow[r] & V_1\mathbb{A}^n \end{tikzcd}$$

Lemma 0.7 (Gheorghe and Isaksen 2017 Stahn 2018, Gant and Williams 2025b.)**.** *Over $k = \mathbb{C}$,*

$$\pi^{s,\mathbb{A}^1}_{a+(b)}(\mathbb{1}) \cong \pi^{s}_{a+b}(S^0)$$

when $a+b>0$ and $0 \le b \le a+2$.

The hard part (i.e., the part for which I refer to Gheorghe and Isaksen 2017 and Stahn 2018) is to compute the p-completed groups. The fact that these agree with the p-completed classical groups in a range depends substantially on a result of Levine 2014.

Theorem 0.8 (A corollary of Asok, Bachmann, and Hopkins 2024, Theorem 6.2.1)**.** *Assume k is a field of characteristic 0:*

$$\pi^{\mathbb{A}^1}_{x+a+(y+b)}(S^{x+(y)}) \xrightarrow{\cong} \pi^{s,\mathbb{A}^1}_{a+(b)}(S^{0,0})$$

provided $x, y \ge 2$ and $b \le \min\{x-2,\, y-2\}$.

(James number)

$$\nu_p(b_t) \;=\; \max\left\{ s + \nu_p(s) \;\middle|\; 1 \le s \le \left\lfloor \tfrac{t-1}{p-1} \right\rfloor \right\} \qquad \text{(0 if the set is empty).}$$

Theorem 0.9 (Gant and Williams 2025b)**.** *Suppose R is a ring containing $\mathbb{C}$. If P is an R-module for which there exists*

$$P \oplus R \;\cong\; R^n, \qquad \textit{where } b_t \mid n,$$

then

$$P \;\cong\; Q \oplus R^{t-1}.$$

Method:

1. (1) Recast as a statement about $V_t\mathbb{A}^n \to V_1\mathbb{A}^n$.
2. (2) Homotopical algebra — recast as a statement about $\pi_{a+(b)}V_1(\mathbb{A}^n)$, using the fibre sequence.
3. (3) Using the results above, compare to classical topology.
4. (4) Problem solved for $W_t(\mathbb{C}^n) \to W_1(\mathbb{C}^n)$ by Adams–Walker (see Adams and Walker 1965) building on work of Toda, James (James 1958), Atiyah–Todd Atiyah and Todd 1960.

References

Adams, J. Frank and G. Walker (1965). "On Complex Stiefel Manifolds". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 61.01, p. 81. ISSN: 0305-0041, 1469-8064. DOI: 10.1017/S0305004100038688. (Visited on 04/26/2017).

Artin, M. and B. Mazur (1969). *Étale Homotopy*. Lecture Notes in Mathematics, No. 100. Berlin: Springer-Verlag.

Asok, Aravind, Tom Bachmann, and Michael J. Hopkins (2024). *On $\mathbb{P}^1$-Stabilization in Unstable Motivic Homotopy Theory*. DOI: 10.48550/arXiv.2306.04631. arXiv: 2306.04631 [math]. (Visited on 07/24/2024).

Asok, Aravind, Marc Hoyois, and Matthias Wendt (2018). "Affine Representability Results in $\mathbb{A}^1$-Homotopy Theory, II: Principal Bundles and Homogeneous Spaces". In: *Geometry & Topology* 22.2, pp. 1181–1225. ISSN: 1465-3060. DOI: 10.2140/gt.2018.22.1181. (Visited on 08/09/2018).

Atiyah, M. F. and J. A. Todd (1960). "On Complex Stiefel Manifolds". In: *Proceedings of the Cambridge Philosophical Society* 56, pp. 342–353. ISSN: 0008-1981. DOI: 10.1017/s0305004100034642. (Visited on 01/20/2022).

Balwe, Chetan, Amit Hogadi, and Anand Sawant (2015). "$\mathbb{A}^1$-Connected Components of Schemes". In: *Advances in Mathematics* 282, pp. 335–361. ISSN: 0001-8708. DOI: 10.1016/j.aim.2015.07.003. (Visited on 08/21/2026).

Bousfield, A. K. and D. M. Kan (1972). *Homotopy Limits, Completions and Localizations*. Lecture Notes in Mathematics, Vol. 304. Berlin: Springer-Verlag.

Dugger, Daniel and Daniel C. Isaksen (2004). "Topological Hypercovers and $\mathbb{A}^1$-Realizations". In: *Mathematische Zeitschrift* 246.4, pp. 667–689. ISSN: 0025-5874. DOI: 10.1007/s00209-003-0607-y.

– (2010). "The Motivic Adams Spectral Sequence". In: *Geometry & Topology* 14.2, pp. 967–1014. ISSN: 1465-3060;1364-0380; DOI: 10.2140/gt.2010.14.967. (Visited on 09/09/2025).

Eisenbud, David and Joe Harris (1999). *Geometry of Schemes*. Springer. ISBN: 978-0-387-98638-8.

Friedlander, Eric M. (1973). "The Etale Homotopy Theory of a Geometric Fibration". In: *manuscripta mathematica* 10.3, pp. 209–244. ISSN: 1432-1785. DOI: 10.1007/BF01332767. (Visited on 08/21/2026).

– (1982). *Étale Homotopy of Simplicial Schemes*. Vol. 104. Annals of Mathematics Studies. Princeton, N.J.: Princeton University Press. ISBN: 0-691-08288-X 0-691-08317-7. (Visited on 08/30/2012).

Gant, Sebastian and Ben Williams (2025a). "Free Summands of Stably Free Modules". In: *Forum of Mathematics, Sigma* 13. ISSN: 2050-5094. DOI: 10.1017/fms.2025.39. (Visited on 07/19/2025).

– (2025b). *Motivic Homotopy Groups of Spheres and Free Summands of Stably Free Modules*. DOI: 10.48550/arXiv.2510.10033. arXiv: 2510.10033 [math]. (Visited on 02/20/2026).

Gheorghe, Bogdan and Daniel C. Isaksen (2017). "The Structure of Motivic Homotopy Groups". In: *Boletín de la Sociedad Matemática Mexicana* 23.1, pp. 389–397. ISSN: 2296-4495. DOI: 10.1007/s40590-016-0094-x. (Visited on 07/31/2025).

Gilmore, Maurice E. (1967). “Complex Stiefel Manifolds, Some Homotopy Groups and Vector Fields”. In: *Bulletin (new series) of the American Mathematical Society* 73.5, pp. 630–633. ISSN: 0273-0979;1088-9485; DOI: 10.1090/S0002-9904-1967-11802-0. (Visited on 07/26/2024).

Hoyois, Marc (2023). “The Étale Symmetric Künneth Theorem”. In: *Mathematische Zeitschrift* 304.1. ISSN: 0025-5874. DOI: 10.1007/s00209-023-03246-1. (Visited on 07/31/2024).

Isaksen, Daniel C. (2004). “Etale Realization on the $\mathbb{A}^1$-Homotopy Theory of Schemes”. In: *Advances in Mathematics* 184.1, pp. 37–63. ISSN: 0001-8708. DOI: 10.1016/S0001-8708(03)00094-X. (Visited on 07/30/2018).

Isaksen, Daniel C., Guozhen Wang, and Zhouli Xu (2023). *Stable Homotopy Groups of Spheres: From Dimension 0 to 90*. DOI: 10.48550/arXiv.2001.04511. arXiv: 2001.04511 [math]. (Visited on 09/10/2025).

James, I. M. (1958). “Cross-Sections of Stiefel Manifolds”. In: *Proceedings of the London Mathematical Society* s3-8.4, pp. 536–547. ISSN: 0024-6115. DOI: 10.1112/plms/s3-8.4.536. (Visited on 07/24/2024).

– (1976). *The Topology of Stiefel Manifolds*. Cambridge, England: Cambridge University Press.

Levine, Marc (2014). “A Comparison of Motivic and Classical Stable Homotopy Theories”. In: *Journal of Topology* 7.2, pp. 327–362. ISSN: 1753-8424. DOI: 10.1112/jtopol/jtt031. (Visited on 12/10/2025).

Lindel, Hartmut (1981). “On the Bass-Quillen Conjecture Concerning Projective Modules Over Polynomial Rings.” In: *Inventiones mathematicae* 65, pp. 319–324. ISSN: 0020-9910; 1432-1297/e. DOI: 10.1007/BF01389017. (Visited on 07/28/2024).

Morel, Fabien (2012). *$\mathbb{A}^1$-Algebraic Topology over a Field*. Vol. 2052. Lecture Notes in Mathematics. Heidelberg: Springer. ISBN: 978-3-642-29513-3. (Visited on 11/04/2012).

Morel, Fabien and Vladimir Voevodsky (1999). “$\mathbb{A}^1$-Homotopy Theory of Schemes”. In: *Publications Mathématiques de L’Institut des Hautes Scientifiques* 90.1, pp. 45–143. ISSN: 0073-8301. DOI: 10.1007/BF02698831.

Ormsby, Kyle M. and Paul Arne Østvær (2014). “Stable Motivic π_1 of Low-Dimensional Fields”. In: *Advances in Mathematics* 265.Journal Article, pp. 97–131. ISSN: 0001-8708. DOI: 10.1016/j.aim.2014.07.024. (Visited on 12/18/2025).

Primozic, Eric (2022). “On Universal Stably Free Modules in Positive Characteristic”. In: *Journal of Pure and Applied Algebra* 226.5, p. 106918. ISSN: 0022-4049. DOI: 10.1016/j.jpaa.2021.106918. (Visited on 08/17/2026).

Raynaud, Michèle (1968). “Modules Projectifs Universels”. In: *Inventiones Mathematicae* 6, pp. 1–26. ISSN: 0020-9910,1432-1297. DOI: 10.1007/BF01389829.

Röndigs, Oliver, Markus Spitzweck, and Paul Arne Østvær (2024). “The Second Stable Homotopy Groups of Motivic Spheres”. In: *Duke Mathematical Journal* 173.6. ISSN: 0012-7094. DOI: 10.1215/00127094-2023-0023. (Visited on 07/29/2024).

Serre, Jean-Pierre (1956). “Géométrie Algébrique et Géométrie Analytique”. In: *Annales de l’institut Fourier* 6, pp. 1–42. ISSN: 0373-0956. DOI: 10.5802/aif.59. (Visited on 12/05/2015).

Stahn, Sven-Torben (2018). “Stable Motivic Homotopy Groups and Periodic Self Maps at Odd Primes”. PhD thesis. Wuppertal: Bergischen Universität Wuppertal. (Visited on 08/07/2025).

Sullivan, Dennis (1974). “Genetics of Homotopy Theory and the Adams Conjecture”. In: *Annals of Mathematics.* Second Series 100.1, pp. 1–79. ISSN: 0003-486X. DOI: 10.2307/1970841. (Visited on 10/14/2014).

Wendt, Matthias (2011). “Rationally Trivial Torsors in $\mathbb{A}^1$-Homotopy Theory”. In: *Journal of K-Theory. K-Theory and its Applications in Algebra, Geometry, Analysis & Topology* 7.3, pp. 541–572. ISSN: 1865-2433. DOI: 10.1017/is011004020jkt157. (Visited on 11/03/2012).