CONNECTIVITY OF MANIFOLD COMPLEMENTS

BEN WILLIAMS

Abstract. It is well known that if M is a connected smooth manifold with a basepoint m_0, then the homotopy groups $\pi_i(M, m_0)$ may be defined using only smooth maps from spheres and smooth homotopies between them. It is also a well-known fact that if M is a smooth manifold and Z is a codimension-d submanifold, then the inclusion $M \setminus Z \to M$ is $d-1$-connected. Proofs of these two results are given.

1. Introduction

The purpose of this note is to prove Theorems 1.7 and 1.11. The first says that the homotopy groups of a manifold may equally well be calculated using continuous maps and homotopies or smooth maps and homotopies. The second says that if M is a manifold and Z a closed submanifold of codimension d, then the inclusion $M \setminus Z \to M$ is $d-1$-connected. The manifolds in question are smooth separable real manifolds without boundary, but they are not assumed to be compact.

These two results are folklore. The first result is mentioned in [BT82], but I have not seen it proved in detail. I do not believe a proof of the second result appears in the peer-reviewed literature, although a proof is sketched in the notes of [Ful07, Appendix A, Proposition 4.1], attributed to D. Speyer. The proof of 1.11 in this note is modelled on the proof there.

Throughout, the term *smooth manifold* means a smooth separable manifold without boundary. The term *smooth manifold with boundary* means a smooth separable manifold with a possibly empty boundary. In general, we do not require our manifolds to be connected, and when we say that Z is of *codimension* d in M, we mean that the minimal codimension of a connected component of Z_i in M is d.

1.1. Smooth homotopy groups.

Definition 1.1. Suppose $f, g : N \to M$ are two smooth maps between smooth manifolds. A *smooth homotopy* from f to g is a map $H : N \times I \to M$ restricting to f (resp. g) at 0 (resp. 1) and such that H extends to a smooth map of some open neighbourhood of $N \times I$ in $N \times \mathbb{R}$.

Lemma 1.2. *Smooth homotopy is an equivalence relation on smooth maps $N \to M$.*

This is [Lee12, Lemma 6.28].

Lemma 1.3. *Suppose $f, g : N \to M$ are two smooth maps that are homotopic relative to some closed $A \subseteq N$ (note that A may be empty). Then f and g are smoothly homotopic relative to A.*

This is [Lee12, Theorem 6.29].

Theorem 1.4 (Whitney Approximation Theorem)**.** *Suppose $f : N \to M$ is a continuous function where the source is a smooth manifold with boundary and the target is a smooth manifold. Suppose $A \subseteq N$ is a closed subset such that $f|_A$ is smooth. Then f is homotopic relative to A to a smooth map $\bar{f} : N \to M$.*

This is [Lee12, Theorem 6.26].

Corollary 1.5 (Extension Lemma)**.** *Suppose N is a smooth manifold with boundary, M a smooth manifold without boundary, $A \subseteq N$ a closed subset and $f : A \to M$ a smooth map. Then f has a smooth extension to N if and only if it has a continuous extension to N.*

This is [Lee12, Corollary 6.27].

Definition 1.6. Let M be a smooth manifold with basepoint m_0. For any integer $n \geq 0$, let $\pi_n^{\text{smooth}}(M, m_0)$ denote the smooth homotopy classes of basepoint-preserving maps $S^n \to M$.

There is a natural transformation $\pi_n^{\text{smooth}}(M, m_0) \to \pi_n(M, m_0)$ between functors defined on the category of smooth manifolds with basepoints, taking values in the category of pointed sets (when $n = 0$) or groups (when $n = 1$) or abelian groups ($n \geq 2$).

Theorem 1.7. *The natural transformation $\pi_n^{smooth}(M, m_0) \to \pi_n(M, m_0)$ is an isomorphism.*

Proof. The set $\pi_n^{\text{smooth}}(M, m_0)$ is functorial for smooth basepoint-preserving maps $M \to M'$. There is a natural map $\pi_n^{\text{smooth}} \to \pi_n$. By Lemma 1.3, this natural map is injective and by Theorem 1.4 it is surjective. □

1.2. **Transversality.**

The following definition is taken from [Lee12, p. 143].

Definition 1.8. If $F : N \to M$ is a smooth map and $Z \subseteq M$ is an embedded submanifold, we say that F is *transverse* to S if for every $x \in F^{-1}(Z)$, the spaces $T_{F(x)}(Z)$ and $dF_x(T_x N)$ together span $T_{F(x)}(M)$.

We use only the most primitive consequence of transversality in this note:

Lemma 1.9. *Suppose N and M are smooth manifolds of dimensions n and m respectively, and $Z \subseteq M$ is a smooth submanifold of dimension z. Suppose $F : N \to M$ is a smooth map that is transverse to Z. Suppose $m > n + z$. Then $F(N) \cap Z = \emptyset$.*

Proof. Suppose for the sake of contradiction that we can find $x \in F^{-1}(Z)$. Then $T_{F(x)}(Z)$ has dimension z and $dF_x(T_x N)$ has dimension no greater than n. In particular

$$m = \dim_R T_{F(x)} M = \dim_{\mathbb{R}}(T_{F(x)}(Z) + dF_x(T_x N)) \leq z + n < m,$$

a contradiction. □

Theorem 1.10 (Extension Theorem)**.** *Let N be a smooth manifold with boundary and M a smooth manifold. Suppose Z is a closed submanifold of M. Suppose C is a closed subset of N, and suppose $f : N \to M$ is a smooth map such that $f|_C$ is transverse to Z and $\partial f|_{C \cap \partial N}$ is transverse to Z. Then there exists a smooth map $g : N \to M$, homotopic to f, such that g is transverse to Z and ∂g is transverse to Z, and such that on a neighbourhood of C, the map g agrees with f.*

This is the "Extension Theorem" on [GP10, p. 72].

1.3. **Dimension and connectivity.**

Theorem 1.11. *Let M be a smooth manifold of dimension m and let Z be an embedded smooth submanifold of codimension d. Let $m_0 \in M \setminus Z$ be a basepoint. Let $i : M \setminus Z \to M$ denote the inclusion. Then*

$$i_* : \pi_n(M \setminus Z, m_0) \to \pi_n(M, m_0)$$

is surjective when $n = d - 1$ and an isomorphism when $n < d - 1$.

Proof. By Theorem 1.7, the sets $\pi_n(M, m_0)$ and $\pi_n(M \setminus Z, m_0)$ admit a description as the set of equivalence classes of smooth basepoint-preserving maps $S^n \to M$ under the equivalence relation of smooth, basepoint-preserving homotopy.

Suppose $n \leq d - 1$. Let $\alpha \in \pi_n(M, m_0)$ be a class, represented by a smooth map $f : S^n \to M$. Using Theorem 1.10 with $N = S^n$ and the basepoint of S^n as C, we may suppose f is transverse to Z, which by counting dimensions and Lemma 1.9 implies that $\operatorname{im}(f)$ is disjoint from Z. The homotopy class of $f : S^n \to M \setminus Z$ gives us a representative for $\bar{\alpha} \in \pi_n(M \setminus Z, m_0)$ mapping to α, so that i_* is surjective.

Suppose $0 < n < d - 1$. Suppose $\beta \in \pi_n(M \setminus Z, m_0)$ has the property that $i_*(\beta)$ is trivial. We will show that β is trivial. Let $f : S^n \to M \setminus Z$ be a smooth representative for β, and let $H'' : D^{n+1} \to M$ be a continuous map restricting to $i \circ f$ on $S^n = \partial D^{n+1}$—H'' exists because $i_*(\beta)$ is trivial. By using the Whitney Approximation Theorem, 1.4, we may replace H'' by a smooth map H', again restricting to $i \circ f$ on ∂D^{n+1}. Then by using Theorem 1.10, with $N = D^{n+1}$ and $C = \partial N = S^N$, we may replace H' by a homotopic smooth map $H : D^{n+1} \to M$ such that $H|_{\partial D^{n+1}} = i \circ f$ and such that H is transverse to Z. By a dimension-counting argument and Lemma 1.9, we know that $\operatorname{im}(H) \cap Z = \emptyset$ since $n < d - 1$. The map H has image in $M \setminus Z$, and is a contraction of f to a constant map in $M \setminus Z$. Since H exists, β is trivial.

The argument to show that $\pi_0(M \setminus Z, m_0) \to \pi_0(M, m_0)$ is injective is similar. Suppose $x, y \in M \setminus Z$ are two points that lie in the same component of M. Then there is a path $\gamma'' : I \to M$ from x to y, and using Theorem 1.4, we may replace γ' by a smooth path γ' from x to y. Then using Theorem 1.10 with $N = I$, and $C = \partial I$, we may replace γ' by a smooth path γ from x to y that meets Z transversely. Provided $d > 1$, this means Z does not meet γ at all, as required. □

Remark 1.12. The map i of the Theorem is said to be $(d-1)$-*connected*.

1.4. **Application to complex varieties.**

Proposition 1.13. *Suppose V is a smooth connected complex variety and Z is a subvariety of (complex) codimension d. Then the inclusion $i : V \setminus Z \to V$ is $(2d-1)$-connected.*

Proof. We may stratify Z into smooth strata of weakly increasing dimensions, and so by an induction argument, it is sufficient to treat the case where $Z \subseteq M$ is a smooth closed subvariety, i.e., a complex-codimension-d smoothly embedded submanifold. The result now follows from Theorem 1.11. ☐

Corollary 1.14. *Suppose Z is a subvariety of $\mathbb{A}^N_{\mathbb{C}}$ of codimension $d > 0$. Let $x_0 \in \mathbb{A}^N_{\mathbb{C}} \setminus Z$ be a basepoint. Then $\pi_n(\mathbb{A}^N_{\mathbb{C}} \setminus Z, x_0)$ is trivial for $n \leq 2d-2$.*

Notation 1.15. One says that $\mathbb{A}^N_{\mathbb{C}} \setminus Z$ is $(2d-2)$-*connected*.

1.5. **Precision of the bound.** If M is a manifold, Z is a closed submanifold and $m_0 \in M \setminus Z$, then there is a long exact sequence of relative homotopy groups (or pointed sets at the right-hand end of the sequence):

(1)

$$\cdots \to \pi_i(M\setminus Z, m_0) \to \pi_i(M, m_0) \to \pi_i(M, M\setminus Z) \to \pi_{i-1}(M\setminus Z, m_0) \to \pi_{i-1}(M, m_0) \to \cdots.$$

See [Whi12, IV, Thm 2.4] for the general theory.

Proposition 1.16. *Suppose M is a manifold and Z is a connected closed submanifold of codimension $d \geq 1$, and that $m_0 \in M \setminus Z$ is a basepoint. Then at least one of the two maps (induced by the inclusion)*

$$\pi_d(M \setminus Z, m_0) \to \pi_d(M, m_0), \quad \pi_{d-1}(M \setminus Z, m_0) \to \pi_{d-1}(M, m_0)$$

is not an isomorphism.

Proof. Let $N \subset M$ be a tubular neighbourhood of Z, the existence of which is proved in [Hir76, Thm 6.2]). In particular:

- N is an open submanifold of M and Z is a closed submanifold of N.
- Z is a deformation retract of N. Let us write $p : N \to Z$ for the retraction map.
- The map $p : N \to Z$ is isomorphic to the structure map $E \to Z$ where E is the normal bundle of Z in M.

There is an excision isomorphism:

$$\mathrm{H}_*(M, M \setminus Z; \mathbb{F}_2) \cong \mathrm{H}_*(N, N \setminus Z; \mathbb{F}_2)$$

The Thom isomorphism theorem gives us: $\mathrm{H}_*(N, N \setminus Z; \mathbb{F}_2) \cong \mathrm{H}_{*-d}(Z; \mathbb{F}_2)$ (see [Dol95, VIII, 7.15]). Therefore

$$\mathrm{H}_i(M, M \setminus Z; \mathbb{F}_2) = 0 \text{ if } i < d \quad \text{and} \quad \mathrm{H}_d(M, M \setminus Z; \mathbb{F}_2) \ncong 0.$$

The universal coefficients theorem for homology implies that

$$\mathrm{H}_d(M, M \setminus Z; \mathbb{F}_2) \cong \mathbb{F}_2 \otimes_{\mathbb{Z}} \mathrm{H}_d(M, M \setminus Z; \mathbb{Z})$$

and therefore that $\mathrm{H}_d(M, M \setminus Z; \mathbb{Z}) \ncong 0$.

Theorem 1.11 and the long exact sequence (1) imply that the relative homotopy groups $\pi_i(M, M \setminus Z)$ all vanish when $i < d-1$. The relative version of the Hurewicz theorem [Whi12, IV, Thm 7.2] tells us that $\mathrm{H}_d(M, M \setminus Z; \mathbb{Z})$ is a quotient of $\pi_d(M, M \setminus Z)$ by an action of $\pi_1(M \setminus Z, m_0)$. In particular, $\pi_d(M, M \setminus Z)$ is not 0. The result now follows from the long exact sequence (1). □

Example 1.17. For all positive integers d, the examples of $Z = \mathrm{pt}$ and $M = S^d$ and $Z = \mathrm{pt}$ and $M = \mathbb{R}^d$ show that either of the two maps in Proposition 1.16 may be an isomorphism, at the expense of the other.

References

[BT82] Raoul Bott and Loring W. Tu, *Differential Forms in Algebraic Topology*, Springer New York, 1982. ↑1
[Dol95] A. Dold, *Lectures on algebraic topology*, Classics in Mathematics, Springer, 1995. ↑4
[Ful07] William Fulton, *Equivariant Cohomology in Algebraic Geometry*, 2007. ↑1
[GP10] Victor Guillemin and Alan Pollack, *Differential topology*, AMS Chelsea Pub, 2010. ↑3
[Hir76] Morris W. Hirsch, *Differential topology*, Graduate Texts in Mathematics, vol. 33., Springer-Verlag, New York, 1976 (English). ↑4
[Lee12] John M. Lee, *Introduction to Smooth Manifolds*, Second, Graduate Texts in Mathematics, vol. 218, Springer-Verlag, New York, 2012. ↑1, 2
[Whi12] George W. Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Mathematics Ser., vol. 61, Springer, 2012. ↑4, 5

Department of Mathematics, University of British Columbia, Vancouver BC, Canada
Email address: tbjw@math.ubc.ca