# THE EQUIVARIANT MOTIVIC COHOMOLOGY OF VARIETIES OF LONG EXACT SEQUENCES

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Thomas Benedict Williams

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Thomas Benedict Williams

I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

(Gunnar Carlsson) Principal Adviser

I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

(Ralph Cohen)

I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

(Ravi Vakil)

Approved for the University Committee on Graduate Studies.

# Abstract

A perfect field, $k$, is fixed throughout. The aim of the present work is to compute the equivariant motivic cohomology of a certain variety, $X$, which represents the space of long exact sequences of given length and with prescribed ranks, all in the category of finite dimensional $k$-vector spaces, equivariant with respect to an action of the multiplicative group scheme of $k$. In order to calculate this, it is necessary to establish a number of results pertaining to the motivic cohomology of the general linear group scheme, and to introduce a spectral sequence for deriving the motivic cohomology of homogeneous varieties. We then present the variety $X$ as a homogeneous variety, and so obtain the cohomology.

As an application, we show that the cohomology furnishes obstructions to equivariant maps from punctured affine $n$-spaces to $X$, which amounts to the same thing as obstructions to the existence of certain differential graded modules over polynomial rings over $k$. The obstructions so found are a generalization of the Herzog-Kühl equations, which are well-known in the particular case where the differential graded module is in fact a resolution of an Artinian module.

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*Is ar scáth a chéile a mhaireann na daoine.*

# Contents

# Introduction

## Overview

This dissertation owes its existence to a conjecture of Gunnar Carlsson's. Let $k$ be a field and let $S = k[x_1, \dots, x_n]$ be a polynomial ring, given a grading by placing $x_i$ in degree $-1$. Let $M$ be a free differential graded $S$-module of finite rank, which is to say $M$ is a finitely generated free $S$-module that is equipped with a map $d : M \to M$ such that $d^2 = 0$. One can define $H_*(M)$ in an obvious way as $\ker d / \operatorname{im} d$. Suppose now that $H_*(M)$ is artinian, or equivalently finite dimensional over $k$, and to avoid trivialities suppose that $H_*(M) \neq 0$. The Carlsson conjecture asserts that then $\operatorname{rank}_S M \geq 2^n$. This conjecture originates in the papers [Car86], [Car87], and in the latter the case of $n = 3$ was settled in the affirmative by ad-hoc methods. Not much progress has been made in the intervening years, for no $n \geq 4$ is the conjecture known in full as of [ABI07].

The motiviating application for the conjecture is the following picture. Given a finite CW-complex $X$ on which the group $G = (\mathbb{Z}/2)^n$ acts freely (that is to say that the stabilizer of every point of $X$ is trivial) and cellularly, then the ordinary $\mathbb{Z}/2$ cellular chain complex, $C_*(X; \mathbb{Z}/2)$ is a free module not only over $\mathbb{Z}/2$, but also over $\mathbb{Z}/2[G]$. In [Car83], a Koszul-duality type functor is constructed

$$\beta : \mathbb{Z}/2[G] - Mod \to \mathbb{Z}/2[x_1, \dots, x_n] - Mod$$

The functor $\beta$ has the property that it takes a chain complex to a differential graded module, and also the two identities

$$H_*(\beta C_*(X;\mathbb{Z}/2)) = H_*(X/G;\mathbb{Z}/2)$$
$$H_*(\beta C_*(X;\mathbb{Z}/2) \otimes_S \mathbb{Z}/2) = H_*(X;\mathbb{Z}/2)$$

The first of these ensures that $\beta C_*(X;\mathbb{Z}/2)$ has artinian homology, the second means we can deduce conditions on $X$ admitting such a $G$-action from conditions on differential graded $S$-modules having finite homology. Since $\operatorname{rank} M \geq \operatorname{rank} H_*(M)$, the conjecture of Carlsson would, if true, imply the following Carlsson-Halperin conjecture: for a finite CW-complex $X$ admitting a free $G$-action, one should have $\sum_{i=0}^{\infty} \dim_{\mathbb{Z}/2} H_*(X;\mathbb{Z}/2) \geq 2^n$. Analogous results are expected to hold for odd primes, the group being $(\mathbb{Z}/p)^n$ and the coefficients being taken in the field $\mathbb{Z}/p$, and also in characteristic 0, the group being the torus $(S^1)^n$ and coefficients being taken in a field of characteristic 0, e.g. $\mathbb{Q}$. These conjectures appear in [Hal85] [Car87]. The reduction of the characteristic-0 case to the Carlsson conjecture is much the same as the characteristic-2 case. The odd primes present a slightly larger challenge, since there are sign problems that intervene, but these could most likely be surmounted if a proof of the algebraic conjecture was found.

It should be mentioned that the conjecture of Carlsson has a close relationship with a conjecture of Buchsbaum-Eisenbud and Horrocks (independently). For a full description of this conjecture see [CE92]. In its aspect that relates to the conjecture of Carlsson most directly, the Buchsbaum-Eisenbud-Horrocks (or BEH) conjecture says that if $F_i \to N$ is a finite free resolution of an artinian $S$-module $N$, then $\operatorname{rank}_S F_i \geq \binom{n}{i}$. Taking $M = \bigoplus_i F_i$, and giving it the evident differential graded module structure arising from the differential in the resolution, the BEH conjecture implies that $\operatorname{rank}_S M \geq 2^n$, which is a special case of Carlsson's conjecture. Progress on the BEH conjecture, now entering its fourth decade, has also been halting, the largest case known in full is $n = 4$, [ABI07], but see [Erm09] for some remarkable recent partial progress. There is moreover a wealth of material regarding a version, indeed the original version, of this conjecture, which was formulated over a regular local ring rather than the graded polynomial ring we are concerned with, but this conjecture also remains open.

The approach we should like to take to the conjecture of Carlsson is the following: a free DGM $M$ can be presented as the rank $m = \operatorname{rank}_S M$, and an $m \times m$-matrix of polynomials $D$ representing the differential $D : M \to M$. The restrictions on $D$ are $D^2 = 0$ and the observation that evaluation at a nonzero $n$-tuple $(\xi_1, \ldots, \xi_n) \in \bar{k}$ yields a matrix $D(\xi_1, \ldots, \xi_n)$ of rank $m/2$. These conditions are derived in [Car87]. If one defines $Y$ as a variety of $m \times m$-matrices of rank $m/2$ (viz. a matrix subject to the non-vanishing of certain minors), then the given $D$ is tantamount to a map $\mathbb{A}^m \setminus \{0\} \to Y$. A version of this argument is presented more fully in the final section of this dissertation.

Since $\mathbb{A}^m \setminus \{0\}$ is a 'sphere', see for instance [DI05], it seems sensible to search for obstructions to maps $\mathbb{A}^m \setminus \{0\} \to Y$ in the algebraic homotopy theory, to wit the $\mathbb{A}^1$-homotopy theory of [MV99]. This theory is quite newly formed, and there is a dearth of concrete results in it. We must therefore spend quite some time establishing analogues of well-worn facts in algebraic topology— this essentially takes us the first three chapters.

To say anything meaningful about the original question is beyond the current work, the key problem is that the variety $Y$ of generic square-free matrices constructed previously is singular, and the methods of $\mathbb{A}^1$-homotopy theory best attuned to the consideration of nonsingular varieties. The variety we actually consider, denoted $X$, in chapter 4, is a nonsingular closed subvariety of $Y$, corresponding to the case where the DGM $M$ has a decomposition $\bigoplus M_i$ with $\operatorname{im} D|_{M_i} \subset M_{i-1}$, or, in other words, the case of chain complexes $M_*$ having artinian homology. The variety $X$ is a homogeneous space, or at least admits a transitive action by a group-scheme, so it follows that $X$ is smooth over $\operatorname{Spec} k$. It is hoped that the variety $Y$ can be stratified into subvarieties which can individually be treated along the same lines as $X$, and so the case of $X$ is an important preliminary to a fuller argument.

We should hope to prove the following: if there is a map $\mathbb{A}^n \setminus \{0\} \to X$, corresponding to a chain complex $M_*$ over $S$ having artinian homology, then $\sum_i \operatorname{rank}_S M_i \geq 2^n$. We are not able to prove this. In chapter 4 we find "Herzog-Kühl equations", which are necessary but insufficient conditions for such a chain complex. These are generalizations of well-known equations in the theory of resolutions of graded artinian $S$-modules, on which the powerful theory of Boij-Soderberg is based, see [BS08] and [ES09]. We should hope perhaps to find an analogue of Boij-Soderberg theory in the general case of DGMs.

The general structure of this dissertation is as follows. In the first chapter, we derive a sequence of elementary results concerning $\mathrm{GL}(n)$ and Stiefel varieties. The headline result, the calculation of the motivic cohomology of $\mathrm{GL}(n)$, is not original, having appeared in [Pus04], but we need a great many minor results on the behaviour of the cohomology with respect to certain standard maps of spaces.

In the second chapter, we modify the bar-construction for topological spaces to apply in the case of simplicial Nisnevich sheaves on the category of smooth $k$-schemes, the category underlying the $\mathbb{A}^1$-homotopy theory of [MV99]. This allows us to derive a fiber-to-base spectral sequence, the Rothenberg-Steenrod spectral sequence of [EKMM97], for a sequence such as the taking of homotopy-orbits $G \to X \to (EG \times X)/G \simeq B(\mathrm{pt}, G, X)$, which should in spirit be a fibration (but is probably not one in the $\mathbb{A}^1$-model structure since the fibrations are highly mysterious). We use this to make a ham-fisted definition of '$G$-equivariant' motivic cohomology, which is not the more geometric construction of [EG98]. We also have of course a ham-fisted definition of classifying space, which is again not the more geometric (and therefore more 'correct') construction of $B_{\acute{e}t}$ in [MV99]. These constructions nevertheless serve our purposes.

The third chapter is purely technical, being a collection of applications of the second chapter to the objects of the first. We are able to use the Rothenberg-Steenrod spectral sequence as a serviceable replacement for Serre spectral sequences in ordinary topology. Heuristically we replace the Serre spectral sequence calculation of

$$X \longrightarrow (EG \times X)/G \longrightarrow BG$$

with the 'fiber-to-base' spectral sequence calculation of

$$G \simeq \Omega BG \longrightarrow EG \times X \longrightarrow (EG \times X)/G$$

It consequently is an exercise in computations in Ext-groups, in order to mimic easier calculations in Serre spectral sequences.

The fourth and final chapter provides the application of the material of the first three chapters to the study of the variety $X$ previously alluded to. Our first goal is to present $X$ as a homogeneous space, a quotient of a group $G$ by a subgeoup $K$. Both are up to $\mathbb{A}^1$-homotopy products of $\mathrm{GL}(n)$s. Since the action of $K$ on $G$ is free, there is a weak equivalence $B(\mathrm{pt}, K, G) \simeq X$, from which we obtain a spectral sequence computing the motivic cohomology of $X$. It happens to collapse immediately. We then use a $\mathbb{G}_m$-action on $X$ to account for the graded nature of the exact sequences it parametrizes, and by considering $\mathbb{G}_m$-equivariant cohomology, we are finally able to deduce the generalized Herzog-Kühl equations, as promised.

## Notational Preliminaries

We shall use the following notational and terminological conventions.

We use $\mathrm{diag}\,(()\, a_1, \dots, a_n)$ as an abbreviation for the $n \times n$ matrix which is 0 off the main diagonal and $a_1, \dots, a_n$ on it. For example

$$\mathrm{diag}\,(()\, 1, 2) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

The letter $k$ will denote a perfect field, that is to say a field which has no algebraic inseperable extensions. All $k$-schemes will be assumed finite type, noetherian and separable. If $R$ is a finitely generated commutative $k$-algebra, and if $X$ is a $k$-scheme, then $X(R)$, the set of *R-points* of $X$ shall mean the set of scheme maps $\mathrm{Spec}\, R \to X$. If $X$ is itself affine, say $X = \mathrm{Spec}\, S$, then the $R$-points of $X$ are precisely the ring homomorphisms $S \to R$, of course. By saying $X$ is a smooth $k$-scheme we mean that the map $X \to \mathrm{Spec}\, k$ is a smooth morphism of schemes.

For the above terminology and most other questions regarding the theory of schemes, we refer to [Har77].

Let *Sets* denote the category of sets. If $C$ is a category, then by $Pre(C)$ we mean the functor category whose objects are contravariant functors $C \to Sets$, and whose morphisms are natural transformations in the obvious way. When $A, B$ are objects in a category $C$, we shall occasionally use the notation $C(A, B)$ to denote the set of maps $A \to B$ in $C$.

We wilfully disregard set-theoretic questions concerning our categories. Our staple references for category-theoretic questions is [ML98].

We shall use $\simeq$ and $\xrightarrow{\sim}$ unless otherwise stated to denote a weak equivalence in the sense of $\mathbb{A}^1$-homotopy. The major exception to this rule is the first part of chapter 2, where the model structure at hand is the prior structure on simplicial sheaves on a cite. Our main reference for the former is [MV99] and for the latter [Jar87].

Let $Sm/k$ denote the category of smooth $k$-schemes, and let $A/k$ denote the category of affine smooth $k$-schemes. We shall frequently make use of the following version of the Yoneda lemma

**Lemma 1.** *The functor $Sm/k \to Pre(A/k)$ given by $X \mapsto h_X$, where $h_X(\operatorname{Spec} R) = X(R)$ is a full, faithful embedding.*

*Proof.* The standard version of Yoneda's lemma is that there is a full, faithful embedding $Sm/k \to Pre(Sm/k)$. The functor we are considering is the composition $Sm/k \to Pre(Sm/k) \to Pre(A/k)$ obtained by restricting the domain of the functors in $Pre(Sm/k)$. One can write any $Y$ in $Sm/k$ as $\operatorname{colim}_{i \in I} \operatorname{Spec} A_i$, where the $A_i$ are finite type $k$-algebras. We have

$$Sm/k(Y, X) = Sm/k(\operatorname{colim} \operatorname{Spec} A_i, X) = \lim Sm/k(\operatorname{Spec} A_i, X) = \lim X(A_i)$$

from which it follows that the functor $Sm/k \to A/k$ inherits fullness and fidelity from the Yoneda embedding by abstract-nonsense arguments. □

In practice this means that rather than specifying a map of schemes $X \to Y$ explicitly, we shall happily exhibit a set-map $X(R) \to Y(R)$, where $R$ is an arbitrary finite-type $k$-algebra, and then observe that this set map is natural in $R$. The result is a map in $Pre(A/k)$, which is therefore (by the fullness and fidelity of Yoneda) also understood as a map of schemes $X \to Y$.

We use this in tandem with the following lemma quite often

**Lemma 2.** *Let $f : A \to B$ be a map in $Sm/k$, and let $g : C \to B$ be a locally closed subscheme of $B$ (to wit the inclusion map factors $B \to U \to C$, where $B \to U$ is a closed subscheme map and $U \to C$ is an open subscheme map). Suppose that on geometric points $A(\bar{k}) \to B(\bar{k})$ actually factors through $g$, i.e. as $A(\bar{k}) \to C(\bar{k}) \to B(\bar{k})$. Then the map $f$ factors as $h : A \to C$ followed by $g : C \to B$.*

*Proof.* By factoring $g : C \to B$ it suffices to prove the cases where $g$ is a closed immersion and $g$ an open immersion separately. Of these, the former is slightly harder, but they are much the same. For instance if $C \to B$ is a closed subscheme map then consider the pull-back

$$\begin{array}{ccc} g^{-1}(A) & \longrightarrow & C \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

Here $g^{-1}(A) \to A$ is also a closed immersion.

Since pull-backs preserve commute with taking geometric points (by abstract non-sense) we have a pull-back diagram

$$\begin{array}{ccc} g^{-1}(A)(\bar{k}) & \longrightarrow & C(\bar{k}) \\ \downarrow & & \downarrow \\ A(\bar{k}) & \longrightarrow & B(\bar{k}) \end{array}$$

so $g^{-1}(A)(\bar{k})$ consists of points $(\alpha, \beta)$ in $A(\bar{k}) \times C(\bar{k})$ with common image in $B(\bar{k})$. Since by hypothesis every $\alpha \in A(\bar{k})$ appears in this way, it follows that the map $g^{-1}(A)(\bar{k}) \to A(\bar{k})$ is surjective. On the other hand, a closed immersion of reduced schemes that is a surjection on geometric points is an isomorphism, so $g^{-1}(A) \cong A$, and the result follows.

The case of an open immersion is much the same, we may even dispense with the hypothesis of reduction. □

We typically employ these results in the following manner: we wish to construct a map between two given $k$-schemes $A \to B$, but rather than doing this by hand we observe that $B \to \mathbb{A}^N$ is locally closed (some other scheme may function equally well in place of $\mathbb{A}^N$). We then construct a natural map $A(R) \to \mathbb{A}^N(R)$ (since the latter is simply the set $R^N$ without conditions, this is generally an easy proposition), and, having done this, we observe that on geometric points $A(\bar{k})$ the map so constructed lands in $B(\bar{k})$. The conditions defining $B$ are going to be linear-algebraic in nature, and we can by these methods avoid having to worry about linear algebra over rings that are not algebraically-closed fields.

# Chapter 1

# The Motivic Cohomology of Stiefel Varieties

## 1.1 Preliminaries

In this chapter we set up some calculations that we shall need later. The main result is theorem 18, which computes the motivic cohomology of varieties of $n \times m$-matrices of of rank $m$, the Stiefel varieties of the title. Beyond this, we establish some technicalities which we shall need later, in part by using express computations with higher Chow groups. This technique is particular to motivic cohomology among all the theories represented in the $\mathbb{A}^1$-homotopy category, and it is the author's opinion that it would be better done away with. On the other hand, it is intrinsically appealing and diverting to work with something so concrete as cycles in a scheme.

The highlight of this chapter, although we do not employ it elsewhere in this work, is the comparison map we construct (in homotopy) $\Sigma_t \mathbb{P}^{n-1}_+ \to \mathrm{GL}(n)$. This is a modification of a well-known result comparing $\mathbb{K}^* \wedge \mathbb{K}P^{n-1}_+$ and $\mathrm{GL}(n, \mathbb{K})$, (where $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$) in the theory of Stiefel manifolds, as expounded in [Jam76]. Features of note are firstly that the underlying $\mathbb{A}^1$-homotopical result, which we are not able to present here but surely exists, may be cleaner than that in ordinary topology, since we are calling for some sort of identity of homotopy types up to the part detected by $K_2$ (independently of $n$), rather than obtaining $m$-connectedness where $m$ varies with $n$. It appears to us that $\Sigma_t \mathbb{P}^{n-1}_+$ is a homotopy-theoretic 'second approximation' to $\mathrm{GL}(n)$. Secondly, by adjunction, we have a comparison map $\mathbb{P}^{n-1}_+ \to \Omega_t \mathrm{GL}(n)$, where the latter is the Tate loop-space, and Tate loop-spaces are notoriously difficult to understand, so this map may be of some general use.

We compute motivic cohomology as a represented cohomology theory in the motivic- or $\mathbb{A}^1$-homotopy category of Morel & Voevodsky, see [MV99] for the construction of this category. The best reference for the theory of motivic cohomology is [MVW06], and the proof that the theory presented there is representable in the category we claim can be found in [Del], subject to the restriction that the field $k$ is perfect. Motivic cohomology, being a cohomology theory (again at least when $k$ is perfect) equipped with suspension isomorphisms for both suspensions, $\Sigma_s, \Sigma_t$, is represented by a motivic spectrum, of course, but we never deal explicitly with such objects.

We therefore fix a perfect field $k$. We shall let $R$ denote a fixed commutative ring of coefficients, generally we shall apply the results in the cases $R = \mathbb{Z}$, $R = \mathbb{Z}[\frac{1}{2}]$ or $R = \mathbb{Z}/p$. We denote the terminal object, $\operatorname{Spec} k$, by pt. If $X$ is a finite type smooth $k$-scheme or more generally an element in the category $\Delta^{\mathrm{op}} Shv_{Nis}(Sm/k))$, we write $H^{*,*}(X;R)$ for the bigraded motivic cohomology ring of $X$. This is graded-commutative in the first grading, and commutative in the second. If $R \to R'$ is a ring map, then there is a map of algebras $H^{*,*}(X;R) \to H^{*,*}(X;R')$. It will be of some importance to us that most of our constructions are functorial in $R$, when the coefficient ring is not specified, therefore, it is to be understood that arbitrary coefficients $R$ are meant and that the result is functorial in $R$.

We write $\mathbb{M}_R$ for the ring $H^{*,*}(\mathrm{pt};R)$. Since pt is a terminal object, the ring $H^{*,*}(X;R)$ is in fact an $\mathbb{M}_R$ algebra. We assume the following vanishing results for a $d$-dimensional smooth scheme $X$: $H^{p,q}(X;R) = 0$ when $p > 2q$, $p > q + d$ or $q < 0$.

At one point we employ the comparison theorem relating motivic cohomology and the higher Chow groups. For all these, see [MVW06].

## 1.2 The Motivic Cohomology of Stiefel Varieties

**Proposition 3.** *Let $X$ be a smooth scheme and suppose $E$ is an $\mathbb{A}^n$-bundle over $X$, and $F$ is a sub-bundle with fiber $\mathbb{A}^\ell$, then there is an exact triangle of graded $H^{*,*}(X)$-modules*

$$\begin{array}{ccc} H^{*,*}(X)\tau & \xrightarrow{\ j\ } & H^{*,*}(X) \\ & {}_{\partial}\nwarrow \quad \swarrow & \\ & H^{*,*}(E \setminus F) & \end{array}$$

*where* $|\tau| = (2n - 2\ell, n - \ell)$

*Proof.* This is the localization exact triangle for the closed subbundle $F \subset E$, where

$$H^{*,*}(E) = H^{*,*}(F) = H^{*,*}(X)$$

arising from the cofiber sequence (see [MV99] for this and all other unreferenced assertions concerning $\mathbb{A}^1$-homotopy)

$$E \setminus F \longrightarrow E \longrightarrow \mathrm{Th}(N)$$

where $N$ is the normal-bundle of $F$ in $E$. There are in two possible choices for $\tau$, since $\tau$ and $-\tau$ serve equally well. We make the convention that in any localization exact sequence associated with a closed immersion of smooth schemes $Z \to X$, viz.

$$\longrightarrow H^{*,*}(Z)\tau \longrightarrow H^{*,*}(X) \longrightarrow H^{*,*}(X \setminus Z) \longrightarrow$$

the class $\tau$ should be the class which, under the natural isomorphism of the above with the localization sequence in higher Chow groups, corresponds to $[Z] \in CH^*(Z, *)$. We suspect this is guaranteed always to be the same class as that arising from the canonical Thom class of [Voe03b], but we shall never need this fact, and so we do not include a proof.

In the case where $F = X$ is the zero-bundle, then $j$ takes $\tau$ to $e(E)$, the Euler class, as proved in [Voe03b]. In general, by identifying $H^{*,*}(X)\tau$ with $CH^{*-n+\ell}(F, *)$, the higher Chow groups of $F$ as a closed subscheme of $E$, and employing covariant functoriality of higher Chow groups for the closed immersions $X \rightarrowtail F \rightarrowtail E$, we see that $j(\tau)e(F) = e(E)$. ☐

As shall be the case throughout, $H^{*,*}(X)$ denotes cohomology with unspecified coefficients, $R$, and the result is understood to be natural in $R$. For the naturality of the localization sequence in $R$, one simply follows through the argument in [MVW06], which reduces it to the computation of $H^{*,*}(\mathbb{P}^d) = \mathbb{M}_R[\theta]/(\theta^{d+1})$ which is natural in $R$ by elementary means, c.f. [Ful84].

We note that $|j(\tau)| = (2c, c)$, so if, as often happens, $H^{2c,c}(X) = 0$, this triangle is a short exact sequence of $H^{*,*}(X)$-modules:

$$0 \longrightarrow H^{*,*}(X) \longrightarrow H^{*,*}(E \setminus F) \longrightarrow H^{*,*}(X)\rho \longrightarrow 0$$

Here $|\rho| = (2n - 2\ell - 1, n - \ell)$

Since $H^{*,*}(X)\rho$ is a free graded $H^{*,*}(X)$-module, this short exact sequence of graded modules splits, and there is an isomorphism of $H^{*,*}(X)$-modules

$$H^{*,*}(E \setminus F) \cong H^{*,*}(X) \oplus H^{*,*}(X)\rho$$

In the first place we remark that $|\rho| = (2c - 1, c)$, so that $2\rho^2 = 0$ by anti-commutativity, we now see that $(a + b\rho)(c + d\rho) = ac + (ad + (-1)^{\deg c} bc)\rho + (-1)^{\deg d} bd\rho^2$, so in many cases (e.g. when $1/2 \in R$) the multiplicative structure is fully determined, and $H^{*,*}(E \setminus F) = H^{*,*}(X)[\rho]/(\rho^2)$

If $H^{2n,n}(X) = 0$ for $n > 0$, as often happens, then the same applies to $H^{*,*}(E \setminus F)$. In this case we say $X$ is *Chow free*.

We will have occasion later to refer to the following two results, which appear here for want of anywhere better to state them

**Proposition 4.** *Let $X$ be a scheme and suppose $E$ is a Zariski-trivializeable fiber bundle with fiber $F \simeq$ pt. Then $E \simeq X$.*

*Proof.* This is standard, see [DHI04]. ☐

The following two propositions allow us to identify such bundles which are not necessarily vector-bundles.

**Proposition 5.** *Suppose $X$ is a scheme, $P$ is a projective bundle of rank $n$ over $X$ and $Q$ is a projective subbundle of rank $n-1$, Then $P \setminus Q$ is a fiber bundle with fiber $\mathbb{A}^{n-1}$.*

*Proof.* This follows immediately by considering an open set $U$ over which $P, Q$ are trivial, for which $(P \setminus Q)|_U \cong U \times \mathbb{A}^{n-1}$. □

Let $k$ be a field. Let $W(n,m)$ denote the variety of full-rank $n \times m$ matrices over $k$, that is to say it is the open subscheme of $\mathbb{A}^{nm}$ determined by the nonvanishing of at least one $m \times m$-minor. Without loss of generality, $m \leq n$. By a *Stiefel Variety* we mean such a variety $W(n,m)$.

**Proposition 6.** *The cohomology of $W(n,m)$ has the following presentation as an $\mathbb{M}_R$-algebra:*

$$H^{*,*}(W(n,m);R) = \frac{\mathbb{M}_R[\rho_n, \ldots, \rho_{n-m+1}]}{I} \qquad |\rho_i| = (2i-1, i)$$

*The ideal $I$ is generated by relations $\rho_i^2 - a\rho_{2i-1}$, where $a \in \mathbb{M}_R^{1,1}$ satisfies $2a = 0$.*

We shall later identify $a$ as $\{-1\}$, the image of the class of $-1$ in $H^{1,1}(\operatorname{Spec} k; \mathbb{Z}) = k^*$ under the map $H^{*,*}(\operatorname{Spec} k; \mathbb{Z}) \to H^{*,*}(\operatorname{Spec} k; R)$.

*Proof.* The proof proceeds on induction on $m$, starting with $m = 1$ (we could start with $W(n,0) = \mathrm{pt}$). In this case $W(n,1) = \mathbb{A}^n \setminus \{0\}$, and $H^{*,*}(W(n,1)) = \mathbb{M}[\rho_n]/(\rho_n^2)$.

$W(n, m-1)$ is a dense open set of $\mathbb{A}^{nm}$, and as such is a smooth scheme. If $m < n$, there is a trivial $\mathbb{A}^n$-bundle over $W(n, m-1)$, the fiber over a matrix $A$ is the set of all $n \times m$-matrices whose first $m-1$ columns are the matrix $A$

$$\begin{pmatrix} & v_1 \\ A & \vdots \\ & v_n \end{pmatrix}$$

As a sub-bundle of this bundle, we find a trivial $\mathbb{A}^{m-1}$-bundle; the fiber of which over a $\bar{k}$-point (i.e. a matrix) $A$ is the set of matrices where $(v_1, \ldots, v_n)$ is in the row-space of $A$. Proposition 3 applies in this setting, and we conclude that there exists an exact triangle

$$\begin{array}{ccc} H^{*,*}(W(n,m-1))\tau & \longrightarrow & H^{*,*}(W(n,m-1)) \\ & {\scriptstyle\partial}\nwarrow \quad \swarrow & \\ & H^{*,*}(W(n,m)) & \end{array}$$

Since $H^{*,*}(W(n,m-1))$ is Chow-free by induction, so this triangle splits to give

$$H^{*,*}(W(m,n)) \cong H^{*,*}(W(m-1,n)) \oplus H^{*,*}(W(m-1,n))\rho_{n-m+1}$$

where $|\rho_{n-m+1}| = (2(n-m+1)-1, n-m+1)$. By graded-commutativity, $\rho^2_{n-m+1} = -\rho^2_{n-m+1}$. In particular, this means that if $\frac{1}{2} \in R$, then $\rho^2_{n-m+1} = 0$, and by induction it follows that

$$H^{*,*}(W(n,m); R) = \frac{\mathbb{M}_R(\rho_n, \ldots, \rho_{n-m+1})}{I}$$

as claimed.

In general, we know that

$$\rho^2_{n-m+1} \in H^{4n-4m+2, 2n-2m+2}(W(n,m-1); R)$$

We identify this cohomology group explicitly in the case $n - m + 1 > 1$. We can describe $H^{*,*}(W(n,m-1); R)$ as an $\mathbb{M}$-module as follows

$$H^{*,*}(W(n,m-1); R) \cong \mathbb{M} \oplus \bigoplus_i \mathbb{M}\rho_i \oplus \bigoplus_{i<j} \mathbb{M}\rho_i\rho_j \oplus \text{ multiples of further products}$$

since $\mathrm{wt}(\rho_i\rho_j) = i + j > 2(n-m+1)$ for all $i, j \geq n-m+2$, it follows the higher product terms are irrelevant to the determination of $H^{4n-4m+2, 2n-2m+2}(W(n, m-1; R))$.

Since the motivic cohomology $H^{p,q}$ of a $d$-dimensional smooth scheme is zero when $p > q + d$, we have $\mathbb{M}^{4n-4m+2, 2n-2m+2} = 0$ unless $n = m$. We postpone treating the case

$n = m$ until the end.

If $i > 2n - 2m + 2$, then $\mathbb{M}\rho_i$ is 0 in bidegrees $(4n - 4m + 2, 2n - 2m + 2)$ by consideration of weight. If $i = 2n - 2m + 2$, then $\mathbb{M}\rho_i$ is 0 in weight $2n - 2m + 2$, except in degree $4n - 4m + 3$, since $\mathbb{M}^{*,0}$ is concentrated in degree 0.

If $i < 2n - 2m + 1$, then $\mathbb{M}\rho_i$ consists of sums of elements of the form $a\rho_i$, where $\mathrm{cht}(a\rho_i) = \mathrm{cht}(a) + 1$ and $\mathrm{wt}(a\rho_i) = \mathrm{wt}(a) + i$. For an element $a \in \mathbb{M}$, $\mathrm{cht}(a) \geq \mathrm{wt}(a)$; it follows that the summands of $\mathbb{M}\rho_i$ in weight $2n - 2m + 2$ are concentrated in Chow height $\geq 3$, and so

$$H^{4n-4m+2,2n-2m+2}(W(n, m-1); R) = \mathbb{M}^{1,1}\rho_{2n-2m+1}$$

For $\mathbb{Z}$-coefficients, $\mathbb{M}^{1,1} = k^*$, and we cannot at this stage determine whether $\rho_{n-m+1}^2 = \{-1\}\rho_{2n-2m+1}$ or $\rho_{n-m+1}^2 = 0$, but at any rate we can say $\rho_{n-m+1}^2 = a\rho_{2n-2m+1}$, where $2a = 0$. □

We denote the cohomology ring $H^{*,*}(W(n, m); R)$ by $\mathbb{M}_R[\rho_n, \ldots, \rho_{n-m+1}]/I$ where the ideal $I$ is understood to depend on $n$, $m$.

We shall need the following technical lemma

**Lemma 7.** *Let $Z \to X$ be a closed immersion of connected smooth schemes, and let $f : X' \to X$ be a map of smooth schemes that is either flat or split by a flat map, in the sense that there exists a flat map $s : X \to X'$ such that $s \circ f = \mathrm{id}_{X'}$ and such that $f^{-1}(Z)$ is again connected. Then there is a map of localization sequences in motivic cohomology*

$$\begin{array}{ccccccc} \longrightarrow & H^{*,*}(Z)\tau & \longrightarrow & H^{*,*}(X) & \longrightarrow & H^{*,*}(X \setminus Z) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & H^{*,*}(f^{-1}(Z))\tau' & \longrightarrow & H^{*,*}(X') & \longrightarrow & H^{*,*}(X' \setminus f^{-1}(Z)) & \longrightarrow \end{array}$$

*such that the last two vertical arrows are the functorial maps on cohomology and such that $\tau \mapsto \tau'$.*

We expect this fact (and presumably some strong generalization—indeed the hypotheses on the map may be entirely unnecessary) is known to the experts, but we are unable to find it anywhere in the literature, and investigating the situation rapidly takes one into the madhouse that is "deformation to the normal cone", [MV99][Chapter 3]. The giving of references for results concerning higher Chow groups is deferred to the beginning of section 1.3.

*Proof.* One begins by observing the existence in general of the following diagram

$$\begin{array}{ccccc} X'\setminus f^{-1}(Z) & \longrightarrow & X' & \longrightarrow & \mathrm{Th}\, N_{f^{-1}(Z)\to X'} \\ \downarrow & & \downarrow & & \vdots \\ X\setminus Z & \longrightarrow & X & \longrightarrow & \mathrm{Th}\, N_{Z\to X} \end{array}$$

where the dotted arrow exists for reasons of general nonsense.

There is in general a map on cohomology arising from the given diagram of cofiber sequences, but we cannot at this stage predict the behaviour of the map represented by the dotted arrow.

$$\begin{array}{ccccccccc} \longrightarrow & H^{*,*}(Z)\tau & \longrightarrow & H^{*,*}(X) & \longrightarrow & H^{*,*}(X\setminus Z) & \longrightarrow \\ & \vdots & & \downarrow & & \downarrow & \\ \longrightarrow & H^{*,*}(f^{-1}(Z))\tau' & \longrightarrow & H^{*,*}(X') & \longrightarrow & H^{*,*}(X'\setminus f^{-1}(Z)) & \longrightarrow \end{array}$$

When the map $X' \to X$ is flat, the pullback $f^{-1}(Z) \to Z$ is too. We identify the motivic cohomology groups with the higher Chow groups, giving the localization sequence

$$\begin{array}{ccccccc} \longrightarrow & CH^*(Z,*) & \longrightarrow & CH^*(X,*) & \longrightarrow & CH^*(X\setminus Z,*) & \longrightarrow \\ & \vdots & & \downarrow & & \downarrow & \\ \longrightarrow & CH^*(f^{-1}(Z),*) & \longrightarrow & CH^*(X',*) & \longrightarrow & CH^*(X'\setminus f^{-1}(Z),*) & \longrightarrow \end{array}$$

and in this case $\tau$, $\tau'$ become the classes of the cycles $[Z]$, $[f^{-1}Z]$. Since the map

$$CH^*(Z,*) \to CH^*(f^{-1}(Z),*)$$

is the contravariant map associated with pull-back along a flat morphism, it follows immediately that $\tau \mapsto \tau'$.

In the case where the map $X' \to X$ is split, the map of localization sequences is also split. When the splitting map, $s$, is flat, we obtain $s^*(\tau') = \tau$, and it follows that $f^*(\tau) = \tau'$, since $s^* f^* = \text{id}$. □

The following results are analogues of classically known facts.

**Proposition 8.** *For $m' \leq m$, there is a projection $W(n, m) \to W(n, m')$ given by ommission of the last $m - m'$-vectors. On cohomology, this yields an inclusion*

$$\mathbb{M}(\rho_n, \ldots, \rho_{n-m'+1})/I \to \mathbb{M}(\rho_n, \ldots, \rho_{n-m'+1}, \ldots, \rho_{n-m+1})/I$$

*Proof.* It suffices to prove the case $m' = m - 1$. In this case, the map $W(n, m) \to W(n, m - 1)$ is the fiber bundle from which we computed the cohomology of $W(n, m)$, and the result on cohomology holds by inspection of the proof. □

**Proposition 9.** *Given a nonzero point, $v \in (\mathbb{A}^n \setminus 0)(k)$, and a complementary $n-1$-dimensional subspace $U$ such that $\langle n \rangle \oplus U = \mathbb{A}^n(k)$, there is a closed immersion $\phi_{v,U} : W(n-1, m-1) \to W(n, m)$ given by identifying $W(n-1, m-1)$ with the space of independent $m-1$-frames in $U$, and then prepending $v$. On cohomology, this yields the surjection $\mathbb{M}[\rho_n, \ldots, \rho_{n-m+1}]/I \to \mathbb{M}[\rho_{n-1}, \ldots, \rho_{n-m+1}]/I$ with kernel $(\rho_n)$.*

*Proof.* We prove this by induction on the $m$, which is to say we deduce the case $(n, m)$ from the case $(n, m-1)$. The base case of $m = 1$ is straightforward, $v$ is a nonzero vector in $\mathbb{A}^n$, the space $W(n-1, 0)$ is trivial, and the map $\text{pt} = W(n-1, 0) \rightarrowtail W(n, 1) = \mathbb{A}^n \setminus 0$ is the inclusion of the point $v$. The result on cohomology is immediately verified.

Recall that we compute the cohomology of $W(n, m)$ by forming a trivial bundle $E_{n,m} \simeq$

$W(n, m-1)$ over $W(n, m-1)$, which on the level of $R$-points consists of matrices

$$\begin{pmatrix} & v_1 \\ A & \vdots \\ & v_n \end{pmatrix}$$

and removing the trivial closed sub-bundle $Z_{n,m}$ where the vector $(v_1, \ldots, v_n)$ is in the span of the columns of $A$ (more precisely we should say we consider the complement of a closed subscheme determined by the simultaneous vanishing of certain evident minors). There is then an open inclusion

$$E_{n,m} \setminus Z_{n,m} \cong W(n,m) \to E_{n,m} \simeq W(n, m-1)$$

The inclusion $\phi_{v,U} : W(n-1, m-1) \to W(n,m)$, without loss of generality can be assumed to act on field-valued points as as

$$B \xmapsto{\phi_{v,U}} \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$$

We abbreviate this map of schemes to $\phi$, and denote the analogous maps $W(n-1, m-2) \to W(n, m-1)$, $Z_{n-1,m-1} \to Z_{n,m}$, $E_{n-1,m-1} \to E_{n,m}$ etc. also by $\phi$ by abuse of notation. It is easily seen that the following are pull-back diagrams

$$\begin{array}{ccc} E_{n-1,m-1} & \longrightarrow & E_{n,m} \\ \uparrow & & \uparrow \\ Z_{n-1,m-1} & \longrightarrow & Z_{n,m} \end{array} \qquad \begin{array}{ccc} E_{n-1,m-1} & \longrightarrow & E_{n,m} \\ \uparrow & & \uparrow \\ W(n-1,m-1) & \xrightarrow{\phi} & W(n,m) \end{array}$$

The second square above is homotopy equivalent to

$$\begin{array}{ccc} W(n-1,m-2) & \xrightarrow{\phi} & W(n,m-1) \\ \downarrow & & \downarrow \\ W(n-1,m-1) & \xrightarrow{\phi} & W(n,m) \end{array}$$

from which we deduce that the map

$$\phi^* : H^{*,*}(W(n,m)) \to H^{*,*}(W(n-1,m-1))$$

satisfies $\phi^*(\rho_j) = \rho_j$ for $n-m+2 \leq j \leq n-1$ and $\phi^*(\rho_n) = 0$, since this holds for $W(n-1,m-2) \to W(n,m-1)$ by induction.

The hard part is the behavior of the element $\rho_{n-m+1}$, which is in the kernel of

$$H^{*,*}(W(n,m)) \to H^{*,*}(W(n,m-1))$$

Recall that $\rho_{n-m+1} \in H^{*,*}(W(n,m))$ is the preimage of the Thom class $\tau$ under the map

$$\partial : H^{*,*}(W(n,m)) \to H^{*,*}(W(n,m-1))\tau = H^{*,*}(Z_{n,m})\tau$$

We should like to assert that the map of localization sequences

$$\begin{array}{ccccccccc} \longrightarrow & H^{*,*}(W(n,m)) & \xrightarrow{\partial} & H^{*,*}(Z_{n,m})\tau & \longrightarrow & H^{*,*}(E_{n,m}) & \longrightarrow \\ & \downarrow{\scriptstyle \phi^*} & & \downarrow & & \downarrow & \\ \longrightarrow & H^{*,*}(W(n-1,m-1)) & \xrightarrow{\partial} & H^{*,*}(Z_{n-1,m-1})\tau' & \longrightarrow & H^{*,*}(E_{n-1,m-1}) & \longrightarrow \end{array} \tag{1.1}$$

one has $\tau \mapsto \tau'$, because then chasing the commutative square of isomorphisms

$$\begin{array}{ccc} \mathbb{Z}\rho_{n-m-1} = H^{2n-2m-1,n-m}(W(n,m)) & \longrightarrow & H^{0,0}(Z_{n,m}\tau = \mathbb{Z}\tau \\ \downarrow & & \downarrow \\ \mathbb{Z}\rho_{n-m-1} = H^{2n-2m-1,n-m}(W(n-1,m-1)) & \longrightarrow & H^{0,0}(Z_{n-1,m-1}\tau = \mathbb{Z}\tau' \end{array}$$

we have $\rho_{n-m-1} \mapsto \rho_{n-m-1}$ as required.

The difficulty is that the map $g : E_{n-1,m-1} \to E_{n,m}$ is a closed immersion, rather than a flat or split map, for which we have deduced this sort of naturality result. We can however factor $g$ into such maps, which we denote only on the level of points, the obvious scheme-theoretic definitions are suppressed. Let $U_{n,m}$ denote the variety of $n, m$-matrices which (on the level of $\bar{k}$-points) have a decomposition as

$$\begin{pmatrix} u & * & * \\ * & A & * \end{pmatrix}$$

where $u \in \bar{k}^*$, and $A \in W(n-1, m-2)(\bar{k})$. It goes without saying that this is a variety, since the conditions amount to the nonvanishing of certain minors. It is also easily seen that $U_{n,m}$ is an open dense subset of $E_{n,m}$. We have a map $E_{n-1,m-1} \to E_{n,m}$, given by

$$B \mapsto \begin{pmatrix} 1 & * \\ * & B \end{pmatrix}$$

and this map is obviously split by the projection onto the bottom-right $n-1 \times m-1$-submatrix. The composition $E_{n-1,m-1} \to U_{n,m} \to E_{n,m}$ is a factorization of the map $E_{n-1,m-1} \to E_{n,m}$ into a split map followed by an open immersion. The splitting of $E_{n-1,m-1}$ is a projection onto a factor, and since all schemes are flat over pt, the splitting is flat as well. We may now use lemma 7 twice to conclude that in diagram (1.1) we have $\tau \mapsto \tau'$, so that $\rho_{n-m+1} \mapsto \rho_{n-m+1}$, as asserted. □

Let $B(n, m)$ denote the scheme of matrices whose $k'$-points are matrices

$$\begin{pmatrix} A & * \\ 0 & 1 \end{pmatrix}$$

Where $A \in W(n, m)(k')$ and $*$ denotes a vector of arbitrary elements. We shall need this scheme later.

**Corollary 9.1.** *The standard map $W(n-1, m-1) \to W(n, m)$ constructed above factors as $W(n-1, m-1) \xrightarrow{\sim} B(n-1, m-1) \to W(n, m)$*

The proof is obvious.

We include the following two propositions here, although they rely on a proposition we prove in the next section. They are in keeping with the material of the section at hand, and we shall need them when it comes to computing equivariant cohomology of $\mathrm{GL}(n)$.

**Proposition 10.** *Let $\gamma : W(n, m) \to W(n, m)$ denote multiplication of the first column by $-1$. Then $\gamma^*$ is identity on cohomology*

*Proof.* By use of the comparison maps $\mathrm{GL}(n) \to W(n, m)$, see proposition 8, we see that it suffices to prove this for $\mathrm{GL}(n)$. By means of the standard inclusion $\mathrm{GL}(n-1) \to \mathrm{GL}(n)$ and induction we see that it suffices to prove $\gamma^*(\rho_n) = \rho_n$, where $\rho_n$ is the highest-degree generator of $H^{*,*}(\mathrm{GL}(n); R)$. By the comparison map again, we see that it suffices to prove that

$$\tau : \mathbb{A}^n \setminus \{0\} \to \mathbb{A}^n \setminus \{0\}$$

has the required property, but this is corollary 14.2 □

**Proposition 11.** *Let $\sigma \in \Sigma_m$, the symmetric group on $m$ letters. Let $f_\sigma : W(n, m) \to W(n, m)$ be the map that permutes the columns of $W(n, m)$ according to $\sigma$. Then $f_\sigma^*$ is the identity on cohomology.*

*Proof.* We can reduce immediately to the case where $\sigma$ is a transposition, and from there we can assume without loss of generality that $\sigma$ interchanges the first two columns. Let $R$ be a finite-type $k$-algebra. We view $W(n, m)$ as the space whose $R$-valued points are $m$-tuples of elements in $R^n$ satisfying certain conditions which we do not particularly need to know. In the case $R = k$, the condition is that the matrix is of full-rank in the usual way.

We can act on $W(n, m)$ by the elementary matrix $e_{ij}(\lambda)$

$$e_{ij}(\lambda) : (v_1, \ldots, v_m, v_{m+1}) \mapsto (v_1, \ldots, v_i + \lambda v_j, \ldots, v_{m+1})$$

The two maps $e_{i,j}(\lambda)$ and $e_{i,j}(0) = \mathrm{id}$ are homotopic, so $e_{i,j}(\lambda)$ induces the identity on cohomology. There is now a standard method to interchange two columns and change the sign of one by means of elementary operation $e_{ij}(\lambda)$, to wit $e_{12}(1)e_{21}(-1)e_{12}(1)$. We therefore know that the map

$$(v_1, v_2, v_3, \ldots, v_m) \mapsto (-v_2, v_1, v_3, \ldots, v_m)$$

induces the identity on cohomology, but now proposition 10 allows us even to undo the multiplication by $-1$. □

## 1.3 Higher Intersection Theory

Let $X$ be a scheme of finite type over a field. The higher Chow groups of $X$, denoted $CH^i(X, d)$ are defined in [Blo86] as the homology of a certain complex:

$$CH^i(X, d) = H_d(z^i(X, *))$$

where $z^i(X, d)$ denotes the free abelian group generated by cycles in $X \times \Delta^d$ meeting all faces of $X \times \Delta^d$ properly. We denote the differential in this complex by $\delta$.

There is a comparison theorem, see [MVW06, lecture 19], [Voe02], which states that for any smooth scheme $X$ over any field $k$, there is an isomorphism between the motivic cohomology groups and the higher Chow groups

$$CH^i(X, d) = H^{2i-d,i}(X, \mathbb{Z})$$

or the equivalent with $\mathbb{Z}$ replaced by a general coefficient ring $R$. The products on motivic cohomology and on higher Chow groups are known to coincide, see[Wei99]. In the difficult paper [Blo94], the following result is proven (the strong moving lemma)

**Theorem 12** (Bloch)**.** *Let $X$ be a quasiprojective variety, let $Y$ be a closed subvariety of pure codimension $c$, and let $U$ denote the open subscheme $X \setminus Y$. For all $i$, there is an exact sequence of* *complexes*

$$0 \longrightarrow z^{i-c}(Y,*) \longrightarrow z^i(X,*) \longrightarrow z^i(U,*) \xrightarrow{\rho} C_* \longrightarrow 0$$

*where $C_*$ is acyclic.*

We rephrase this result slightly

**Corollary 12.1.** *Let $X$, $Y$, $U$ and $c$ be as above. Then for all $i$, there is an exact sequence of complexes*

$$0 \longrightarrow z^{i-c}(Y,*) \longrightarrow z^i(X,*) \longrightarrow z^i_a(U,*) \longrightarrow 0$$

*where $z^i_a(U,*)$ is the subcomplex of $z^i(U,*)$ generated by subvarieties $\gamma$ whose closure $\overline{\gamma} \rightarrowtail X \times \Delta^*$ meet all faces properly. The inclusion of complexes $z^i_a(U,*) \subset z^i(U,*)$ induces an isomorphism on homology groups.*

*Proof.* We take $z^i_a(U,*) = \ker \rho$, where $\rho$ is as in the theorem. The assertions follow immediately by considering the long exact sequence of homology. □

For a cycle $\alpha \in z^i(U,d)$, we can write $\alpha = \sum_{i=1}^N n_i A_i$ for some subvarieties $A_i$ of $U \times \Delta^d$, and $n_i \in \mathbb{Z} \setminus \{0\}$. We can form the scheme-theoretic closure of $A_i$ in $X \times \Delta^d$, denoted $\overline{A_i}$. We remark that $\overline{A_i} \times_{X \times \Delta^d} (U \times \Delta^d) = A_i$ [Har77, II.3]. We define

$$\overline{\alpha} \sum_{i=1}^N n_i \overline{A_i}$$

We say that $\overline{\alpha}$ meets a subvariety $K \rightarrowtail X$ properly if every $\overline{A_i}$ meets $K$ properly. Suppose $\alpha$ is such that $\overline{\alpha}$ meets the faces of $X \times \Delta^d$ properly, then $\alpha = (U \to X)^*(\overline{\alpha})$, so $\alpha \in z^i_a(U,d)$.

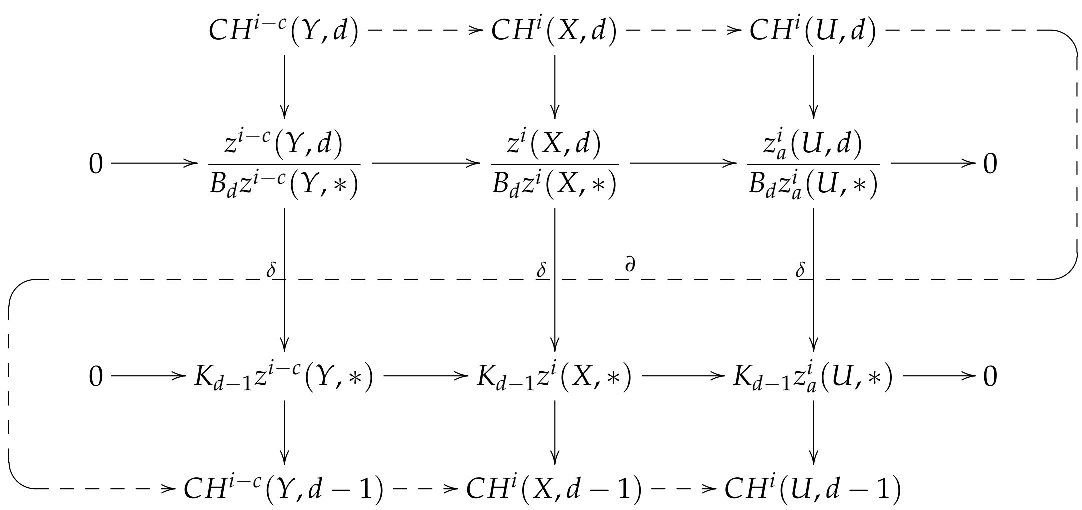
**Proposition 13.** *As before, let $X$ be a quasiprojective variety, let $Y$ be a closed subvariety of pure codimension $c$, let $U = X - Y$ and let $\iota : U \to X$ denote the open embedding. Suppose $\alpha \in z^i(U,d)$ is such that $\overline{\alpha}$ meets the faces of $X \times \Delta^d$ properly, then the connecting homomorphism $\partial : CH^i(U,d) \to CH^i(Y,d-1)$ takes the class of $\alpha$ to the class of $\delta(\overline{\alpha})$ in $z^{i-c}(Y,d-1)$.*

*Proof.* First, since $\alpha$ is such that $\overline{\alpha}$ meets the faces of $X \times \Delta^d$ properly, it follows that $\alpha = \iota^*(\overline{\alpha})$, so $\alpha \in z^i_a(U,d)$.

The localization sequence arises from the short exact sequence of complexes

$$0 \longrightarrow z^{i-c}(Y,*) \longrightarrow z^i(X,*) \longrightarrow z^i_a(U,*) \longrightarrow 0$$

via the snake lemma:




Here $B_d$ and $Z_d$ denote the $d$-th boundaries and $d$-th cycles of the complex, respectively. A diagram chase now completes the argument. □

**Proposition 14.** *Consider $\mathbb{A}^n \setminus \{0\}$ as an open subscheme of $\mathbb{A}^n$ in the obvious way, so there is a localization sequence in higher Chow groups for* pt, $\mathbb{A}^n$ *and* $\mathbb{A}^n \setminus \{0\}$. *The higher Chow groups* $CH(\mathrm{pt}) = \mathbb{M}$ *are given an explicit generator,* $v$. *Write* $H^{2n-1,n}(\mathbb{A}^n \setminus \{0\}, \mathbb{Z}) = CH^n(\mathbb{A}^n \setminus \{0\}, 1) = \mathbb{Z}\gamma \oplus Q$, *where* $Q = 0$ *for* $n \geq 2$ *and* $Q = k^*$ *for* $n = 1$, *and where* $\gamma$ *is such that the boundary map*

$$\partial : CH^n(\mathbb{A}^n \setminus \{0\}, 1) \to CH^0(\mathrm{pt}, 0)$$

*maps* $\gamma$ *to* $v$. *The element* $\gamma$ *may be represented by any curve in*

$$\mathbb{A}^n \times \Delta^1 = \operatorname{Spec} k[x_1, \ldots, x_n, t]$$

*which fails to meet the hyperplane* $t = 0$ *and meets* $t = 1$ *with multiplicity one at* $x_1 = x_2 = \cdots =$

$x_n = 0$ *only.*

*Proof.* The low-degree part of the localization sequence is

$$CH^0(\mathrm{pt}, 1) = 0 \longrightarrow CH^n(\mathbb{A}^n, 1) = Q \longrightarrow CH^n(\mathbb{A}^n \setminus 0, 1) \xrightarrow{\partial} CH^0(\mathrm{pt}, 0)$$
$$\longrightarrow CH^0(\mathbb{A}^n, 0) = \mathbb{Z} \longrightarrow CH^0(\mathbb{A}^n \setminus 0, 0) = \mathbb{Z} \longrightarrow 0$$

Suppose $C$ is a curve which does not meet $t = 0$, and which meets $t = 1$ with multiplicity one at $x_1 = \cdots = x_n = 0$ only, then by proposition 13 the cycle $[C] \in CH^n(A^n - 0, 1)$ maps to the class of a point in $CH^0(\mathrm{pt}, 0) = \mathbb{Z}$, which is a generator, [Ful84]. The assertion now follows from straightforward homological algebra. □

**Corollary 14.1.** *Suppose* $p \in \mathbb{A}^n \setminus \{0\}$ *is a k-valued point. Write* $p = (p_1, \ldots, p_n)$*. The curve given by the equation*

$$\gamma_p : \quad (x_1 - p_1)t + p_1 = (x_2 - p_2)t + p_2 = \cdots = (x_n - p_n)t + p_n = 0$$

*represents a canonical generator of* $CH^n(\mathbb{A}^n \setminus \{0\}, 1)$*.*

*Proof.* One verifies easily that the proposition applies. □

**Corollary 14.2.** *Consider the map* $\mathbb{A}^n \setminus \{0\} \to \mathbb{A}^n \setminus \{0\}$ *given by multiplication by* $-1$*. This map induces the identity on cohomology.*

*Proof.* The preimage of the curve $\gamma_p$ is the curve $\gamma_{-p}$, but both represent the same generator of $CH^n(\mathbb{A}^n \setminus \{0\}, 1)$, so the result follows. □

## 1.4 The Comparison Map: $\mathbb{G}_m \wedge \mathbb{P}^{n-1}_+ \to \mathrm{GL}(n)$

It will be necessary in this section to pay attention to basepoints. The group schemes $\mathbb{G}_m$ and $\mathrm{GL}_n$ will be pointed by their identity elements. When we deal with pointed spaces, we compute reduced motivic cohomology for preference.

We establish a map in homotopy $\mathbb{G}_m \times \mathbb{P}^{n-1} \to \mathrm{GL}(n)$, in fact we have a map from the half-smash product

$$\mathbb{G}_m \wedge \mathbb{P}^{n-1}_+ \to \mathrm{GL}(n) \tag{1.2}$$

and we show this latter map induces isomorphism on a range of cohomology groups.

We view $\mathbb{P}^{n-1}$ as being the space of lines in $\mathbb{A}^n$, and $\check{\mathbb{P}}^{n-1}$ the space of hyperplanes in $\mathbb{A}^n$. Define a space $F^{n-1}$ as being the subbundle of $\mathbb{P}^{n-1} \times \check{\mathbb{P}}^{n-1}$ consisting of pairs $(L, U)$ where $L \cap U = 0$, or equivalently, where $L + U = \mathbb{A}^n$. More precisely, we take $S = \mathbb{P}^{n-1}$ and construct $\mathbb{P}^{n-1} \times \mathbb{P}^{\check{n}-1} = \mathrm{Proj}_S(S[y_0, \ldots, y_{n-1}])$. Let $Z$ denote the closed subscheme of $\mathbb{P}^{n-1} \times \check{\mathbb{P}}^{n-1}$ determined by the bihomogeneous equation $x_0y_0 + x_1y_1 + \cdots + x_{n-1}y_{n-1} = 0$. Then $F^{n-1}$ is the complement $\mathbb{P}^{n-1} \times \mathbb{P}^{\check{n}-1} \setminus Z$. Taking $\mathbb{A}^{n-1}$ to be a coordinate open subscheme of $S = \mathbb{P}^{n-1}$ determined by $x_0 \neq 0$, we obtain the following pull-back diagram

$$\begin{array}{ccc} U = \mathrm{Proj}_{\mathbb{A}^{n-1}}(y_0, \ldots, y_{n-1}) \setminus Z|_{\mathbb{A}^{n-1}} & \longrightarrow & F^{n-1} \\ \downarrow & & \downarrow \\ \mathbb{A}^{n-1} & \longrightarrow & \mathbb{P}^{n-1} \end{array}$$

The scheme $U$ is the complement of a hyperplane in $\mathrm{Proj}_{\mathbb{A}^{n-1}}(y_0, \ldots, y_{n-1})$, and so takes the form $\mathrm{Spec}_{\mathbb{A}^{n-1}}(t_1, \ldots, t_{n-1})$. The projection $U \to \mathbb{A}^{n-1}$ is consequently an $\mathbb{A}^1$-equivalence. Since the coordinate open subschemes $\mathbb{A}^{n-1}$ form an open cover of $\mathbb{P}^{n-1}$, it follows that $F^{n-1} \simeq \mathbb{P}^{n-1}$.

The set of $k$-valued points $F^{n-1}(k)$ is the set of pairs of $n$-tuples of elements of $k$, modulo action by nonzero scalars, $[a_0; \ldots; a_{n-1}], [b_0; \ldots; b_{n-1}]$ satisfying $\sum_i a_i b_i \neq 0$, the former representing the line $L$, and the latter representing the hyperplane $U$. In fact, if $R$ is any $k$-algebra, then the set of $R$-points $F^{n-1}(R)$ is the set of pairs of $n$-tuples of elements of $R$, modulo scalar action by $R^\times$, denoted $[a_0; a_1; \ldots; a_{n-1}]$ and $[b_0; b_1; \ldots; b_{n-1}]$, such that $\sum a_i b_i \in R^\times$.

**Lemma 15.** *For any $n \geq 1$, let $X_n$ denote the motivic space*

$$X_n = \mathbb{G}_m \wedge (F^{n-1}_+) \tag{1.3}$$

*Then there is a map* $f_n : X_n \to B(n,n) \xrightarrow{\sim} \mathrm{GL}(n)$*, which makes the following diagram commute*

$$\begin{array}{ccccc} & & \mathrm{GL}(n) & & \\ & & \uparrow & & \\ X_n & \xrightarrow{f_n} & B(n,n) & \longrightarrow & \mathrm{GL}(n+1) \\ \downarrow & & & & \downarrow \\ X_{n+1} & \xrightarrow{f_{n+1}} & B(n+1,n+1) & \longrightarrow & \mathrm{GL}(n+2) \\ & & \downarrow & & \\ & & \mathrm{GL}(n+1) & & \end{array} \tag{1.4}$$

*Where the maps* $X_n \to X_{n+1}$ *are induced by a standard inclusion* $\mathbb{P}^{n-1} \to \mathbb{P}^n$ *as the first n coordinates.*

*Proof.* This map is surprisingly difficult to construct, but the general idea is quite straightforward. We consider the map $\tilde{f}_n : \mathbb{G}_m(k) \times F^{n-1}(k) \to \mathrm{GL}(n,k)$ by sending the triple $(\sigma, L, U)$ to the transformation

$$L \oplus U \xrightarrow{\times\sigma,\mathrm{id}} L \oplus U$$

That is to say, the transformation should act like scalar multiplication by $\sigma$ on the line $L$ and should act as the identity on $U$. Consequently, if $\sigma = 1$, the map is guaranteed to be the identity, so the maps factors through $\mathbb{G}_m \times F^{n-1} \to \mathbb{G}_m F^{n-1}_+$.

Unfortunately this appealing map may does not seem to be a map of schemes, we must work a little harder and construct a map that may not quite fix $U$. Let $R$ be an arbitrary $k$-algebra. We attempt to construct a map of sets

$$\mathbb{G}_m(R) \times F^{n-1}(R) \to \mathrm{GL}(n,R)$$

That is to say, we start with the data $\sigma \in R^\times$ and $[a_0; \ldots; a_{n-1}]$, $[b_0; \ldots; b_{n-1}] \in \mathbb{P}^{n-1}(R)$, with $\Pi = \sum a_i b_i \in R^\times$, and we try to construct a transformation on $R^n$ that is multiplication by $\sigma$ on $\mathbf{a} = (a_0, \ldots, a_{n-1})$ and the identity on the hyperplane in $R^n$ given by $b_0 x_0 + \cdots +$

$b_{n-1}x_{n-1} = 0$. It is not possible in general to extend $\mathbf{a}$ to a basis of $R^n$, which impedes the naive construction we might attempt.

Instead, we consider an isomorphism $R^{n+1} \cong R^n \oplus R\mathbf{c}$, where $R^n \subset R^{n+1}$ are given standard bases, the latter being an extension of the former by appending a vector we call $\mathbf{c}$. Let $D$ be the matrix

$$D = \begin{pmatrix} \mathbf{a}^t & I_n \\ 0 & (-1)^{n+1}\mathbf{b} \end{pmatrix}$$

The determinant of $D$ is $\Pi \in R^\times$, so $D$ is invertible, or in other words the columns of $D$ form a basis of $R^{n+1}$, the first element of which is $\mathbf{a}$, the others are denoted $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$. We see by considering the trailing coefficient that $\mathbf{w} = \alpha\mathbf{a} + \sum \beta_i \mathbf{v}_i$ is in $R^n \subset R^{n+1}$ if and only if $\sum b_i \beta_i = 0$. We define a linear transformation on $R^{n+1}$ by

$$\overline{T}\left(\alpha\mathbf{a} + \sum \beta_i \mathbf{v}_i\right) = t\alpha\mathbf{a} + \sum \beta_i \mathbf{v}_i$$

It is now easy to see that $R^n \subset R^{n+1}$ is $\overline{T}$-invariant, and as a consequence we can define $T : R^n \to R^n$ as the restriction of $\overline{T}$, for when we want an honest map to $\mathrm{GL}(n)$. In the case of an algebraically closed field $R = k = \overline{k}$, we make the following additional observations. First, $\overline{T}$ has eigenvalues $t$ and 1, the latter with (geometric) multiplicity $n$. Since $R^n \subset R^{n+1}$ is $\overline{T}$-invariant, any eigenvector of the restriction, $T$, is also an eigenvactor of $\overline{T}$. It follows that $T$ has eigenvalues 1 and $t$, the latter with geometric multiplicity 1.

It will also be useful to factor the map $\overline{T}$ as

$$\mathbb{G}_m(R) \times F^{n-1}(R) \xrightarrow{T_0} B(n,n)(R) \longrightarrow \mathrm{GL}(n+1,R)$$

where $B(n,n)(R)$ is the subgroup of $\mathrm{GL}(n+1,R)$ consisting of elements, $A$, for which $R^n \subset R^{n+1}$ is $T$-invariant and for which the projection along the standard basis satisfies $\mathrm{Proj}_{\mathbf{c}}(A\mathbf{c}) = \mathbf{c}$. One verifies directly that $\overline{T}$ actually lies in $B(n,n)(R)$ by first observing that $\mathbf{c} = \Pi^{-1}(\mathbf{a} + (-1)^n \sum a_i \mathbf{v}_i)$, so that $\overline{T}\mathbf{c} = \Pi^{-1}(t\mathbf{a} + (-1)^n \sum a_i \mathbf{v}_i)$, and the projection onto $\mathbf{c}$ depends only on the coefficients of $\mathbf{v}_i$, and not on that of $\mathbf{a}$.

This $B(n,n)(R)$ is in fact a representable functor on rings, represented by the group-scheme $B(n,n)$ of matrices of the form

$$\begin{pmatrix} * & * & \dots & * & * \\ * & * & \dots & * & * \\ \vdots & \vdots & & \vdots & \vdots \\ * & * & \dots & * & * \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

which we considered in corollary 9.1. We have $B(n,n) \simeq \mathrm{GL}(n)$.

This completes our definition of a map $\mathbb{G}_m(R) \times F^{n-1}(R) \to \mathrm{GL}(n,R)$. We remark that the definition could be made without passing to the larger space $R^{n+1}$, but the cost is introducing many more unmotivated equations. The essential obstruction to extending $\mathbf{a}$ to a basis of $\mathbb{R}^n$ is akin to the nontriviality of a tangent bundle on a sphere, and the workaround we employ is based on the stable triviality of the tangent bundle, viz. the problem can be solved after the addition of a trivial summand.

Given an arbitrary homomorphism $R \to S$ of $k$-algebras, there is a commuting diagram

$$\begin{array}{ccc} \mathbb{G}_m(R) \times F^{n-1}(R) & \longrightarrow & \mathrm{GL}(n,R) \\ \downarrow & & \downarrow \\ \mathbb{G}_m(S) \times F^{n-1}(S) & \longrightarrow & \mathrm{GL}(n,S) \end{array}$$

because the horizontal maps are effected by polynomials, as can easily be verified. In short, we have a natural transformation of functors $\mathbb{G}_m(\cdot) \times F^{n-1}(\cdot) \to \mathrm{GL}(n,\cdot)$ from $k$-algebras to *Sets*. By Yoneda's lemma we have a map of schemes $\mathbb{G}_m \times F^{n-1} \to \mathrm{GL}(n)$.

We remark that if $t = 1$, then $T = I_n$, so that we do indeed obtain the claimed factorization $\mathbb{G}_m \wedge F^{n-1}_+ \to \mathrm{GL}(n)$ as claimed.

The commutative diagram (1.4) follows without great difficulty. In

$$\begin{array}{ccccc} X_n & \xrightarrow{f_n} & B(n,n) & \longrightarrow & \mathrm{GL}(n+1) \\ \downarrow & & & & \downarrow{\scriptstyle \phi_{e_1,k^n}} \\ X_{n+1} & \xrightarrow{f_{n+1}} & B(n+1,n+1) & \longrightarrow & \mathrm{GL}(n+2) \end{array}$$

the first vertical map arises from the evident closed inclusion $F^{n-1} \to F^n$ given by the homogeneous equations $a_n = b_n = 0$, the latter is the standard inclusion $\mathrm{GL}(n+1) \to \mathrm{GL}(n+2)$. The diagram is simply a matter of book-keeping. □

The construction $\mathbb{G}_m \wedge X$ is denoted by $\Sigma^1_t X$ and is called the Tate suspension, [Voe03b]. We have

$$H^{*,*}(\mathbb{G}_m; R) \cong \frac{\mathbb{M}[\sigma]}{\sigma^2 - \{-1\}\sigma}$$

the relation being derived in loc. cit. It is easily seen that as rings, we have

$$H^{*,*}(\mathbb{G}_m \times X; R) \cong H^{*,*}(X; R) \otimes_{\mathbb{M}} \frac{\mathbb{M}[\sigma]}{(\sigma^2 - \{-1\}\sigma)}$$

and that $\tilde{H}(\Sigma^1_t X; R)$ is the split submodule (ideal) generated by $\sigma$, leading to a peculiar feature of the Tate suspension

**Proposition 16.** *Suppose $x, y \in \tilde{H}^{*,*}(X; R)$, and that $\sigma x, \sigma y$ are their isomorphic images in $\tilde{H}^{*,*}(\Sigma^1_t X; R)$. Then $(\sigma x)(\sigma y) = \{-1\}\sigma(xy)$.*

**Proposition 17.** *The map $f_n$ induces an isomorphism on cohomology*

$$H^{2j-1,j}(\mathrm{GL}(n); R) \to H^{2j-1,j}(\Sigma^1_t \mathbb{P}^{n-1}_+)$$

*in dimensions $(2j-1, j)$ where $j \geq 1$.*

*Proof.* We first remark that when $j$ is large, $j > n$,

$$H^{2j-1,j}(\mathrm{GL}(n); R) = H^{2j-1,j}(\Sigma^1_t \mathbb{P}^{n-1}; R) = 0$$

so the result holds trivially in this range, so we restrict to the case $1 \leq j \leq n$.

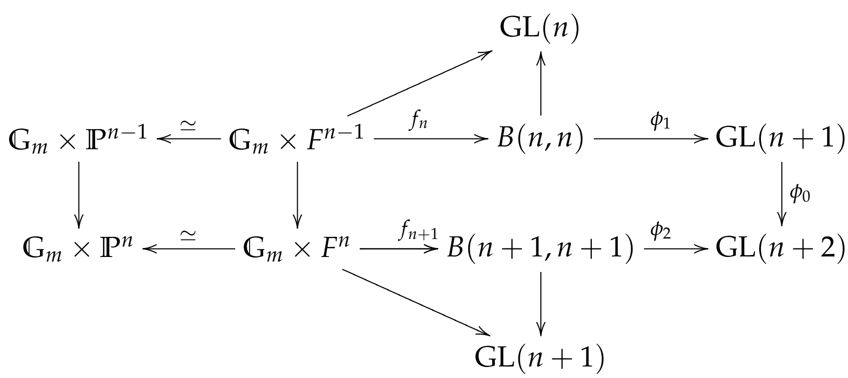
The following are known:

$$H^{*,*}(\mathrm{GL}_n) = \mathbb{M}[\rho_n, \dots, \rho_1]/I \quad |\rho_i| = (2i-1, i)$$
$$H^{*,*}(\Sigma^1_t \mathbb{P}^{n-1}; R) = \bigoplus_{i=0}^{n-1} \sigma\eta^i \mathbb{M} \quad |\sigma| = (1,1), \quad |\eta| = (2,1)$$

It suffices to show that $f_n^*(\rho_i) = \sigma\eta^{i-1}$. Since $f_n^*(\rho_i) = \tilde{f}_n^*(\rho_i)$, it suffices to prove this for the map $\tilde{f}_n : \mathbb{G}_m \times F^{n-1}$, which has the benefit of being more explicitly geometric

We prove this by induction on $n$. In the case $n = 1$, the space $F^{n-1}$ is trivial, and $X_n = \mathbb{G}_m = \mathrm{GL}(n)$. The map $f_1$ is the identity map, so the result holds in this case.

View the matrices in $\mathrm{GL}(n)$ as acting on a given space, $k^n$. Identify $k^n = \langle e_1 \rangle \oplus k^{n-1}$ There is a diagram of varieties




as previously constructed. We understand the vertical map on the left since we can rely on the theory of ordinary Chow groups, [Ful84, chapter 1], we know that the induced map $H^{2j,j}(\mathbb{P}^{n-1}) \to H^{2j,j}(\mathbb{P}^{n-2})$ is an isomorphism for $j < n-1$, and so $i^*$ is an isomorphism $H^{2j-1,j}(\mathbb{G}_m \times F^{n-1}) \cong H^{2j-1,j}(\mathbb{G}_m \times F^{n-2})$ for $j < n$.

The maps $\phi_i^*$ are also isomorphisms in this range, by proposition 9 and its corollary, so the diagram implies that the result holds except possibly for $f_n^*(\rho_n)$.

The argument we use to prove $f_n^*(\rho_n) = \sigma\eta^{n-1}$ is based on the composition

$$\mathbb{G}_m \times F^{n-1} \longrightarrow \mathrm{GL}(n) \xrightarrow{\pi} \mathbb{A}^n \setminus \{0\}$$

where the map $\pi$ is projection on the first column. We write $g$ for the composition of the two maps. Since $H^{*,*}(\mathbb{A}^n \setminus \{0\}) \cong \mathbb{M}[\iota]/(\iota^2)$, with $|\iota| = (2n-1, n)$, and $\pi^*(\iota) = \rho_n$, it suffices to prove that $g^*(\iota) = \sigma\eta^{n-1}$.

For the sake of carrying out computations, it is helpful to identify motivic cohomology and higher Chow groups, e.g. identify $H^{2n-1,n}(\mathbb{A}^n \setminus \{0\})$ and $CH^n(\mathbb{A}^n \setminus \{0\}, 1)$. We can write down an explicit generator for $CH^n(\mathbb{A}^n \setminus \{0\}, 1)$, see corollary 14.1, for instance the curve $\gamma$ in $\mathbb{A}^n \setminus \{0\} \times \Delta^1$ given by $t(x_1 - 1) = -1$, $x_2 = x_3 = \cdots = x_n = 0$. We can also write down an explicit generator, $\mu$, for a class $\sigma\eta_0^{n-1} \in \mathbb{G}_m \times \mathbb{P}^{n-1}$, writing $x$ and $a_0, \ldots, a_{n-1}$ for the coordinates on each, and $t$ for the coordinate function on $\Delta^1$, $\sigma\eta^{n-1}$ is explicitly represented by the product of the subvariety of $\mathbb{P}^{n-1}$ given by $a_0 = 1, a_2 = 0 \ldots, a_{n-1} = 0$, which represents $\eta^{n-1}$, with the cycle given by $t(x-1) = -1$ on $\mathbb{G}_m \times \Delta^1$.

We claim that the pullback $g^*(\gamma)$ to $\mathbb{G}_m \times F^{n-1}$ coincides with the pullback of $\mu$ along $\mathbb{G}_m \times F^{n-1} \to \mathbb{G}_m \times \mathbb{P}^{n-1}$. Since this is a statement concerning the coincidence of two subvarieties of $\mathbb{G}_m \times F^{n-1}$, it suffices to prove it after a faithfully flat base-change to the algebraic closure, $\bar{k}$. We therefore can assume $k = \bar{k}$.

We now consider the curve $\gamma \in \mathbb{A}^n \setminus \{0\}$. The pullback of this curve to $\mathrm{GL}_n \times \Delta^1$ is the closed subscheme whose closed points are matrices satisfying

$$\begin{pmatrix} x_1 & * & \ldots & * \\ 0 & * & \ldots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \ldots & * \end{pmatrix}$$

where $t(x_1 - 1) = -1$. In the interest of brevity do not write down the equations defining this scheme.

If such a matrix is in the image of the map $f : \mathbb{G}_m \times F^{n-1}$, since $e_1$ is an eigenvalue with eigenvector $x_1 \neq 1$, it follows that $L = (e_1)$, the first standard basis vector, $\sigma = x_1$. But any quadruple $(\sigma, L, U, t)$ such that $t(\sigma - 1) = -1$, and $L = \langle e_1 \rangle$ maps to such a matrix under $f$. We have identified the preimage $g^*(\gamma)$ with the closed subvariety $\Gamma \rightarrowtail \mathbb{G}_m \times F^{n-1} \times \Delta^1$ of such quadruples. $\Gamma$ is, furthermore, the product of the cycle $\gamma_1$ given by $t(\sigma - 1) = -1$ and the cycle $\gamma_2$ given by $L = \langle e_1 \rangle$ — the intersection theory here is particularly easy since the latter is a cycle in ordinary Chow groups. It is easy to see now that this is exactly the pullback of the cycle $\mu$ along $\mathbb{G}_m \times F^{n-1} \to \mathbb{G}_m \times \mathbb{P}^{n-1}$, which represents $\sigma\eta^{n-1}$. In summary, we have shown $f_n^*(\rho_n) = \sigma\eta^{n-1}$. □

We can incidentally compute the action of the Steenrod operations on the cohomology $H^{*,*}(W(n,m); \mathbb{Z}/2)$ by means of the inclusions

**Corollary 17.1.** *Represent* $H^{*,*}(W(n,m); \mathbb{Z}/2)$ *as* $\mathbb{M}_2[\rho_n, \dots, \rho_{n-k+1}]/I$. *The even motivic Steenrod squares act as*

$$\mathrm{Sq}^{2i}(\rho_j) = \begin{cases} \binom{j-1}{i}\rho_{j+i} & \text{if } i + j \leq n \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* There is an inclusion of $H^{*,*}(W(n,m); \mathbb{Z}/2) \subset H^{*,*}(\mathrm{GL}(n); \mathbb{Z}/2)$ arising from the projection map, see proposition 8. It suffices therefore to compute the action of the squares on $H^{*,*}(\mathrm{GL}(n); \mathbb{Z}/2)$. Using the previous proposition and the decomposition in equation (1.3), we have isomorphisms $H^{2n-1,n}(\mathrm{GL}(n); \mathbb{Z}/2) \cong H^{2n-1,n}(\Sigma^1_t \mathbb{P}^{n-1}; \mathbb{Z}/2)$. To be precise, we have

$$f_n^*(\mathrm{Sq}^{2i}\, \rho_j) = \mathrm{Sq}^{2i}\, f_n^*(\rho_j) = \mathrm{Sq}^{2i}\, \sigma\eta^{j-1}$$

$$= \sigma\, \mathrm{Sq}^{2i}\, \eta^{j-1} = \begin{cases} \binom{j-1}{i}\sigma\eta^{j+i-1} = \binom{j-1}{i} f_n^*(\rho_{j+i}) & \text{if } i + j \leq n \\ 0 & \text{otherwise} \end{cases}$$

Since $f_n^*$ is an isomorphism on these groups, the result follows. □

We are also in a position to pay off at last the debt we owe regarding the product structure of $H^{*,*}(W(n,m); R)$.

**Theorem 18.** *The cohomology of $W(n,m)$ has the following presentation as a graded-commutative $\mathbb{M}_R$-algebra:*

$$H^{*,*}(W(n,m);R) = \frac{\mathbb{M}_R[\rho_n, \dots, \rho_{n-m+1}]}{I} \qquad |\rho_i| = (2i-1, i)$$

*The ideal $I$ is generated by relations $\rho_i^2 - \{-1\}\rho_{2i-1}$, where $\{-1\} \in \mathbb{M}_R^{1,1}$ is the image of $-1 \in k^* = \mathbb{M}_\mathbb{Z}$ under the map $\mathbb{M}_\mathbb{Z} \to \mathbb{M}_R$.*

*Proof.* It sufficies to deal with the case $R = \mathbb{Z}$. It suffices also to consider only the case $m = n$, since we can use the inclusion $H^{*,*}(W(n,m);R) \subset H^{*,*}(\mathrm{GL}(n,m);R)$ to deduce it for all $n, m$.

We have proved everything already in proposition 6, except that in the relation $\rho_i^2 - a\rho_{2i-1}$, we were unable to show $a$ was nontrivial. We consider the map $\mathbb{G}_m \times F_+^{n-1} \to \mathrm{GL}(n)$, which induces a map of rings on cohomology. In the induced map, we have $\rho_i \mapsto \sigma\eta^{i-1}$, and so $\rho_i^2 \mapsto -\sigma^2\eta^{2i-2} = \{-1\}\sigma\eta^{2i-2}$. Since this is nontrivial if $2i - 2 \leq n-1$, it follows that $\rho_i^2$ is similarly nontrivial. □

In the case $n = m$ this result, although computed by a different method, appears in [Pus04].

## 1.5 Hopf Algebra Structure

Since $\mathrm{GL}(n)$ is a group scheme, the algebra $H^{*,*}(\mathrm{GL}(n))$ is in fact a Hopf algebra. In particular there is of course a product map $\mathrm{GL}(n) \times \mathrm{GL}(n) \to \mathrm{GL}(n)$, which gives rise to a map on cohomology

$$H^{*,*}(\mathrm{GL}(n)) \to H^{*,*}(\mathrm{GL}(n) \times \mathrm{GL}(n)) \to H^{*,*}(\mathrm{GL}(n)) \otimes_{\mathbb{M}} H^{*,*}(\mathrm{GL}(n))$$

we denote the composition by $\mu^*$ by an abuse of notation. It satisfies a number of well-known axioms, including

1. $\mu^*$ is an algebra homomorphism.

2. There is an augmentation map $\eta^* : H^{*,*}(\mathrm{GL}(n)) \to \mathbb{M}$

3. The composition $\mathrm{GL}(n) \times \mathrm{pt} \to \mathrm{GL}(n) \times \mathrm{GL}(n) \to \mathrm{GL}(n)$ given by the inclusion of the basepoint gives on the one hand $(\eta^* \otimes \mathrm{id})\mu^*$, and on the other it is simply the identity, and similarly for the composition $(\mathrm{id} \otimes \eta^*)\mu^*$.

From these axioms we derive the following

**Proposition 19.** *For $\rho_i \in \Lambda_{\mathbb{M}}(\rho_1, \ldots, \rho_n) = H^{*,*}(\mathrm{GL}(n))$, we have $\mu^*(\rho_i) = 1 \otimes \rho_i + \rho_i \otimes 1$, and this determines the action of $\mu^*$ on $H^{*,*}(\mathrm{GL}(n))$ in full.*

*Proof.* The fact that $\mu^*$ is an algebra homomorphism means that it suffices to compute it on a generating set of $H^{*,*}(\mathrm{GL}(n))$. We know that $|\mu^*\rho_i| = (2i-1, i)$, but it is immediately seen that only elements of the form $a(\rho_i \otimes 1) + b(1 \otimes \rho_i) \in H^{*,*}(\mathrm{GL}(n)) \otimes H^{*,*}(\mathrm{GL}(n))$ have this bidegree. Since $\rho_i$ is in the kernel of the augmentation map, it follows from axiom 3 above that $a = b = 1$. □

From this we determine the following useful result, which will see action when we consider equivariant cohomology of $\mathrm{GL}(n)$.

**Proposition 20.** *The involution $\iota : \mathrm{GL}(n) \to \mathrm{GL}(n)$ given by the group-theoretic inverse induces the map $\rho_i \mapsto -\rho_i$ on the motivic cohomology of $\mathrm{GL}(n)$. In fact, if we let $s_j : \mathrm{GL}(n) \to \mathrm{GL}(n)$ denote the map that sends $A$ to $A^j$ for $j \in \mathbb{Z}$, then $s_j^*(\rho_i) = j\rho_i$.*

*Proof.* Consider the following diagram in which $\Delta$ denotes the diagonal

$$\mathrm{GL}(n) \xrightarrow{\quad\Delta\quad} \mathrm{GL}(n) \times \mathrm{GL}(n) \xrightarrow{\mathrm{id}\times\iota} \mathrm{GL}(n) \times \mathrm{GL}(n) \xrightarrow{\ \mu\ } \mathrm{GL}(n)$$

The composition of the maps is the map $\mathrm{GL}(n) \to \mathrm{pt} \to \mathrm{GL}(n)$, which sends $\rho_i$ to 0 for all $i$. On the other hand, the map factors as

$$\rho_i \mapsto \rho_i \otimes 1 + 1 \otimes \rho_i \mapsto \rho_i \otimes 1 + 1 \otimes \iota^*(\rho_i) \mapsto \rho_i + \iota^*(\rho_i)$$

The result for $\iota = s_{-1}$ follows immediately.

The result for $s_0, s_1$ being trivial, we consider now the case of $j \geq 2$.

$$\mathrm{GL}(n) \xrightarrow{\Delta} \mathrm{GL}(n) \times \mathrm{GL}(n) \xrightarrow{\iota \times s_j} \mathrm{GL}(n) \times \mathrm{GL}(n) \xrightarrow{\mu} \mathrm{GL}(n)$$

The composition being $s_{j-1}$, and the map on cohomology factors as

$$\rho_i \mapsto \rho_i \otimes 1 + 1 \otimes \rho_i \mapsto -\rho_i \otimes 1 + 1 \otimes s_j^*(\rho_i) \mapsto -\rho_i + s_j^*(\rho_i)$$

and the result for all positive $j$ follows by induction. For negative $j$, we obtain the result by observing that $s_j = \iota s_{-j}$. □

The following observation is of great utility

**Proposition 21.** *Let $\mathbb{G}_m$ act on $\mathrm{GL}(n)$ by multiplication of a column (or row). Then the co-action on cohomology*

$$H^{*,*}(\mathrm{GL}(n)) \to H^{*,*}(\mathbb{G}_m) \otimes H^{*,*}(\mathrm{GL}(n))$$

*is given by $\rho_1 \mapsto 1 \otimes \rho_1 + \rho_1 \otimes 1$, and $\rho_i \mapsto 1 \otimes \rho_i$ for $i \geq 2$.*

*Proof.* There is a split homomorphism $\mathbb{G}_m \to \mathrm{GL}(n)$ given by (on $R$-valued points)

$$z \mapsto \operatorname{diag}(1, \ldots, 1, z, 1, \ldots, 1)$$

which is split by the determinant map $\mathrm{GL}(n) \to \mathbb{G}_m$. The result for $\rho_1$ now follows from the Hopf algebra structure on $H^{*,*}(\mathbb{G}_m)$ and the diagram

$$\begin{array}{ccccc} \mathbb{G}_m \times \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \times \mathrm{GL}(n) & \longrightarrow & \mathbb{G}_m \times \mathbb{G}_m \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{G}_m & \longrightarrow & \mathrm{GL}(n) & \longrightarrow & \mathbb{G}_m \end{array}$$

The result for $\rho_i$ with $i \geq 2$ follows for dimensional reasons. □

# Chapter 2

# The Rothenberg-Steenrod Spectral Sequence

## 2.1 Preliminaries

We establish in proposition 43 a spectral sequence computing the motivic cohomology of a homogeneous variety $X = Y/G$ in terms of the motivic cohomology of $Y$ and $G$, provided a local trivializability condition is satisfied. This is an analogue of a spectral sequence in classical algebraic topology that goes by several names, "Rothenberg-Steenrod" or "fiber-to-base Eilenberg-Moore". The local trivializability condition can be thought of as a translation to algebraic geometry of an assumption on $\pi_1$-actions in classical algebraic topology.

In algebraic topology, because $Y$ is a principal $G$-bundle over $X$, the map $Y \to X$ is a fibration, which is not the case in the $\mathbb{A}^1$-homotopy theory. In classical topology, the fiber sequence can be continued by delooping to give another fiber sequence $Y \to EG \times_G Y \simeq X \to BG$, in which the space of interest is the total space, and so the computation may be carried out by the Serre spectral sequence. In the $\mathbb{A}^1$-homotopy theory where we do not have the Serre spectral sequence at our disposal, the sequence established here is much more useful.

In the case where a space $Y$ has a non-free $G$-action, the sequence we construct still converges, to the cohomology of a homotopy type $B(\mathrm{pt}, G, Y)$ which depends functorially on $Y$. This is one of several definitions of the Borel construction for $G$ acting on $Y$. We are able to demonstrate that, at least in ideal cases, the equivariant motivic cohomology computed by our methods agrees with the equivariant motivic cohomology defined by [EG98].

## 2.2 Bisimplicial Sheaves and $\mathbb{A}^1$-homotopy

It will be necessary to work with several categories and notions of weak equivalence. Our starting point is the category $Sm/k$ of smooth schemes of finite type over a field $k$, as in [MV99]. We will eventually want to work in the category $Spaces = sSh_{Nis}(Sm/k)$ of simplicial Nisnevich sheaves on this site, with $\mathbb{A}^1$-weak equivalences. Along the way, however, we shall consider also the category $sPre(Sm/k)$ of simplicial presheaves on the category $Sm/k$, which we will denote by $P$ for convenience. The former is a full subcategory of the latter, $Spaces \subset P$. Each category can be given one of many model structures, see [Jar87], [Bla01], [Isa05]. We use the model structure presented in [Jar87], which is 'standard' among model structures, being that of [MV99]. It seems to be most often known as the *injective* model structure, [Isa05], although it is not always so named. In this model category, the cofibrations are exactly the monomorphisms of presheaves, and the weak equivalences are detected stalkwise. The initial object in both categories is the empty presheaf, and as a consequence all objects are cofibrant. All model categories considered here are simplicial.

In general, weak equivalences are detected locally, which is why the associated-sheaf functor yields an equivalence, but in what follows we shall frequently make use of the following blunt proposition.

**Proposition 22.** *Suppose $f : X \to Y$ is a map of simplicial sheaves or presheaves, and suppose that for all $U \in Sm/k$, the induced map $X(U) \to Y(U)$ is a weak equivalence. Then $f : X \to Y$ is a weak equivalence.*

*Proof.* This follows quite easily from the presentation of the model structure in [Jar87], it is somewhat obscured in [MV99]. □

Our references for the homotopy theory of simplicial objects in a simplicial model category are [Hir03], especially chapter 18, and [BK72]. Let $X_\bullet$ be a simplicial object in *Spaces* (or in $P$), which one views either as a bisimplicial sheaf in $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to Sh_{Nis}(Sm/k)$ (in $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to Pre(Sm/k)$ respectively), or as a $\Delta^{\mathrm{op}}$-shaped diagram in the category *Spaces* (in the category $P$ respectively). There are therefore two ways to view $X_\bullet$ as an element of *Spaces*, first by taking the realization, in the notation of [Hir03]

$$|X_\bullet| = \mathrm{coeq}\Big( \coprod_{(\sigma : [n] \to [k]) \in \Delta} X_n \otimes \Delta[k] \rightrightarrows \coprod_{[n] \in \mathrm{Ob}(\Delta)} X_n \otimes \Delta[n] \Big)$$

and second by taking the homotopy colimit, which is given by

$$\operatorname*{hocolim}_{\Delta} X_\bullet = \mathrm{coeq}\Big( \coprod_{(\sigma : [n] \to [k]) \in \Delta} X_n \otimes B([k] \downarrow \Delta)^{\mathrm{op}} \rightrightarrows \coprod_{[n] \in \mathrm{Ob}(\Delta)} X_n \otimes B([n] \downarrow \Delta)^{\mathrm{op}} \Big)$$

From this description it is clear that if $X_\bullet$ is a simplicial object in *Spaces*, then it does not matter whether the realization or the homotopy colimit is taken in the category *Spaces* or $P$, the answer in either case is the same. This applies indeed to arbitrary homotopy colimits of diagrams in the category *Spaces*, for which the general construction is

$$\operatorname*{hocolim}_{I} X = \mathrm{coeq}\Big( \coprod_{(\sigma : \alpha \to \alpha') \in I} X(\alpha) \otimes B(\alpha' \downarrow I)^{\mathrm{op}} \rightrightarrows \coprod_{\alpha \in \mathrm{Ob}(I)} X(\alpha) \otimes B(\alpha \downarrow I)^{\mathrm{op}} \Big) \tag{2.1}$$

**Proposition 23.** *Let $X_\bullet$ be a simplicial object in either Spaces or in $P$, then there is a weak equivalence*

$$\phi_* : \operatorname*{hocolim}_{\Delta} X_\bullet \xrightarrow{\sim} |X_\bullet|$$

*Proof.* The bulk of the heavy lifting is carried out by the results of chapter 18 of [Hir03]

In both *Spaces* and $P$ ordinary colimits are taken objectwise, viz., for a diagram $\mathcal{D}$ indexed by a category $C$ and $U \in Sm/k$, one has

$$\big(\operatorname*{colim}_C \mathcal{D}\big)(U) = \operatorname*{colim}_C \mathcal{D}(U)$$

as a consequence, it follows that if $C = \Delta$, the latching maps of, loc. cit. chapter 15, are monomorphisms. We conclude that any simplicial object $X_\bullet$ in $P$ or *Spaces* is Reedy cofibrant.

We may apply theorem 18.7.4 of loc. cit. so there is a weak equivalence

$$\phi_* : \operatorname*{hocolim}_\Delta X_\bullet \xrightarrow{\sim} |X_\bullet|$$

as promised. □

We recall from [MV99] that a point of a site $p : S \to Sets$ is a functor that commutes with finite limits and all colimits. A map $f : X \to Y$ in *Spaces* or in $P$ is a weak equivalence if and only if it induces a weak equivalence at all points. Let $I$ be a small category and let $X$ be an $I$-shaped diagram in *Spaces* (respectively in $P$). By construction of hocolim, one has

$$p\big(\operatorname*{hocolim}_I X\big) = \operatorname*{hocolim}_I pX$$

This is given in [MV99] as lemma 2.1.20.

**Corollary 23.1.** *Let $D : \mathcal{I} \to \Delta^{op}Spaces$ be a diagram (a functor from a small category) in the category of simplicial spaces. One has*

$$|\operatorname*{hocolim}_{i\in I} D(i)| \simeq \operatorname*{hocolim}_{i\in I} |D(i)|$$

*Proof.* The given equation admits rephrasing as

$$\operatorname*{hocolim}_{\Delta^{\mathrm{op}}} \operatorname*{hocolim}_{i\in I} D(i) \simeq \operatorname*{hocolim}_{i\in I} \operatorname*{hocolim}_{\Delta^{\mathrm{op}}} D(i)$$

this is a consequence of the Fubini theorem for hocolim, [BK72, Ch. XIII, 3.3], proved there for simplicial sets, which can be promoted to the current setting by arguing at points. □

An $\mathbb{A}^1$-weak equivalence between spaces, $X \simeq_{\mathbb{A}^1} Y$ is a weaker relation than a simplicial weak equivalence, in that $X \simeq Y$ implies $X \simeq_{\mathbb{A}^1} Y$. The $\mathbb{A}^1$-model structure on *Spaces*, [MV99], inherits the cofibrations of the simplicial model structure, which are simply the monomorphisms of presheaves and which coincide with the monomorphisms of sheaves. It follows from this that if $\mathcal{D}$ is a diagram in *Spaces*, then hocolim $\mathcal{D}$ may be constructed as in equation (2.1) even for the $\mathbb{A}^1$-model structure.

The following is a special case of [Hir03, Theorem 18.5.3], but it bears mentioning

**Proposition 24.** *If $I$ is a small category and $f : X \to Y$ is a natural transformation of $I$-shaped diagrams in Spaces which is an objectwise weak equivalence (respectively an objectwise $\mathbb{A}^1$ weak equivalence), then the induced map* $\operatorname{hocolim}_I X \to \operatorname{hocolim}_I Y$ *is a weak equivalence (resp. an $\mathbb{A}^1$-weak equivalence). In particular, if $f : X_\bullet \to Y_\bullet$ is a weak equivalence (resp. $\mathbb{A}^1$-weak equivalence) of simplicial objects in Spaces, then $|X_\bullet| \to |Y_\bullet|$ is a weak equivalence (resp. an $\mathbb{A}^1$-weak equivalence).*

## 2.3 Computing Represented Cohomology Theories

Since the Nisnevich topology is subcanonical on $Sm/k$, the Yoneda embedding $Sm/k \hookrightarrow P$ factors through an embedding $Sm/k \hookrightarrow Spaces$. We shall generally denote a scheme and the simplicial sheaf it represents by the same letter. We shall be careful to write $X/G$ only when a quotient of sheaves or of presheaves is intended, and not for a quotient taken in some other sense, e.g. in the category of schemes.

We use $\mathbb{A}^1$-weak equivalence because motivic cohomology is represented in the $\mathbb{A}^1$-homotopy category, viz.

$$H^{p,q}(X;R) = [X, \mathbb{H}R(p,q)]_{\mathbb{A}^1}$$

where $\mathbb{H}R(p,q)$ are a bigraded family of Eilenberg-MacLane spaces, as has been asserted already.

We denote by pt the sheaf represented by Spec $k$, it is also the constant sheaf with value the one-point simplicial set. We denote the motivic cohomology of pt by $\mathbb{M}_R = H^{*,*}(\mathrm{pt}; R)$, it is a bigraded ring over which the cohomology of any motivic space is a module.

We remark also that for schemes $X$, $Y$, there is no ambiguity in the construction $X \times Y$, since the product of schemes represents the product of sheaves. In fact, we have the following useful fact

**Proposition 25.** *The embedding $Sm/k \to$ Spaces preserves pull-back diagrams*

*Proof.* The argument is direct. Given a diagram

$$\begin{array}{ccc} & & B \\ & & \downarrow \\ C & \longrightarrow & A \end{array}$$

of smooth $k$-schemes, and a smooth $k$-scheme $U$, it is immediately verified that as sets

$$Sm/k(U, B \times_A C) = Sm/k(U, B) \times_{Sm/k(U,A)} Sm/k(U, C)$$

and since the image of $A$, for instance, in *Spaces* is the functor $Sm(\cdot, A)$ viewed as a trivial simplicial set, the result follows. □

The following proposition allows us to compute the motivic cohomology of the realization of a simplicial object in *Spaces*; similar spectral sequences exist for any homology or cohomology theory represented in the motovic homotopy category, provided issues of convergence can be resolved.

**Proposition 26.** *Suppose $A_\bullet$ is a simplicial object in Spaces. Associated naturally to $A_\bullet$ there is a chain complex of bigraded motivic cohomology groups*

$$H^{*,*}(A_{p-1}; R) \to H^{*,*}(A_p; R) \to H^{*,*}(A_{p+1}; R)$$

*the differential in which is the alternating sum of the maps induced on cohomology by the face maps* *of the simplicial structure. There is a spectral sequence converging to $H^{*,*}(|A_\bullet|;R)$ the $E_1$-page of which is this complex.*

*Proof.* For technical reasons we occasionally prefer to work with reduced cohomology of pointed simplicial sheaves. We may replace $A_n$ by $(A_n)_+$ by addition of a disjoint basepoint, and so obtain a bisimplicial sheaf $(A_\bullet)_+$. We have $|(A_\bullet)_+| = |A_\bullet|_+$ and also $H^{*,*}(X;R) = \tilde{H}^{*,*}(X_+;R)$. Henceforth in this proof we assume all sheaves pointed.

The existence of the complex given in the statement of the proposition comes from the well-known equivalence of simplicial objects and nonnegatively graded chain complexes in an abelian category.

The simplicial object $A_\bullet$ is filtered by the skeleta, $\mathrm{sk}_i(A_\bullet)$. Write $B_i$ for the cofiber $\mathrm{cone}(|\,\mathrm{sk}_{i-1}(A_\bullet)| \to |\,\mathrm{sk}_i(A_\bullet)|)$, we have a cofiber sequence

$$|\,\mathrm{sk}_{i-1}(A_\bullet)| \xrightarrow{\iota_i} |\,\mathrm{sk}_i(A_\bullet)| \longrightarrow B_i$$

Since $\iota_i$ is an inclusion, it follows that $B_i = |\,\mathrm{sk}_i(A_\bullet)|/|\,\mathrm{sk}_{i-1}(A_\bullet)|$. By arguing at an arbitrary $U \in Sm/k$, and employing the usual simplicial methods, we can establish

$$B_i \simeq \Sigma^i(A_i/A_i^d)$$

where $A_i^d$ denotes the image of the degeneracies in $A^i$, and $\Sigma^i$ denotes the $i$-fold reduced simplicial suspension functor.

The filtration by skeleta leads, as in [Boa99], to an unrolled exact couple

$$\begin{array}{ccccc} \longrightarrow \tilde{H}^{*,*}(|\,\mathrm{sk}_{-s+1}(A_\bullet)|;R) & & \longrightarrow & & \tilde{H}^{*,*}(|\,\mathrm{sk}_{-s}(A_\bullet)|;R) \longrightarrow \\ & \nwarrow & & \swarrow & \\ & & \tilde{H}^{*,*}(B_{-s+1};R) & & \end{array}$$

We note that $\tilde{H}^{*,*}(|\,\mathrm{sk}_{-s}(A_\bullet)|;R) = 0$ for $s \gg 0$, so that

$$\lim_s \tilde{H}^{*,*}(|\,\mathrm{sk}_{-s}(A_\bullet)|;R) = \mathrm{R}\lim_s \tilde{H}^{*,*}(|\,\mathrm{sk}_{-s}(A_\bullet)|;R) = 0$$

and in the terminology of that paper, we immediately have conditional convergence of the spectral sequence to $\lim_{s\to-\infty}\tilde{H}^{*,*}(|\operatorname{sk}_{-s}(A_\bullet)|;R)$. We show that in fact we have strong convergence to $\tilde{H}^{*,*}(|A_\bullet|;R)$.

Write $\iota_n$ for the map $\iota_n : |\operatorname{sk}_{-s}(A_\bullet)| \to |\operatorname{sk}_{-s+1}(A_\bullet)|$. Since $\iota_n$ is a cofibration for all $n$, one has

$$|A_\bullet| = \operatorname*{colim}_s |A_s| = \operatorname*{hocolim}_s |A_s| = \operatorname{tel}|A_s|$$

where tel denotes the mapping telescope construction. A standard argument, as in [May99, Chapter 19.4], gives

$$\tilde{H}^{*,*}(|A_\bullet|;R) \cong \lim_s \tilde{H}^{*,*}(|\operatorname{sk}_{-s}(A_\bullet)|;R)$$

contingent on the vanishing of the Rlim-term in

$$\begin{array}{l} 0 \longrightarrow \lim_s \tilde{H}^{*,*}(|\operatorname{sk}_{-s}(A_\bullet)|;R) \longrightarrow \\ \prod_{s=0}^{\infty}\tilde{H}^{*,*}(|\operatorname{sk}_s(A_\bullet)|;R) \xrightarrow{id-\iota^*} \prod_{s=0}^{\infty}\tilde{H}^{*,*}(|\operatorname{sk}_s(A_\bullet)|;R) \longrightarrow \\ \operatorname{Rlim}_s \tilde{H}^{*,*}(|\operatorname{sk}_{-s}(A_\bullet)|;R) \longrightarrow 0 \end{array}$$

By restricting to a particular grading $\tilde{H}^{p,*}(\cdot;R)$, we obtain

$$\begin{array}{l} 0 \longrightarrow \lim_s \tilde{H}^{p,*}(|\operatorname{sk}_{-s}(A_\bullet)|;R) \longrightarrow \\ \prod_{s=0}^{\infty}\tilde{H}^{p,*}(|\operatorname{sk}_s(A_\bullet)|;R) \xrightarrow{id-\iota^*} \prod_{s=0}^{\infty}\tilde{H}^{p,*}(|\operatorname{sk}_s(A_\bullet)|;R) \longrightarrow \\ \operatorname{Rlim}_s \tilde{H}^{p,*}(|\operatorname{sk}_{-s}(A_\bullet)|;R) \longrightarrow 0 \end{array}$$

but since $\iota_i^* : \tilde{H}^{p,*}(|\operatorname{sk}_i(A_\bullet)|;R) \to \tilde{H}^{p,*}(|\operatorname{sk}_{i-1}(A_\bullet)|;R)$ is an isomorphism when $i > p$, it follows that the derived limit $\operatorname{Rlim}_s \tilde{H}^{p,*}(|\operatorname{sk}_{-s}(A_\bullet)|;R)$ vanishes. It follows that

$$\tilde{H}^{*,*}(|A_\bullet|;R) \cong \lim_s \tilde{H}^{*,*}(|\operatorname{sk}_{-s}(A_\bullet)|;R)$$

as required.

It is easily seen that the convergence is strong, since, for example, any particular $E_r^{p,q}$

admits only finitely many nonvanishing differentials.

To identify the $E_1$-page of the spectral sequence, we employ [Seg68, proposition 5.1]. Although that paper deals with (semi-)simplicial topological spaces, and we trade in simplicial simplicial sheaves, the portion of proposition 5.1 that deals with identification of the $E_1$ and $E_2$ pages holds without major modification. One verifies this by first verifying that the proposition holds for bisimplicial sets, and then by promoting to the setting of bisimplicial sheaves by arguing at points.

Also by arguing at points, it is also possible to recast lemma 5.4 of loc. cit. in the category of simplicial *Spaces*. That is to say, there are diagonal maps $|A_\bullet| \to |A_\bullet| \times |A_\bullet|$ and $|A_\bullet| \to |A_\bullet \times A_\bullet|$, the former being the standard diagonal of *Spaces* the latter being the diagonal map of bisimplicial sheaves, composed with a realization functor. There is a natural map $|A_\bullet \times A_\bullet| \to |A_\bullet| \times |A_\bullet|$, which is a weak equivalence at all points by loc. cit. and therefore a weak equivalence in *Spaces*.

The diagonal map $A_\bullet \to A_\bullet \times A_\bullet$ is simplicial and so respects the simplicial filtration. Combined with an Alexander-Whitney map, we obtain in a standard way a product pairing $H^{*,*}(A_*) \otimes H^{*,*}(A_*) \to H(A_*)$, and so a product on the $E_1$-page and all subsequent pages of the spectral sequence. With respect to this product, the differentials are derivations in the usual way, and the product, by construction, converges to the product induced by the diagonal $A_\bullet \to A_\bullet \times A_\bullet$, which is the standard product induced by the diagonal map on $|A_\bullet|$, by our previous observation. □

## 2.4 The Bar Construction

Suppose $G$ is a group object in *Spaces* and $X, Y \in$ *Spaces* admit left- and right- actions by $G$ respectively. We can form the two-sided bar construction $B(X, G, Y)_\bullet$. It is a simplicial object in *Spaces*, i.e. a bisimplicial sheaf, but we suppress the simplicial indexes arising from the intrinsic structure of objects in *Spaces*. One has

$$B(X, G, Y)_n = X \times \overbrace{G \times \cdots \times G}^{n\text{-times}} \times Y$$

Bar constructions exhibit a wealth of desirable properties in algebraic topology [May75], and some of these results also hold in the context of $\mathbb{A}^1$-homotopy theory. We shall frequently prefer to work with an object of *Spaces*, rather than a simplicial object, so we adopt the notational convention that $B(X, G, Y) = |B(X, G, Y)_\bullet|$.

**Proposition 27.** *The constructions $B(X, G, Y)_\bullet$, $B(X, GY)$ are natural in all three variables, in the sense that, if $G \to G'$ is a homomorphism of group-objects, and if $X$, $Y$ are right- and left-$G$-spaces, and $X'$ and $Y'$ are right- and left-$G'$-spaces, such that there are maps $X \to X'$ and $Y \to Y'$ of right- and left-$G'$-spaces, then there is a map $B(X, G, Y)_\bullet \to B(X', G', Y')_\bullet$, and similarly for the realization. Moreover, if each of the maps in question is an $\mathbb{A}^1$-weak-equivalence (resp. a simplicial weak equivalence), then the map $B(X, G, Y) \to B(X', G', Y')$ is an $\mathbb{A}^1$-weak-equivalence (resp. a simplicial weak equivalence), and a similar result holds for the realization.*

*Proof.* The proof is entirely straightforward for bisimplicial objects, and the passage to realizations is immediately effected by proposition 24. □

We remark also that for any $U \in Sm/k$, and $X, G, Y$ as before, one has the identity

$$B(X, G, Y)(U) = B(X(U), G(U), Y(U))$$

Define the *(simplicial) G-Borel construction* on $Y$ to be the object $B(\mathrm{pt}, G, Y)$ and the *(simplicial) equivariant motivic cohomology* of $X$ to be the represented cohomology of $B(\mathrm{pt}, G, Y)$,

$$H^{n,i}_G(X; R) = [B(\mathrm{pt}, G, Y), \mathbb{H}R(n, i)]_{\mathbb{A}^1}$$

Which is different from the construction of [EG98], even for a smooth scheme $X$. This difference is responsible for our use of the term 'simplicial' to modify our terms, the point being that for our construction, very little of the underlying algebraic geometry is involved, and so we obtain a construction that is much more simplicial in character than the 'geometric' étale construction of [EG98] or [MV99].

Of course, everything goes through equally well if the $G$-action is on the right, and we obtain the construction $B(X, G, \mathrm{pt})$. We understand all subsequent results in this chapter as asserting also the equivalent result for such an action *mutatis mutandis*.

We remark that our construction is functorial in both $Y$ and $G$. Indeed, any other extraordinary cohomology theory represented on the motivic homotopy category admits of such a definition. If such a theory is graded and bounded below, so that $E^p(X) = 0$ for all $p$ sufficiently negative and all $X$, then the arguments concerning convergence in 26 still apply, and so the obvious spectral sequence converges strongly.

The utility in the construction we have given lies in its computability, at least in easy cases, the nature of which we outline below.

**Proposition 28.** *Suppose $G$ is a group object in Spaces acting on the left on $Y \in$ Spaces. There is a convergent spectral sequence of algebras*

$$E_1^{p,q} = H^{q,*}(B(\mathrm{pt}, G, Y)_p; R) \Rightarrow H^{p+q,*}(B(\mathrm{pt}, G, Y); R)$$

*which is natural in both $G$ and $Y$, in that a map $(G, Y) \to (G', Y')$ induces a map of spectral sequences.*

*Proof.* We simply apply proposition 26 to the case of $B(\mathrm{pt}, G, Y)$. □

We suppose a group scheme $G$ is motivically cellular in the sense of [DI05]– this might not be strictly necessary, it would suffice to know a Künneth spectral sequence

$$\mathrm{Tor}_{H^{*,*}(\mathbb{M};R)}(H^{*,*}(G; R), H^{*,*}(X; R)) \Longrightarrow H^{*,*}(G \times X; R)$$

obtains for all $X$— and that the motivic cohomology $H^{*,*}(G; R)$ is a free $\mathbb{M}_{\mathbb{R}}$-module of finite rank. We call such group scheme *benign*. Examples of benign groups include finite groups, $\mathrm{GL}(n)$ (for which see the previous chapter), $\mathrm{SL}(n)$ (a case we never use) and finite products of benign groups. For any $\mathbb{M}_R$-module, $N$, we use the notation $\hat{N}$ for $\mathrm{Hom}_{\mathbb{M}_R}(N, \mathbb{M}_R)$. For a free $\mathbb{M}_R$-module of finite rank, one has $\hat{\hat{N}} = N$.

We remark that for two modules, $N_1$, $N_2$, there is a natural map $\hat{N}_1 \otimes_{\mathbb{M}_R} \hat{N}_2 \to (N_1 \otimes_{\mathbb{M}_R} N_2)\hat{}$, which is an isomorphism when both modules are finitely generated and free.

We fix a benign group scheme $G$, and write $S = H^{*,*}(G; R)$. The module $\hat{S}$ is in fact a ring, due to the Hopf-algebra structure on $S$. If $G$ acts on a motivic space $Y$ on the left, then the action map $G \times Y \to Y$, along with compatibility diagrams, imbues $H^{*,*}(Y; R)$ with an $H^{*,*}(G; R)$ comodule structure. Alternatively, the dual of $H^{*,*}(Y; R)$ is a module over the dual of $H^{*,*}(G; R)$.

We shall need the notion of *relative* Ext-groups for the $\mathbb{M}$-algebra $\hat{H}^{*,*}(G)$. These can be defined via a bar construction as follows. Fix a (graded) commutative ring $\mathbb{M}$ and an $\mathbb{M}$-algebra $S$. Let $N, M$ be $S$-modules, then we can form the bar complex $\perp_* N$ whose $p$-th term is $N \otimes_{\mathbb{M}} R^{\otimes p}$, then

$$\mathrm{Ext}^*_{S/\mathbb{M}}(N, M) = H_*(\mathrm{Hom}_R(\perp_* N, M))$$

For a fuller definition and some properties of such groups we refer to [ML63]. [Wei94, Chapter 8].

We cull the following results from [Wei94], although strictly speaking the dual case of relative Tor is proved there, and that for the case case of a commutative base ring. Neither the passage from Tor to Ext nor from commutative to graded-commutative ground-ring, $\mathbb{M}$, presents any difficulty.

**Proposition 29.** *Let $R$ be a k-algebra, and let*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

*be a short exact sequence of R-modules that is split when considered as a sequence of k-modules. Then there is a long exact sequence of groups*

$$\longrightarrow \mathrm{Ext}^*_{R/k}(M_3, N) \longrightarrow \mathrm{Ext}_{R/k}(M_2, N) \longrightarrow \mathrm{Ext}_{R/k}(M_1, N) \xrightarrow{\partial} \mathrm{Ext}^{*+1}_{R/k}(M_3, N) \longrightarrow$$

**Proposition 30.** *Let $R_1$, $R_2$ be k-algebras, with k graded-commutative and let $M_i, N_i$ be $R_i$-modules for $i = 1, 2$. Then $M_1 \otimes_k M_2$, $N_1 \otimes_k N_2$ are $R_1 \otimes_k R_2$-modules, there is an external* *product*

$$\mathrm{Ext}_{R_1/k}(M_1, N_1) \otimes_k \mathrm{Ext}_{R_2/k}(M_2, N_2) \to \mathrm{Ext}_{R_1 \otimes_k R_2/k}(M_1 \otimes_k M_2, N_1 \otimes_k N_2)$$

*which is natural in all four variables and commutes with the connecting homomorphism of proposition 29.*

We remark in passing that the product arises from a standard Alexander-Whitney construction on $\perp_* M_1 \otimes \perp_* M_2$.

In general we shall be dealing with group actions in the category of *Spaces*, which is to say a group object $G$, an object $Y$, and a map $G \times Y \to Y$.

In the general situation we encounter, the ring $\hat{S} = \hat{H}^{*,*}(G)$ is in fact a Hopf algebra over $\mathbb{M}$, so there is an algebra homomorphism $\hat{S} \to \hat{S} \otimes_{\mathbb{M}} \hat{S}$. Write $\hat{N}$ for $\hat{H}^{*,*}(Y)$, since we shall be treating of $\mathrm{Ext}_{\hat{S}}(\hat{N}, \mathbb{M})$, and there is a coalgebra map

$$\hat{N} \to \hat{N} \otimes_{\mathbb{M}} \hat{N}$$

arising from the diagonal $Y \to Y \times Y$. The upshot is that for such data, there is a product

$$\begin{array}{ccc} \mathrm{Ext}_{\hat{S}/\mathbb{M}}(\hat{N}, \mathbb{M}) \otimes_{\mathbb{M}} \mathrm{Ext}_{\hat{S}/\mathbb{M}}(\hat{N}, \mathbb{M}) & \longrightarrow & \mathrm{Ext}_{\hat{S}\otimes\hat{S}/\mathbb{M}}(\hat{N} \otimes_{\mathbb{M}} \hat{N}, \mathbb{M} \otimes_{\mathbb{M}} \mathbb{M}) \\ & & \downarrow \\ & & \mathrm{Ext}_{\hat{S}/\mathbb{M}}(\hat{N} \otimes_{\mathbb{M}} \hat{N}, \mathbb{M} \otimes_{\mathbb{M}} \mathbb{M}) \\ & & \downarrow \\ & & \mathrm{Ext}_{\hat{S}/\mathbb{M}}(\hat{N}, \mathbb{M}) \end{array}$$

where the first map is the external product, and the other maps are those arising from the functoriality of $\mathrm{Ext}_{\hat{S}/\mathbb{M}}(\hat{N}, M)$.

We synopsize

**Proposition 31.** *Suppose $\hat{S}$ is a Hopf algebra over $\mathbb{M}$, $\hat{N}$ is a coalgebra over $\hat{S}$, then there is a ring structure on* $\mathrm{Ext}_{\hat{S}/\mathbb{M}}(\hat{N}, \mathbb{M})$

We shall not have use for the following observation until the next chapter, but it seems to be of a kind with the other results in this section.

**Proposition 32.** *Let $\hat{S}$ be a Hopf algebra over $\mathbb{M}$, let*

$$0 \longrightarrow \hat{N}_1 \longrightarrow \hat{N}_2 \longrightarrow \hat{N}_3 \longrightarrow 0$$

*be a short exact sequence of $\hat{S}$-modules that splits as a sequence of $\mathbb{M}$-modules. Then the long exact sequence of relative* Ext*-groups*

$$\begin{array}{ccc} \mathrm{Ext}_{\hat{S}/\mathbb{M}}(\hat{N}_3, \mathbb{M}) & \longrightarrow & \mathrm{Ext}_{\hat{S}/\mathbb{M}}(\hat{N}_2, \mathbb{M}) \\ {\scriptstyle \partial}\nwarrow & & \swarrow \\ & \mathrm{Ext}_{\hat{S}/\mathbb{M}}(\hat{N}_1, \mathbb{M}) & \end{array}$$

*is in fact a long exact sequence of $\mathrm{Ext}_{\hat{S}/\mathbb{M}}(\mathbb{M}, \mathbb{M})$-modules.*

*Proof.* There is a (trivial) map $\hat{N}_i \to \mathbb{M} \otimes_{\mathbb{M}} \hat{N}_i$. Both $\hat{N}_i$ and $\mathbb{M}$ are $\hat{S}$-modules, and consequently we can use the external product on relative Ext as before to obtain a product $\mathrm{Ext}_{\hat{S}/\mathbb{M}}(\mathbb{M}, \mathbb{M}) \otimes_{\mathbb{M}} \mathrm{Ext}_{\hat{S}/\mathbb{M}}(\hat{N}_i, \mathbb{M})$. By proposition 30 the $\mathrm{Ext}_{\hat{S}/\mathbb{M}}(\mathbb{M}, \mathbb{M})$-action is compatible with the long exact sequence of proposition 29. □

We shall need one other observation before proving the main theorem

**Proposition 33.** *Suppose, in addition to the hypotheses of proposition 31, $\hat{N}$ is a free $\mathbb{M}$-module, then $\mathrm{Ext}_{\hat{S}/\mathbb{M}}(\hat{N}, \mathbb{M}) = \mathrm{Ext}_{\hat{S}}(\hat{N}, \mathbb{M})$,*

*Proof.* In this case, the complex $\perp_* \hat{N}$ is in fact a free $R$-resolution of $\hat{N}$, see [Wei94, chapter 8.5]. □

**Theorem 34.** *Let $G$ be a benign group scheme, and let $Y \in Spaces$ be a motivic space on which $G$ acts on the left. Suppose $N = H^{*,*}(Y; R)$ is a free $\mathbb{M}$-module and write $S$ for the Hopf algebra $H^{*,*}(G; R)$. There is a convergent spectral sequence*

$$E_2^{p,q} = \mathrm{Ext}_{\hat{S}}^{p,q}(\hat{N}, \mathbb{M}_R) \Longrightarrow H^{p+q}(B(\mathrm{pt}, G, Y); R)$$

*which is natural in both $Y$ and $G$, in the sense that if $\phi : G \to G'$ is a group homomorphism, and $f : Y \to Y'$ is a map from a $G$-space to a $G'$-space so that the following diagram commutes*

$$\begin{array}{ccc} G \times Y & \longrightarrow & G' \times Y' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

*then there is a map of spectral sequences*

$$\begin{array}{ccc} E_2^{p,q} = \mathrm{Ext}_{\hat{S}'}^{p,q}(\hat{N}', \mathbb{M}_R) & \Longrightarrow & H^{p+q}(B(\mathrm{pt}, G', Y'); R) \\ \downarrow & & \downarrow \\ E_2^{p,q} = \mathrm{Ext}_{\hat{S}}^{p,q}(\hat{N}, R) & \Longrightarrow & H^{p+q}(B(\mathrm{pt}, G, Y); R) \end{array}$$

*Moreover, there is a product structure on this spectral sequence, which is given on the $E_2$-page by the natural product structure on $\mathrm{Ext}_{\hat{S}/\mathbb{M}}(\hat{N}, \mathbb{M}_R)$ constructed above, and which converges to the ordinary product on $H^{*,*}(|B(\mathrm{pt}, G, Y)_\bullet|; R)$. This product is also functorial in $G, Y$.*

*Proof.* Since $H^{*,*}(G; R)$ is a free $\mathbb{M}_R$-module, and since $G$ is motivically cellular, a Künneth isomorphism holds

$$H^{*,*}(B(\mathrm{pt}, G, Y)_p; R) = H^{*,*}(G; R)^{\otimes p} \otimes_{\mathbb{M}_R} H^{*,*}(Y; R) = S^{\otimes p} \otimes_{\mathbb{M}_R} N$$

The face maps are

$$\delta_i : \overbrace{G \times G \times \cdots \times G}^{p+1} \times Y \to \overbrace{G \times \cdots \times G}^{p} \times Y$$

where $\delta_0$ is projection onto the last $p+1$ spaces, $\delta_i$ for $1 \leq i \leq p-1$ restricts to the multiplication map $G \times G \to G$ on the $i-1$- and $i$-th copies of $G$, and is the identity elsewhere. The last face-map $\delta_p$ restricts to the map $G \times Y \to Y$ on the last $G$, and to the identity elsewhere. Functoriality of the Künneth isomorphism means that the induced maps on cohomology restrict to $\mathbb{M}_R \to S$, the comultiplication $\Delta : S \to S \otimes_{\mathbb{M}_R} S$, and the coaction map $\Gamma : N \to S \otimes_{\mathbb{M}_R} N$.

Elements in $H^{*,*}(B(\mathrm{pt}, G, Y)_p; R)$ may be written as sums of elements of the form $g_1 \otimes g_2 \otimes \ldots \otimes g_p \otimes y$. The $d_1$ differential can be written explicitly in terms of these elements, see proposition 26, and the comultiplication maps.

$$\begin{aligned} d_1(g_1 \otimes g_2 \otimes \ldots \otimes g_p \otimes y) = 1 \otimes g_1 \otimes g_2 \otimes \ldots \otimes g_p \otimes y + \\ \sum_{i=1}^{p} (-1)^1 g_1 \otimes \ldots \otimes \Delta(g_i) \otimes \ldots \otimes g_p \otimes y + \\ g_1 \otimes g_2 \otimes \ldots \otimes g_p \otimes \Gamma(y) \end{aligned}$$

We now consider the complex of left $S$-modules whose $p$-th term is

$$\overbrace{\hat{S} \otimes_{\mathbb{M}_R} \hat{S} \otimes_{\mathbb{M}_R} \ldots \otimes_{\mathbb{M}_R} \hat{S}}^{p+1} \otimes_{\mathbb{M}_R} \hat{N}$$

and in which the differentials are given as

$$d_1(s_0 \otimes \ldots \otimes s_p \otimes n) = \sum_{i=0}^{p-1} s_0 \otimes \ldots \otimes s_i s_{i+1} \otimes \ldots \otimes s_p \otimes n + (-1)^p s_0 \otimes \ldots \otimes s_p n$$

This is the usual bar resolution of $\hat{N}$ as a $\hat{S}$-module. It is well-known, see [Wei94, chapter 8], that the bar resolution is in fact a resolution of $\hat{N}$, in this case by free $\hat{S}$-modules.

We observe that applying the functor $\mathrm{Hom}_{\hat{S}}(\cdot, \mathbb{M}_{\mathbb{R}})$ to the bar resolution gives exactly the $E_1$-page of the spectral sequence under consideration and its attendant differentials. It follows that the $E_2$-page is precisely $\mathrm{Ext}_{\hat{S}/\mathbb{M}}(\hat{N}, \mathbb{M}_R)$, as claimed.

We remark on the naturality of the sequence. In the first place, given a map of $G$-spaces, $Y \to Y'$, we have a map $B(\mathrm{pt}, G, Y) \to B(\mathrm{pt}, G, Y')$, so we have naturality in $Y$. In the second, suppose we have a group homomorphism $G' \to G$, and a space $Y$ with a $G$ action, then we have an induced $G'$ action on $Y$ and a map $B(\mathrm{pt}, G', Y) \to B(\mathrm{pt}, G, Y)$. In both cases we obtain maps of spectral sequences, so we have naturality in both $G$ and $Y$.

We remark also that the maps on $E_2$-pages obtained in this way are exactly the maps one obtains by functoriality of Ext in the ring $\hat{S}$ and in the module $\hat{N}$.

One has a diagonal map $B(\mathrm{pt}, G, Y) \to B(\mathrm{pt}, G, Y) \times B(\mathrm{pt}, G, Y)$, and the diagonal on $B(\mathrm{pt}, G, Y)$ may be replaced by a weakly equivalent map so as to be compatible with this filtration, as in [Seg68, Lemma 5.4], so the map is induced by the usual coproduct on $B(\mathrm{pt}, G, Y)_\bullet$, that given by

$$B(\mathrm{pt}, G, Y)_n \xrightarrow{\text{diag}} B(\mathrm{pt}, G, Y)_n \times B(\mathrm{pt}, G, Y)_n \xrightarrow{f_{pq}} B(\mathrm{pt}, G, Y_n)_p \times B(\mathrm{pt}, G, Y_q)$$

where $f_{pq}$ is the Alexander-Whitney map. After passage to cohomology, one takes the sum over all possible $f_{pq}^*$ with $p + q = n$.

The resulting product on the $E_1$-page coincides with that of proposition 31, since both are obtained by the Alexander-Whitney map applied to the same bisimplicial object.

The functoriality of the product is entirely routine, since both the diagonal map

$$B(\mathrm{pt}, G, Y) \to B(\mathrm{pt}, G, Y) \times B(\mathrm{pt}, G, Y)$$

and the Alexander-Whitney maps are natural in all arguments. □

For the most part now we devote ourselves to understanding means to compute this spectral sequence or its abutment. Our first result towards this end is the following, which will allow us to compute $H^{*,*}(B(\mathrm{pt}, G, Y); R)$ by decomposition of $Y$.

**Proposition 35.** *Let G be a group object in Spaces. Let I be a small category and let $F : I \to$ Spaces be a diagram in which all objects $F(i)$ are equipped with a G-action and such that the morphisms $F(i \to j)$ are G-equivariant. Then there is a weak equivalence* $\operatorname{hocolim} |B(\mathrm{pt}, G, F(I))_\bullet| \simeq_s |B(\mathrm{pt}, G, \operatorname{hocolim} F(I))_\bullet|$

*Proof.* There is a homeomorphism

$$B(\mathrm{pt}, G, \operatorname{hocolim} F(i))_n \cong \operatorname{hocolim} B(\mathrm{pt}, G, F(i))_n$$

as a result we have

$$\operatorname{hocolim} |B(\mathrm{pt}, G, F(i))_\bullet| \simeq_s |\operatorname{hocolim} B(\mathrm{pt}, G, F(i))_\bullet| \simeq_s |B(\mathrm{pt}, G, \operatorname{hocolim} F(i))_\bullet|$$

The first weak equivalence being an application of corollary 23.1. □

## 2.5 The Case of a Free Action

When the action of $G$ on $Y$ is free, one might hope that the simplicial Borel construction and the homotopy type of the scheme-quotient agree. This is the case when a scheme-quotient exists and is particularly well-behaved, the condition being essentially that we avoid complications arising from a $\pi_1^{\acute{e}t}$ action. We first treat of the easiest possible case, that of a product.

**Lemma 36.** *Let $G$ be a group object in Spaces and let $X \in$ Spaces. Then $G \times X$ is a left $G$-space, and one has a map $\varepsilon : |B(\mathrm{pt}, G, G \times X)_\bullet| \xrightarrow{\sim} X$. This is natural in the sense that if $\phi : X \to X'$ is a map of motivic spaces, and if $\tilde{\phi} : G \times X \to G \times X'$ is a map of $G$-spaces lying over $\phi$, then the following diagram commutes*

$$\begin{array}{ccc} |B(\mathrm{pt}, G, X)_\bullet| & \longrightarrow & |B(\mathrm{pt}, G, X')_\bullet| \\ \downarrow \simeq & & \downarrow \simeq \\ X & \xrightarrow{\phi} & X' \end{array}$$

*Proof.* For any $U \in Sm/k$, one has the following weak equivalence of simplicial sets $|B(\mathrm{pt}(U), G(U), (G \times X)(U))_\bullet| \xrightarrow{\sim} (G \times X)(U)/G(U)$, see [May75, Chapter 8]. This map also has the required naturality. Since this holds at any $U$, it holds at points and there is a weak equivalence of simplicial presheaves

$$\begin{array}{ccc} |B(\mathrm{pt}, G, G \times X)_\bullet| & \xrightarrow{\simeq} & (G \times X)/G \\ \downarrow & & \downarrow \\ |B(\mathrm{pt}, G, G \times X')_\bullet| & \xrightarrow{\simeq} & (G \times X')/G \end{array}$$

There is an evident isomorphism of presheaves $(G \times X)/G \cong X$, so the result, including the naturality assertion, follows. $\square$

The main result of this section is to extend the above to nontrivial $G$-bundles. We are trying to imitate the following fact, true in the context of simplicial sets

**Proposition 37.** *Let $X$ be a simplicial set, and $G$ a simplicial group, acting freely on $X$. Then there is a map $B(\mathrm{pt}, G, X) \to X/G$ which is a weak equivalence. This map is natural, in that $(G, X) \to (G', X')$ induces a diagram*

$$\begin{array}{ccc} B(\mathrm{pt}, G, X) & \longrightarrow & X/G \\ \downarrow & & \downarrow \\ B(\mathrm{pt}, G', X') & \longrightarrow & X'/G' \end{array}$$

See [May75, Chapter 8] for the proof. It is not possible to translate this directly into the language of schemes, since there are always difficulties with the formation of group quotients in that category. We will eventually restrict our attention therefore to a particularly well-behaved case, in which the obvious definitions of homogeneous space coincide.

We specify that if $G$ is a group object in *Spaces* and if $X$ is an object in *Spaces* on which $G$ acts, then the notation $X/G$ is to mean the 'orbit sheaf' in *Spaces*. This is the quotient sheaf associated to the presheaf

$$U \longmapsto \frac{X(U) \times X(U)}{X(U) \times G(U)}$$

the latter quotient being of course that of simplicial sets.

Even when $X$, $G$ are represented by schemes, we shall never write $X/G$ for any other quotient than this sheaf-theoretic quotient.

We will need the following sheaf-theoretic lemma

**Lemma 38.** *Let : $\phi : A \to B$ be a map in Spaces, then there are two maps $A \times_B A \to A$, being the projection on the first and second factor respectively. Suppose $A \to B$ is a sheaf epimorphism, then the following*

$$A \times_B A \rightrightarrows A \longrightarrow B$$

*is a coequalizer diagram.*

*Proof.* We appeal first to the case of sets. Suppose given a surjection of sets $f : A \to B$, then $A \times_B A$ is the set of pairs $(a, a')$ with $f(a) = f(a')$. A model for the coequalizer $C\pi$ of the projection maps $\pi_i : A \times_B A \to A$ is the quotient of $A$ by the relation $a \sim a'$ if $f(a) = f(a')$. It is clear that $f : A \to B$ factors through $C\pi \to B$, and also that the latter map is injective, since $f(a) = f(a')$ implies that $a$, $a'$ represent the same element in $C\pi$. Since $f$ is surjective also, we see that $f : A \to B$ is a coequalizer as claimed.

We remark also that this construction is natural in the sense that if

$$\begin{array}{ccc} A & \twoheadrightarrow & B \\ \downarrow & & \downarrow \\ A' & \twoheadrightarrow & B' \end{array}$$

is a commutative square of sets and the horizontal arrows are set surjections, then the diagram

$$\begin{array}{ccccc} A \times_B A & \rightrightarrows & A & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ A' \times_{B'} A' & \rightrightarrows & A' & \longrightarrow & B' \end{array}$$

is a map of coequalizer diagrams.

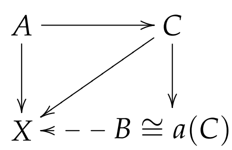
It is easy to promote this result to a category of simplicial presheaves of sets, since objects of this category are diagrams in the category of sets, where the fiber product and coequalizers are constructed objectwise, and the epimorphisms are detected objectwise.

The forgetful functor from presheaves to sheaves has a left-adjoint, the associated sheaf functor, which we denote $a(\cdot)$. As a result, the formation of limits of sheaves commutes with the forgetful functor, that is to say the limit agrees with the limit of presheaves.

In particular, suppose we are given an epimorphism in *Spaces* (an epimorphism of simplicial sheaves) $f : A \to B$. We form the simplicial presheaf $C$ where $C(U) = \mathrm{im}(f(U) : A(U) \to B(U))$. Note first that $A \times_B A = A \times_C A$, since $(A \times_B A)(U) = A(U) \times_{B(U)} A(U)$ by proposition 25, and this is simply the set of pairs $(a, a') \in A(U) \times A(U)$ such that

$f(a) = f(a')$. Secondly, to say $f : A \to B$ is an epimorphism is equivalent to asserting that the map $C \to B$ is the natural map from $C$ to its associated sheaf.

Suppose there is a map of simplicial sheaves $g : A \to X$ such that the compositions $g \circ \pi_i : A \times_B A \to X$ for $i = 1, 2$ agree. In particular, there is a unique map from the presheaf coequalizer, $C \to X$, making the upper triangle commute in




but since $X$ is a sheaf, there exists a unique map, represented by the dotted arrow above, making the lower triangle commute. It follows that $B$ is indeed the coequalizer of the two maps $A \times_B A \to A$ in the category of simplicial sheaves, as claimed.

□

**Proposition 39.** *Let $X$ be an object in Spaces, let $G$ be a group object in Spaces, and suppose $G$ acts freely on $X$ in the sense that $X \times G \to X \times X$ is a monomorphism. Then there is a weak equivalence $|B(\mathrm{pt}, G, X)| \simeq_s X/G$ in Spaces. This weak equivalence is natural in both $G$ and $X$.*

*Proof.* For any $U \in Sm/k$, we have $B(\mathrm{pt}, G, X)(U) = B(\mathrm{pt}, G(U), X(U))$, and so it follows that

$$B(\mathrm{pt}, G, X)(U) \simeq X(U)/G(U)$$

but $X(U)/G(U)$ is simply shorthand for the presheaf quotient $(X \times X)(U)/(X \times G)(U) = (X \times X/X \times G)(U)$, so $X(U)/G(U) = (X/G)^{\mathrm{pre}}(U)$, the latter being the presheaf quotient. It follows that $|B(\mathrm{pt}, G, X)| \simeq_s (X/G)^{\mathrm{pre}}$ in the model category of presheaves. By [Jar87], we know that $(X/G)^{\mathrm{pre}} \simeq_s X/G$, the latter being the Nisnevich quotient sheaf, and by composition of the two weak equivalences, the result follows. □

In [MFK94] a strong notion of principal bundle is defined. There, $\pi : Y \to X$ is a principal $G$-bundle under the assumption that $\pi : Y \to X$ is a geometric quotient of $Y$ by a $G$-action, $\pi$ is a flat morphism of finite type and the natural map $G \times Y \to X \times_Y X$ is an isomorphism. Of all these conditions, it is the last that has most utility for us.

Let $Y \to X$ be a map in $Sm/k$ and let $G$ be a group object in $Sm/k$ such that there is an action of $G$ on $Y$, denoted $\alpha : G \times Y \to Y$. Suppose further that the following square commutes

$$\begin{array}{ccc} G \times Y & \xrightarrow{\alpha} & Y \\ \downarrow{\scriptstyle \pi_2} & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

Under these assumptions there exists a natural map $\Psi : G \times Y \to Y \times_X Y$. If $\Psi$ is an isomorphism, then we say that $Y \to X$ is a *principal $G$-pseudobundle*. The data of a pseudobundle, although not the conditions, are encapsulated in a diagram $G \times Y \to Y \to X$.

Naturally, the state of being a principal $G$-bundle as defined in loc. cit is a special case of being a principal $G$-pseudobundle.

A map of pseudobundles is defined to be a commutative diagram

$$\begin{array}{ccccc} G \times Y & \longrightarrow & Y & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ G' \times Y' & \longrightarrow & Y' & \longrightarrow & X' \end{array}$$

**Proposition 40.** *If $G \times Y \to Y \to X$ is a principal $G$-pseudobundle, then the natural map $G \times Y \to Y \times Y$ is a sheaf monomorphism, in particular the action of $G$ on $Y$ is sheaf-theoretically free.*

*Proof.* Since $G \times Y$ is by hypothesis equivalent to $Y \times_X Y$, it suffices to prove that $Y \times_X Y \to Y \times Y$ is a sheaf monomorphism. We test against $U \in Sm/k$: $(Y \times_X Y)(U)$ is equivalent to a pair of maps $U \rightrightarrows Y$ so that the compositions $U \rightrightarrows Y \to X$ coincide. In particular this is a subset of the set of pairs of maps $U \rightrightarrows Y$ without restriction, which is $(Y \times Y)(U)$, so $Y \times_X Y \to Y \times Y$ is indeed a monomorphism. □

One font of psuedobundles is the following. If $G \in Sm/k$ is a group object, and $G$ acts on $X \in Sm/k$, and if $X$ has a $k$-point $x_0 : \mathrm{pt} \to X$, then there is a map $G \times \mathrm{pt} \to G \times X \to X$.

In the composition, we denote this by $f : G \to X$. Let $H$ be the pull-back as given below

$$\begin{array}{ccc} H & \longrightarrow & G \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{pt} & \longrightarrow & X \end{array}$$

then $H$ is called the *stabilizer* of $x_0$ in $G$. It is easily shown (by arguing at $R$-points) that $H$ is a subgroup object of $G$. For any $k$-algebra $R$, the group $H(R)$ is exactly the group-theoretic stabilizer of the $R$-point of $X$ given by $\operatorname{Spec} R \to \operatorname{Spec} k \xrightarrow{x_0} X$, which we denote $x_0$ by an abuse of notation.

As a result, there is a group action (on the right) of $H$ on $G$ given by multiplication $G \times H \to G$.

**Proposition 41.** *With notation as above, $G \times H \to G \to X$ is a principal $H$-pseudobundle.*

*Proof.* Let $R$ be a $k$-algebra. Let $(g, h)$ be an $R$-point of $G \times H$, and consider the two maps $G \times H \to G$. These take $(g, h)$ to $g$ and $gh$ respectively. The image of these in $X(R)$ is the same in either case, being $g(x_0)$. In particular, since it holds for $R$-points for any $R$, the square

$$\begin{array}{ccc} G \times H & \longrightarrow & G \\ \downarrow & & \downarrow \\ G & \longrightarrow & X \end{array}$$

commutes.

Now we consider $G \times H \to G \times_X G$. The map is given on $R$-points by $(g, h) \mapsto (g, gh)$, which is plainly injective. Suppose $(g, g')$ is a pair in $(G \times_X G)(R)$, which is to say $g(x_0) = g'(x_0)$, then $g^{-1}g' \in H(R)$, but observe that $(g, g^{-1}g') \in (G \times H)(R)$ maps to $(g, g')$, so the map is also surjective. Since the map of points is an isomorphism for all $R$ the map of schemes is also an isomorphism as claimed. □

**Proposition 42.** *Let $X$ be a scheme, and let $c : U \to X$ be a Nisnevich cover of $X$. Let $G$ be a group scheme, and let $\pi : Y \to X$ be a principal $G$-pseudobundle over $X$. Suppose further that* *there exists a Nisnevich cover $c : U \to X$, so that one has a section, $s$, in*

$$\begin{array}{ccc} \pi^{-1}(U) & \longrightarrow & Y \\ {\scriptstyle s}\uparrow\downarrow & & \downarrow{\scriptstyle \pi} \\ U & \longrightarrow & X \end{array} \tag{2.2}$$

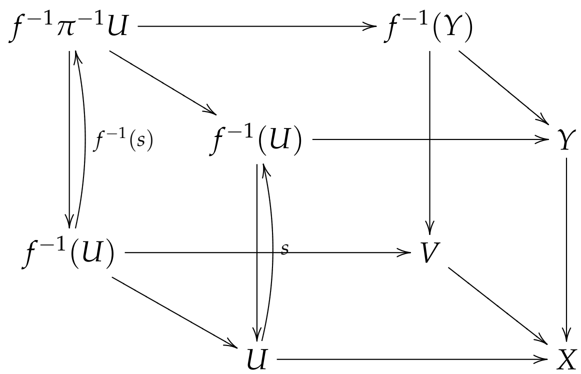
*There are weak equivalences $|B(\mathrm{pt}, G, Y)_\bullet| \simeq Y/G \simeq X$, which are natural in $Y \to X$, in the sense that if $(G, Y) \to (G', Y')$ is a map of pseudobundles in the sense defined above then there is a map $|B(\mathrm{pt}, G, Y)_\bullet| \to |B(\mathrm{pt}, G', Y')_\bullet|$ weakly equivalent to $X \to X'$.*

*Proof.* Since the action of $G$ on $Y$ is free, by proposition 40, there exists a natural equivalence of the sheaf quotient $Y/G \xrightarrow{\sim} |B(\mathrm{pt}, G, Y)|$. It will therefore suffice to exhibit a natural isomorphism of sheaves $X \simeq Y/G$.

The existence of local sections implies of $\pi : Y \to X$ that the map of represented sheaves which we also denote $\pi : Y \to X$ is a sheaf epimorphism, which we now prove. In particular, suppose $V \in Sm/k$, then $X(V) = Sm/k(V, X)$, and similarly for $Y(V)$. Let $f \in X(V)$, and consider the pullback of the diagram (2.2).

$$\begin{array}{ccc} f^{-1}\pi^{-1}(U) & \longrightarrow & f^{-1}(Y) \\ \downarrow & & \downarrow \\ f^{-1}(U) & \longrightarrow & V \end{array}$$

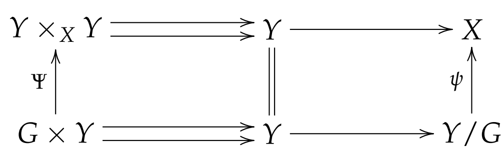
The section $s : U \to \pi^{-1}(U)$ pulls back, as can be seen from a diagram chase, and we obtain a section $f^{-1}(s)$ of the map $f^{-1}\pi^{-1}(U) \to f^{-1}(U)$. The following commutative cubical diagram encapsulates the situation




Given a map $f : V \to X$, we obtain a map $c^{-1}(f) : f^{-1}(U) \to V \to X$. There exists a map $f^{-1}(U) \to f^{-1}\pi^{-1}(U) \to \pi^{-1}(U) \to Y$, which when composed with $\pi : Y \to X$ gives $c^{-1}(f)$.

In particular, although the map $Y(V) \to X(V)$ may not be surjective, for a particular $f \in X(V)$, there is some cover $f^{-1}(U) \to V$ such that the image, $c^{-1}(f)$ of $f$ in $X(f^{-1}(U))$ is in the image of the map $Y(f^{-1}(U))$. Since by taking sufficient refinements, any $f \in X(V)$ is in the image of $Y \to X$, it follows $Y \to X$ is a sheaf epimorphism.

By assumption $\Psi : G \times Y \to Y \times_X Y$ is an isomorphism. Using lemma 38, we see that in the diagram




both sequences are coequalizer sequences, and by categorical uniqueness of coequalizers, the map $\psi : Y/G \to X$ is an isomorphism. The asserted naturality results all follow immediately by considering diagrams of coequalizer sequences, and are routine. $\square$

Recall that $H^{*,*}(\mathrm{pt}; R)$ is denoted $\mathbb{M}_R$ and that we write $\hat{N}$ for $\mathrm{Hom}_{\mathbb{M}_R}(N, \mathbb{M}_R)$. Combining the above with proposition 34 gives the following tidy result.

**Proposition 43.** *Let $G$ be a benign group scheme, and let $S = H^{*,*}(G; R)$ denote its motivic* *cohomology. Suppose we have a map of schemes, $\pi : Y \to X$, which makes $Y$ a principal $G$-pseudobundle over $X$. Suppose that the map $Y \to X$ admits Nisnevich-local sections. Suppose further that the motivic cohomology of $N$, denoted $H^{*,*}(Y; R)$ of $Y$ is free and finitely generated as an $\mathbb{M}_R$-module. There exists a spectral sequence of algebras*

$$E_2^{p,q} = \mathrm{Ext}_{\hat{S}}^{p,q}(\hat{N}, \mathbb{M}_R) \Longrightarrow H^{p+q}(X; R)$$

*and this spectral sequence is functorial in $(G, Y)$*

It is worth mentioning that the functoriality is exactly the usual functoriality of Ext.

In general, even if $\pi : Y \to X$ is a principal $G$-bundle in the sense of [MFK94], so that the diagram

$$G \times Y \rightrightarrows Y \longrightarrow X$$

is a coequalizer in the category of schemes (this is what it means for $X$ to be the categorical quotient of $Y$ by $G$) and $Y \to X$ is surjective as a map of schemes, $X$ still may not be the quotient $Y/G$ considered as Nisnevich sheaves— for an example see section 2.6. On the other hand, it should not surprise us that it suffices to check such a diagram Nisnevich-locally.

## 2.6 The Case of $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$

The map $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$ appears to be the minimal example not exhibiting the behavior described above. We describe two failures.

In this section we abbreviate $\mathbb{Z}/2$ as $k$. We take cohomology with $k$-coefficients tacitly throughout. In the first place, we know the motivic cohomology of $\operatorname{Spec} \mathbb{C}$, $\operatorname{Spec} \mathbb{R}$ with $k$-coefficients. They are, see [Voe03a][Parts 6 & 7] but also [DI09],

$$H^{*,*}(\operatorname{Spec} \mathbb{C}) \cong k[\tau], \qquad H^{*,*}(\operatorname{Spec} \mathbb{R}) \cong k[\tau, \rho], \qquad |\tau| = (0,1), \quad |\rho| = (1,1)$$

and the map $H^{*,*}(\operatorname{Spec} \mathbb{R}) \to H^{*,*}(\operatorname{Spec} \mathbb{C})$ is the evident quotient map.

One also has the pullback diagram

$$\begin{array}{ccc} \operatorname{Spec}\mathbb{C} \times \mathbb{Z}/2 & \longrightarrow & \operatorname{Spec}\mathbb{C} \\ \downarrow & & \downarrow \\ \operatorname{Spec}\mathbb{C} & \longrightarrow & \operatorname{Spec}\mathbb{R} \end{array} \tag{2.3}$$

The first thing to note is that $\operatorname{Spec}\mathbb{C} \times \mathbb{Z}/2 = \operatorname{Spec}\mathbb{C} \amalg \operatorname{Spec}\mathbb{C}$, so we can compute the motivic cohomology of this too.

We should like to have a spectral sequence

$$\operatorname{Tor}^{p,q}_{H^{*,*}(\operatorname{Spec}\mathbb{R})}(H^{*,*}(\operatorname{Spec}\mathbb{C}), H^{*,*}(\operatorname{Spec}\mathbb{C})) \Rightarrow H^{*,*}(\operatorname{Spec}\mathbb{C})^2$$

but it turns out that this does not hold. To prove this, consider a free resolution of $B = H^{*,*}(\operatorname{Spec}\mathbb{C})$ over $A = H^{*,*}(\operatorname{Spec}\mathbb{R})$. We have $B = A/(\rho)$, so a resolution can be constructed, as standard for a quotient of a polynomial ring by the ideal generated by an indeterminate

$$\begin{array}{ccccc} 0 & \longrightarrow & A(1,1) & \xrightarrow{1\mapsto\rho} & A \\ & & & & \downarrow \\ & & & & B \end{array}$$

which when we apply $\cdot \otimes_A B$ yields

$$0 \longrightarrow B(1,1) \xrightarrow{0} B \longrightarrow 0$$

And so we obtain an $E_2$-page consisting of two copies of $B$. In weight-0, however, we find only a single copy of $k$, rather than the two that convergence to $B^2$ would require.

In [DI05] it is proved that any product of $\mathbb{R}$-schemes either one of which is suitably "cellular" gives rise to a Künneth spectral sequence. We have proved

**Proposition 44.** *The scheme* $\operatorname{Spec}\mathbb{C}$ *is not stably cellular as a* $\operatorname{Spec}\mathbb{R}$*-scheme.*

This appears to be the simplest example of a non-cellular scheme.

There is a Galois action of $\mathbb{Z}/2$ on $\operatorname{Spec}\mathbb{C}$ over $\operatorname{Spec}\mathbb{R}$. By referring to [MFK94], we know that the map $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$ is a geometric quotient map. By reference to (2.3), we see that in fact this is a principal $\mathbb{Z}/2$-bundle, where $\mathbb{Z}/2$ is presented as $\operatorname{Spec}\mathbb{R} \amalg \operatorname{Spec}\mathbb{R}$.

There is a quotient as (simplicial) Nisnevich sheaves, which is by definition the coequalizer

$$\mathbb{Z}/2 \times \operatorname{Spec} C \rightrightarrows \operatorname{Spec}\mathbb{C} \longrightarrow A$$

From the statement that $\operatorname{Spec}\mathbb{C} \to A$ is a Nisnevich sheaf epimorphism we conclude that $\operatorname{Spec}\mathbb{C}(\mathbb{R}) \to A(\mathbb{R})$ is an epimorphism, there being no nontrivial connected covers of $\operatorname{Spec}\mathbb{R}$ we can pass to if $\operatorname{Spec}\mathbb{C}(\mathbb{R}) \to A(\mathbb{R})$ is not an epimorphism on the nose. On the other hand there are no maps $\operatorname{Spec}(\mathbb{R}) \to \operatorname{Spec}(\mathbb{C})$, so it follows $A(\mathbb{R})$ is the empty set. In particular, $A \neq \operatorname{Spec}\mathbb{R}$, since $\operatorname{Spec}(\mathbb{R}) \to \mathbb{R}$ contains the identity map.

In [MV99], it is proved that having a $k$-rational point is an $\mathbb{A}^1$-invariant of a sheaf, so that in fact $A \not\simeq \mathbb{R}$. This example shows that some assumption is necessary to ensure that the map of sheaves $\pi : Y \to X$ in proposition 42 is surjective.

As it happens, $H^{*,*}(\operatorname{Spec}\mathbb{C};\mathbb{Z})$ is not finitely generated projective over $H^{*,*}(\operatorname{Spec}\mathbb{R};\mathbb{Z})$, so that our spectral sequence methods would not apply anyway. It is possible to modify the example slightly, replacing it with the standard double-cover $\mathbb{G}_m \to \mathbb{G}_m$. This is a principal $\mathbb{Z}/2$-bundle in the sense of [MFK94] but the spectral sequence converges to the 'wrong' answer, indicating that the [MFK94] quotient and the Nisnevich sheaf quotient do not even share an $\mathbb{A}^1$-homotopy type. Of course, on fraction fields, this map is a quadratic extension $k(x) \to k(\sqrt{x})$, so it is not in essence different from the example of $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$.

# Chapter 3

# The Equivariant Motivic Cohomology of $\mathrm{GL}(n)$

## 3.1 Preliminaries

The theme of this technical chapter is the application the tools of the previous chapter to the objects of the first. The signal results are the complete calculation (by which we mean the identification of the $E_\infty$-page of a convergent spectral sequence) of the $\mathbb{G}_m$-equivariant cohomology of $\mathrm{GL}(n)$ with a general $\mathbb{G}_m$-action in proposition 51, and the partial calculation of the $\mathbb{G}_m$ equivariant cohomology of the other Stiefel varieties previously considered in theorem 54.

Of course, in classical topology, one would consider the fibration $X \to X \times_G EG \to BG$ and then employ a Serre spectral sequence to go from knowledge of $H^*(X)$ and $H^*(BG)$ to $H^*(X \times_G EG)$. This is what we do in spirit, since if one were to take that fiber sequence and start to extend it to a Puppe sequence, we should arrive at $G \simeq \Omega BG \to X \to X \times_G EG$, which is really also the fiber sequence $G \to EG \times X \to EG \times_G X$. For this last, the Rothenberg-Steenrod sequence computes the cohomology of $EG \times_G X$ given the cohomology of the other two, and this is what we are doing. As a consequence of our having to go this slightly roundabout way, the spectral sequences we obtain have $E_2$-pages resembling the $E_3$-pages of the Serre spectral sequences of which they are analogues.

In this chapter and the next $H^{*,*}(X)$ shall denote the motivic cohomology of $X$ taken with coefficients in some unspecified commutative ring $A$ such that $\frac{1}{2} \in A$. In general $\mathbb{M}$ shall denote $H^{*,*}(\operatorname{Spec} k; A)$, as usual, but in the first section we do not demand this, instead allowing it to be an arbitrary graded-commutative algebra.

There are a number of different definitions of Ext given in [Wei94]. We generally use the most common definitions, viz. the homology of the result of applying $\operatorname{Hom}(\cdot, A)$ of $\operatorname{Hom}(B, \cdot)$ to projective or injective resolutions, at one point we use the definition of $\operatorname{Ext}^i(A, B)$ as being the set of maps to a shift of $B$, $P_\bullet \to B[i]$ — where $P_\bullet$ is a projective resolution of $A$ — taken modulo homotopy equivalence.

We shall also in this chapter and the next need to deal with symmetric polynomials. In the polynomial ring $R[x_1, \ldots, x_n]$ there are distinguished elementary symmetric polynomials, $\sigma_1, \ldots, \sigma_n$, which one may define as the coefficients of the polynomial in $R[x_1, \ldots, x_n, t]$ given by the expansion

$$\prod_{i=1}^{n}(t + x_i) = t^n + \sum_{i=1}^{n} \sigma_i(x_1, \ldots, x_n) t^{i-1}$$

## 3.2 Homological Algebra

Recall that if $X \to X \otimes X$ is a coproduct in a coalgebra, then $x \in X$ is called *primitive* if $x \mapsto x \otimes 1 + 1 \otimes x$.

**Proposition 45.** *Let* $\mathbb{M}$ *be a graded-commutative algebra, and let*

$$\hat{S} = \Lambda_{\mathbb{M}}(\hat{\alpha}_1, \ldots, \hat{\alpha}_n, \hat{\beta}_1, \ldots, \beta_m)$$

*be an exterior algebra over* $\mathbb{M}$*, with grading given by* $|\hat{\alpha}_i| = a_i$ *(the grading on the* $\hat{\beta}_i$ *is immaterial). Give* $\hat{S}$ *a Hopf-algebra structure by making* $\hat{\alpha}_i$*,* $\hat{\beta}_i$ *all primitive. Write* $N = \hat{S}/(\hat{\beta}_1, \ldots, \hat{\beta}_m)$*, which inherits a* $\hat{S}$*-linear coproduct map* $N \to N \otimes_{\mathbb{M}} N$*, the action of* $\hat{S}$ *on* $N \otimes_{\mathbb{M}} N$ *being via*

$\hat{S} \to \hat{S} \otimes_{\mathbb{M}} \hat{S}$. *There is an isomorphism of bigraded* $\mathbb{M}$*-algebras*

$$\mathrm{Ext}^{*,*}_{\hat{S}}(N, \mathbb{M}) \cong \mathbb{M}[\theta_1, \ldots, \theta_m]$$

*with* $|\theta_i| = (1, b_i)$, *and this isomorphism is natural in* $\hat{S}$, $N$ *and* $\mathbb{M}$.

*Proof.* All tensor products are taken over $\mathbb{M}$. The naturality of the isomorphism follows from the naturality of all constructions carried out below, and we shall not mention it again.

We consider first the case when $n = 0$, so $N = \mathbb{M}$. The result

$$\mathrm{Ext}_{\hat{S}}(\mathbb{M}, \mathbb{M}) = \mathbb{M}[\theta_1, \ldots, \theta_m]$$

is a standard one, but in the generality we want, references are not particularly easy to come by. We outline a proof.

First, we resolve $N = \mathbb{M}$ by taking the associated complex, $\perp_* \mathbb{M}$ to the simplicial object $\perp_\bullet \mathbb{M}$, whose $p$-simplices are $(\hat{S})^{\otimes p}$. An element $\mathrm{Ext}^i_{\hat{S}}(\mathbb{M}, \mathbb{M})$ is represented by a chain map $\perp_* \mathbb{M} \to \mathbb{M}[i]$, and the product of two such elements in $\mathrm{Ext}^i_{\hat{S}}(\mathbb{M}, \mathbb{M}) \otimes_{\mathbb{M}} \mathrm{Ext}^j_{\hat{S}}(\mathbb{M}, \mathbb{M})$ is represented by a map $\mathrm{Tot}(\perp_* \mathbb{M} \otimes \perp_* \mathbb{M}) \to M[i+j]$.

We remark however that that $\mathrm{Tot}(\perp_* \mathbb{M} \otimes \perp_* \mathbb{M})$ is quasi-isomorphic as a complex of $\hat{S} \otimes \hat{S}$-modules to the chain complex $\perp_*^{\hat{S} \otimes \hat{S}} \mathbb{M}$ associated with the simplicial resolution of $\mathbb{M}$ as an $\hat{S} \otimes \hat{S}$-module having $q$-simplices $(\hat{S} \otimes \hat{S})^q$, [Wei94, Chapter 8.6]. This quasi-isomorphic map

$$\perp_*^{\hat{S} \otimes \hat{S}} \mathbb{M} \to \mathrm{Tor}(\perp_* \mathbb{M} \otimes \perp_* \mathbb{M})$$

is effected by Alexander-Whitney maps, see loc. cit., and so we can compute it explicitly if need be.

Having therefore a map $\mathrm{Ext}_{\hat{S} \otimes \hat{S}}(\mathbb{M}, \mathbb{M})^{\otimes 2} \to \mathrm{Ext}_{\hat{S} \otimes \hat{S}}(\mathbb{M}, \mathbb{M})$, we arrive at a product by composing with the map $\perp_* \mathbb{M} \to \perp_*^{\hat{S} \otimes \hat{S}} \mathbb{M}$.

Explicitly, the maps $\perp_* \mathbb{M} \to M[i]$ are given by maps $\hat{S}^{\otimes i} \to M$ which are generated by the duals of the elements $\beta_{j_1} \otimes \ldots \otimes \beta_{j_i}$. We denote such a dual by $[\theta_{j_1} \ldots \theta_{j_i}]$, anticipating the identity $[\theta_{j_1}\dots\theta_{j_i}] = \prod_{k=1}^{i}[\theta_{j_k}]$. We remark that the nature of the differential in $\perp_*\mathbb{M}$ implies the element represented by $[\theta_{j_1}\dots\theta_{j_i}]$ after taking homology is equivalent to that obtained by permuting the $\theta_{j_k}$s. Following through the Alexander-Whitney maps gives a product $[\theta_{j_1}\dots\theta_{j_i}]\otimes[\theta_{\ell_1}\dots\theta_{\ell_k}]\mapsto[\theta_{j_1}\dots\theta_{j_i}\theta_{\ell_1}\dots\theta_{\ell_k}]$, which justifies the notation.

We return to the case of general $N$. First we remark that, in the set-up of the proposition, we have $\mathrm{Ext}_{\hat{S}}(N,N) = N[\theta_1,\dots,\theta_m]$ as algebras, since $\hat{S}\cong\Lambda_N(\hat{\beta}_1,\dots,\beta_m)$

There is an easier way to compute $\mathrm{Ext}_{\hat{S}}(N,\mathbb{M})$, which is to resolve $N$ by a standard resolution

$$\longrightarrow \bigoplus_{1\le i,j\le n}\hat{S}\beta_i\beta_j \longrightarrow \bigoplus_{1\le i\le n}\hat{S}\beta_i \longrightarrow \hat{S} \longrightarrow N$$

If we write $F_j$ for the $j$-th term, then the differentials have image lying always in the submodule $(\beta_1,\dots,\beta_n)F_j$. In particular, application of $\mathrm{Hom}_{\hat{S}}(\cdot,N)$ kills the differentials, and it follows that the map $\mathrm{Ext}_{\hat{S}}(N,N)\to\mathrm{Ext}_{\hat{S}}(N,\mathbb{M})$ is actually the map (of $\mathbb{M}$-modules

$$N[\theta_1,\dots,\theta_m]\to N[\theta_1,\dots,\theta_m]\otimes_N\mathbb{M}\cong\mathbb{M}[\theta_1,\dots,\theta_m]$$

Since this is actually a ring map by the naturality of the ring structure on Ext, for the proof of which see [Wei94][Chapter 8], the result follows. □

**Proposition 46.** *Let $\mathbb{M}$ be a bigraded ring, let $S$ be an exterior algebra*

$$S=\Lambda_{\mathbb{M}}(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_m)$$

*that is also a Hopf algebra with the elements $\alpha_i,\beta_i$ primitive and homogeneous. Let $B$ be a ring*

$$B=\Lambda_R(\alpha'_1,\dots,\alpha'_n,\gamma_1,\dots,\gamma_p)$$

*with $\alpha'_i$, $\beta_i$ homogeneous which is equipped with a comodule structure over $A$, where $B\to A\otimes_R B$ is given by*

$$\alpha'_i\mapsto\alpha'_i\otimes_R 1+1\otimes\alpha_i,\quad \gamma_i\mapsto\gamma_i\otimes 1$$

*Let $\hat{S}$ denote the dual algebra of $S$ over $\mathbb{M}$, and $\hat{B}$ the dual coalgebra over $\mathbb{M}$. Then $\hat{B}$ is an $\hat{S}$-module, and there is an isomorphism*

$$\mathrm{Ext}^*_{\hat{S}}(\hat{B}, \mathbb{M}) = \Lambda_{\mathbb{M}}(\gamma'_1, \ldots, \gamma'_p)[\theta_1, \ldots, \theta_m]$$

*which is again natural in $\hat{S}, \hat{B}, \mathbb{M}$. There is a map*

$$\mathrm{Ext}^*_{\hat{S}}(\hat{B}, \mathbb{M}) \to \mathrm{Ext}^*_{\mathbb{M}}(\hat{B}, \mathbb{M}) = B$$

*mapping $\gamma'_i$ to $\gamma_i$. The element $\theta_i$ corresponds to $\beta_i$; if $\beta_i$ has bidegree $(r, s)$, then $\theta_i$ has bidegree $(r, s)$ in $\mathrm{Ext}^1_{\hat{S}}(\hat{B}, \mathbb{M})$.*

*Proof.* Again, the naturality goes by the book and we do not mention it again.

We observe that $B$ is free and finitely generated as an $\mathbb{M}$-module, so $\mathrm{Hom}_{\mathbb{M}}(B, \mathbb{M}) = \hat{B}$ is too, and as a result $\mathrm{Ext}_{\mathbb{M}}(\hat{B}, \mathbb{M}) = \mathrm{Hom}_{\mathbb{M}}(\hat{B}, \mathbb{M}) = \hat{\hat{B}} = B$.

For a subset $J \subset \{1, \ldots, p\}$, write $\gamma_J$ for the product $\prod_{i \in J} \gamma_i$. We can decompose $\hat{B}$ as a direct sum indexed over products

$$\bigoplus_{J \subset \{1,\ldots,p\}} \frac{\hat{S}}{(\hat{\beta}_1, \ldots, \hat{\beta}_m)} \hat{\gamma}_J$$

For convenience, we write $T = \frac{\hat{S}}{(\hat{\beta}_1,\ldots,\hat{\beta}_m)}$. With the given decomposition and by use of the previous proposition, we have

$$\mathrm{Ext}_{\hat{S}}(\hat{B}, \mathbb{M}) = \bigoplus_{J \subset \{1,\ldots,p\}} \mathrm{Ext}_{\hat{S}} \left( \frac{\hat{S}}{(\hat{\beta}_1, \ldots, \hat{\beta}_m)}, \mathbb{M} \right) \hat{\hat{\gamma}}_J = \bigoplus_{J \subset \{1,\ldots,p\}} \mathbb{M}[\theta_1, \ldots, \theta_m] \hat{\hat{\gamma}}_J$$

The indeterminates $\theta_i$ lie in $\mathrm{Ext}^1_{\hat{S}}(\hat{B}, \mathbb{M})$. What remains to be determined is the multiplication $\hat{\hat{\gamma}}_J \hat{\hat{\gamma}}_{J'}$, but the ring map $\mathrm{Ext}_{\hat{S}}(\hat{B}, \mathbb{M}) \to B$ takes $\hat{\hat{\gamma}}_i$ in the former to $\gamma_i$ in the latter, and since

$$\hat{\hat{\gamma}}_i \in \mathrm{Ext}^0_{\hat{S}}(\hat{B}, \mathbb{M}) = \bigoplus_{J \subset \{1,\ldots,p\}} \mathbb{M} \hat{\hat{\gamma}}_J$$

it follows quite easily that $\hat{\hat{\gamma}}_i\hat{\hat{\gamma}}_j = \widehat{\widehat{\gamma_i\gamma_j}}$. We are therefore justified in dropping the distinction and write $\hat{\hat{\gamma}}_i = \gamma_i$. □

We must account for one additional complexity in our calculation of Ext-rings

**Proposition 47.** *Let* $\mathbb{M}$ *be a bigraded ring, let* $A$ *be an exterior algebra*

$$A = \Lambda_{\mathbb{M}}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)$$

*that is also a Hopf algebra with the elements* $\alpha_i, \beta_i$ *primitive and homogeneous. Let* $B$ *be a ring*

$$B = \Lambda_{\mathbb{M}}(\alpha'_1, \ldots, \alpha'_n, \gamma_1, \ldots, \gamma_p, \eta)$$

*with* $\alpha'_i, \gamma_i$ *homogeneous which is equipped with a comodule structure over* $A$*, where* $B \to A \otimes_{\mathbb{M}} B$ *is given by*

$$\alpha'_i \mapsto \alpha'_i \otimes_R 1 + 1 \otimes \alpha_i, \quad \gamma_i \mapsto \gamma_i \otimes 1, \quad \eta \mapsto \eta \otimes 1 + 1 \otimes \sum_{i=1}^{m} b_i\beta_i$$

*with* $\sum_{i=1}^m b_i \neq 0$*. Let* $\hat{A}$ *denote the dual algebra of* $A$ *over* $R$*, and* $\hat{B}$ *the dual coalgebra over* $R$*. Then* $\hat{B}$ *is an* $\hat{A}$*-module, and we can compute the ring*

$$\mathrm{Ext}^*_{\hat{A}}(\hat{B}, R) = \frac{\Lambda_{\mathbb{M}}(\gamma'_1, \ldots, \gamma'_p)[\theta_1, \ldots, \theta_m]}{(\sum b_i\theta_i)}$$

*As before, there is a natural map*

$$\mathrm{Ext}^*_{\hat{A}}(\hat{B}, \mathbb{M}) \to \mathrm{Ext}^*_R(\hat{B}, \mathbb{M}) = \mathbb{M}$$

*mapping* $\gamma'_i$ *to* $\gamma_i$*. The element* $\theta_i$ *corresponds to* $\beta_i$*; if* $\beta_i$ *has bidegree* $(r, s)$*, then* $\theta_i$ *has bidegree* $(r, s)$ *in* $\mathrm{Ext}^1_{\hat{A}}(\hat{B}, \mathbb{M})$*.*

*Proof.* As before, we can reduce the question by decomposing the $\hat{A}$-module $\hat{B}$ into direct summands generated by monomials in the $\hat{\gamma}_i$. It suffices to consider the ring $S =$

$\Lambda_{\mathbb{M}}(\hat{\alpha}_1, \ldots, \hat{\alpha}_m)$, the ring $\hat{A}$ over it, and the module $M = \Lambda_{\hat{A}}(\hat{\eta})$ over $\hat{A}$. By the same argument as before, it will suffice to calculate $\operatorname{Ext}_{\hat{A}}(M, S)$, from which $\operatorname{Ext}_{\hat{A}}(M, \mathbb{M})$ can be deduced.

There is a short exact sequence of $\hat{A}$-modules which splits as a sequence of $S$-modules

$$0 \longrightarrow S \xrightarrow{1 \mapsto \hat{\eta}} M \longrightarrow S \longrightarrow 0$$

The $\hat{A}$-action on $M$ is given by $\hat{\beta}_i \cdot 1_M = b_i \eta$. In fact, $M \to S$ is a map of coalgebras. From this one obtains a long exact sequence of Ext-groups which is, by dint of the $S$-comodule structure on the original exact sequence, a sequence of $\operatorname{Ext}_{\hat{A}}(S, S) = S[\theta_1, \ldots, \theta_m]$-modules.

$$\begin{array}{ccc} \operatorname{Ext}_{\hat{A}}(S,S) & \longrightarrow & \operatorname{Ext}_{\hat{A}}(M,S) \\ {\scriptstyle\partial}\nwarrow & & \swarrow \\ & \operatorname{Ext}_{\hat{A}}(S,S) & \end{array}$$

By explicit calculation in the snake lemma, the boundary map

$$\operatorname{Hom}_{\hat{A}}(S, S) = S \to \operatorname{Ext}^1_{\hat{A}}(S, S) = S[\theta_1, \ldots, \theta_m]^{(1)}$$

takes 1 to $\sum b_i \theta_i$. Our assertion now follows, since the long exact sequence splits as

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ext}_{\hat{A}}(S,S) & \xrightarrow{1 \mapsto \sum b_i \theta_i} & \operatorname{Ext}_{\hat{A}}(S,S) & \longrightarrow & \operatorname{Ext}_{\hat{A}}(M,S) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & & & \\ & & S[\theta_1, \ldots, \theta_m] & & S[\theta_1, \ldots, \theta_m] & & & & \end{array}$$

and this is in fact an exact sequence of $\operatorname{Ext}_{\hat{A}}(S, S)$-modules, by proposition 32 so $\operatorname{Ext}_{\hat{A}}(M, S)$ is $S[\theta_1, \ldots, \theta_m]/(\sum b_i \theta_i)$.

The argument we used in the previous proposition works equally well here for the progression to $\operatorname{Ext}_{\hat{A}}(\hat{B}, R)$, and so this has precisely the structure we claim. □

## 3.3 Torus Actions on GL($n$)

We compute a handful of examples of the Motivic Rothenberg-Steenrod spectral sequence.

This section makes extensive use of the tri-graded nature of the motivic spectral sequences, and it is therefore convenient to have a notational convention for that grading. An element $\alpha$ in the $j$-th page of a spectral sequence will be said to have tri-degree $|\alpha| = (p, q, r)$ or $(p, (q, r))$ if it is in homological degree $p$, motivic degree $q$ and weight $r$. This element corresponds to one that would classically be understood to be in bidegree $(p, q)$, that is to say $p$ 'across' and $q$ 'up'. The differential $d_j$ invariably will take $\alpha$ in of tridegree $(p, q, r)$ to $d_j\alpha$ in tridegree $(p + j, q - j + 1, r)$. We define the *total Chow height* of $\alpha$ to be $\operatorname{tch}\alpha = 2r - p - q$, and we note that

$$\operatorname{tch} d_j\alpha = \operatorname{tch}\alpha - 1 \tag{3.1}$$

Since total Chow height is linear in each grading, we also have $\operatorname{tch}\alpha\beta = \operatorname{tch}\alpha + \operatorname{tch}\beta$. In general, equation (3.1) allows us to discount a great many potential differentials in the motivic spectral sequence, which is a great advantage over the classical case, where we do not have the crutch of the weight filtration.

The first, and easiest, of the spectral sequences is the following

**Proposition 48.** *Let $\mathbb{G}_m$ act on $\mathbb{A}^n \setminus \{0\}$ via the diagonal action, that is to say the map $\mathbb{G}_m(R) \times (\mathbb{A}^n \setminus \{0\})(R) \to (\mathbb{A}^n \setminus \{0\})(R)$ is given by $r \circ (a_1, \ldots, a_n) = (ra_1, \ldots, ra_n)$.*

*Suppose $n > 1$. Then the $E_2$-page of the associated spectral sequence in motivic cohomology is the $\mathbb{M}$-algebra $\mathbb{M}[\rho_n, \theta]/(\rho_n^2)$, with $|\rho_n| = (0, 2n - 1, n)$ and $|\theta| = (1, 1, 1)$. There is a single nonvanishing differential of note, on the $n$-th page, satisfying $d_n\rho_n = \theta^n$. All other differentials are determined by this one.*

*Suppose $n = 1$, then $\mathbb{G}_m \cong \mathbb{A}^1 \setminus \{0\}$ and the spectral sequence is trivial.*

*Proof.* We consider only the case $n > 1$, the case $n = 1$ being trivial.

The group action $\mathbb{G}_m$ on $\mathbb{A}^n \setminus \{0\}$ gives rise to the principal $\mathbb{G}_m$-bundle

$$\begin{array}{c} \mathbb{A}^n \setminus \{0\} \\ \downarrow \\ \mathbb{P}^{n-1} \end{array}$$

which is Zariski-locally trivial, as one can see by taking e.g. the open set $U \subset \mathbb{P}^{n-1}$, which is isomorphic to $\mathbb{A}^{n-1}$, corresponding to the nonvanishing of the first coordinate. We have the following cartesian diagram

$$\begin{array}{ccc} \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^{n-1} & \longrightarrow & \mathbb{A}^n \setminus \{0\} \\ \downarrow\uparrow & & \downarrow \\ U & \longrightarrow & \mathbb{P}^{n-1} \end{array}$$

in which the section is simply given by the isomorphism $\mathbb{A}^{n-1} \cong U$ and the inclusion of the identity $\mathrm{pt} \to \mathbb{G}_m$. We conclude, using proposition 42, that $|B(\mathrm{pt}, \mathbb{G}_m, \mathbb{A}^n \setminus \{0\})| \simeq \mathbb{P}^{n-1}$.

The calculation of the $E_2$-page of the spectral sequence is straightforward. Since

$$H^{*,*}(\mathbb{G}_m; R)' = \Lambda_R(\hat{\hat{\rho}}_1)$$

and $H^{*,*}(\mathbb{A}^n \setminus \{0\})' = \Lambda_R(\hat{\hat{\rho}})$, with $|\hat{\rho}_i| = (2i-1, i)$, we can refer to proposition 46. Given that the $E_2$-page is therefore $\mathbb{M}[\rho_n, \theta]/(\rho_n^2)$, with $|\rho_n| = (0, 2n-1, n)$ and $|\theta| = (1,1,1)$, the only questions of note are the differentials potentially supported by $\rho_n$ and $\theta$. For dimensional reasons, $\theta$ cannot support any nonzero differentials. The total Chow height of $\rho_n$ is 1, and so if it is to support a differential, the image must be of total Chow height 0. We have $\operatorname{tch}\theta = 0$, and so $\operatorname{tch}\theta^i = 0$, whereas for any nonzero degree element, $\mu$, of $\mathbb{M}$, one has $\operatorname{tch}\mu\theta^i > 0$, so that if $\rho_n$ is to support a differential, it must take the form $d_j\rho_n = \ell\theta^i$, where $\ell \in \mathbb{M}^{0,0}$. Considering degrees we should have $(j, 2n-j, n) = (i,i,i)$, so $i = j = n$. Since $\rho_n$ can support no other differential, the spectral sequence collapses by the $n$-th page at the latest. On the diagonal $p + q = 2n$, in weight $n$, there is only one nonzero group, a copy of $R/\ell$ in degree $(n, n, n)$. Since the sequence converges to the motivic cohomology of $\mathbb{P}^{n-1}$, for which the corresponding group $H^{2n,n}(\mathbb{P}^{n-1}; R)$ is 0, it follows that $\ell$ is a unit. Without loss of generality, we can choose generators for the cohomology of $\mathbb{G}_m$, $\mathbb{A}^{n-1} \setminus \{0\}$ so that this unit is in fact 1 □

**Proposition 49.** *Suppose $n > 1$. Let $T_n = (\mathbb{G}_m)^n$ act on $\mathbb{A}^n \setminus \{0\}$ via the action $(\lambda_1, \ldots, \lambda_n) \cdot (\mu_1, \ldots, \mu_n) = (\mu_1\lambda_1, \ldots \mu_n\lambda_n)$. Then the $E_2$-page of the associated spectral sequence for motivic cohomology is the $\mathbb{M}$-algebra $\mathbb{M}[\rho_n, \theta_1, \ldots, \theta_n]/(\rho_n^2)$. There is a single nonvanishing differential of note, on the n-th page, satisfying $d_n\rho_n = \prod_{i=1}^n \theta_i$. All other differentials are determined by this one.*

*Proof.* The proof proceeds by induction on $n$. In the case $n = 1$ this has already been handled above.

The determination of the $E_2$-page is again straightforward, being a special case of proposition 46. It goes without saying that for dimensional reasons the elements $\theta_i$ cannot support nonzero differentials themselves. There is a map of group-schemes $\Delta : \mathbb{G}_m \to T_n$ given by the diagonal. There is a commutative diagram of group actions

$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{A}^n \setminus \{0\} & \xrightarrow{\Delta\times\mathrm{id}} & T_n \times \mathbb{A}^n \setminus \{0\} \\ \downarrow & & \downarrow \\ \mathbb{A}^n \setminus \{0\} & = & \mathbb{A}^n \setminus \{0\} \end{array}$$

from which it follows that there is a map of spectral sequences, which we denote by $\Delta^*$. We refer to the spectral sequence for the $\mathbb{G}_m$-action as the first spectral sequence, that for the $T_n$-action as the second, and write ${}_IE_*^{*,*,*}$, ${}_{II}E_*^{*,*,*}$ to distinguish them. The map $\Delta^*$ goes from ${}_{II}E_*$ to ${}_IE_*$. We have $\Delta^*(\hat{\rho}_n) = \hat{\rho}_n$ and $\Delta^*(\theta_i) = \theta$, the second conclusion following from the effect of the map $\Delta^* : H^{*,*}(T_n; \mathbb{Z}) \to H^{*,*}(\mathbb{G}_m; \mathbb{Z})$ on cohomology along with some rudimentary homological algebra.

In ${}_{II}E_2$, an $m$-fold product of $\theta_i$s has tridegree $(m, m, m)$, and a nontrivial multiple of this by a positively graded element of $\mathbb{M}$ has Chow height greater than 1. It follows for dimensional reasons that $\rho_n$ cannot support any differential before $d_n(\rho_n)$.

Since $\Delta^*(d_n\rho_n) = d_n(\Delta^*\rho_n) = d_n\rho_n = \theta^n$, it follows that $d_n\rho_n$ is nonzero in the second spectral sequence. Since the term ${}_{II}E_n^{n,n,n}$ is the $n$-graded part of the ring $\mathbb{Z}[\theta_1, \ldots, \theta_n]$, we must have $d_n\rho_n = p(\theta_1, \ldots, \theta_n)$ for some homogeneous polynomial $p$ of degree $n$. We also know that

$$\Delta^*(p(\theta_1, \ldots, \theta_n)) = p(\theta, \ldots, \theta) = \theta^n \tag{3.2}$$

In order to determine the polynomial $p$ precisely, we decompose $\mathbb{A}^n \setminus \{0\}$ with the diagonal $T_n$ action into two open subschemes, $U_1 = \mathbb{A}^{n-1} \setminus \{0\} \times \mathbb{A}^1$ and $U_2 = \mathbb{A}^{n-1} \times \mathbb{A}^1 \setminus \{0\}$. We obtain in this way a homotopy pushout square

$$\begin{array}{ccc} U_1 \cap U_2 & \longrightarrow & U_1 \\ \downarrow & & \downarrow \\ U_2 & \longrightarrow & \mathbb{A}^n \setminus \{0\} \end{array}$$

the maps in which are evidently $\mathbb{G}_m^n$ equivariant, since the action is diagonal. The functor $B(\mathrm{pt}, \mathbb{G}_m^n, \cdot)$ preserves homotopy colimits by proposition 35 so it follows that we have a homotopy pushout square

$$\begin{array}{ccc} B(\mathrm{pt}, \mathbb{G}_m^n, U_1 \cap U_2) & \longrightarrow & B(\mathrm{pt}, \mathbb{G}_m^n, U_1) \\ \downarrow & & \downarrow \\ B(\mathrm{pt}, \mathbb{G}_m^n, U_2) & \longrightarrow & B(\mathrm{pt}, \mathbb{G}_m^n, \mathbb{A}^n \setminus \{0\}) \end{array} \tag{3.3}$$

which in turn gives rise to a long exact sequence of Mayer-Vietoris type for the equivariant motivic cohomology

$$\begin{array}{ccc} & & H^{*,*}(B(\mathrm{pt}, \mathbb{G}_m^n, U_1)) \oplus H^{*,*}(B(\mathrm{pt}, \mathbb{G}_m^n, U_2)) \\ & \nearrow & \downarrow f^* \\ H^{*,*}(B(\mathrm{pt}, \mathbb{G}_m^n, \mathbb{A}^n \setminus \{0\})) & \xleftarrow{\ \partial\ } & H^{*,*}(B(\mathrm{pt}, \mathbb{G}_m^n, U_1 \cap U_2)) \end{array}$$

where the map marked $\partial$ is a coboundary map and thus shifts simplicial degree by 1.

Recall that we write $R^{(n)}$ for the $n$-graded part of the ring $R$. We have

$$H^{2n,n}(B(\mathrm{pt}, \mathbb{G}_m^n, \mathbb{A}^n \setminus \{0\})) = \frac{\mathbb{Z}[\theta_1, \dots, \theta_n]^{(n)}}{(p(\theta_1, \dots, \theta_n))}$$

where $p(\theta_1, \dots, \theta_n)$ is the homogeneous polynomial of degree $n$ we wish to determine. It is a matter of some elementary algebra to determine the $2n, n$-graded part of $\ker f^*$, which is important since it is the target of a map in the Mayer-Vietoris sequence

$$H^{2n,n}(B(\mathrm{pt}, \mathbb{G}_M^n, \mathbb{A}^n \setminus \{0\})) \twoheadrightarrow (\ker f^*)^{(2n,n)}$$

By induction on $n$ and straightforward comparison, we have

$$\begin{aligned} H^{*,*}(B(\mathrm{pt}, T_n, U_1)) &= \frac{\mathbb{M}[\theta_1, \dots, \theta_n]}{(\prod_{i=1}^{n-1} \theta_i)} \\ H^{*,*}(B(\mathrm{pt}, T_n, U_2)) &= \frac{\mathbb{M}[\theta_1, \dots, \theta_n]}{(\theta_n)} \\ H^{*,*}(B(\mathrm{pt}, T_n, U_1 \cap U_2)) &= \frac{\mathbb{M}[\theta_1, \dots, \theta_n]}{(\prod_{i=1}^{n-1} \theta_i, \theta_n)} \end{aligned}$$

so that a reading of the Mayer-Vietoris sequence shows that $(\ker f^*)^{2n,n}$ is the subgroup of

$$\left( \frac{\mathbb{M}[\theta_1, \dots, \theta_n]}{\prod_{i=1}^{n-1} \theta_i} \oplus \frac{\mathbb{M}[\theta_1, \dots, \theta_n]}{\theta_n} \right)^{(2n,n)}$$

consisting of pairs of polynomials $f, g$ the reductions of which to $H^{*,*}(B(\mathrm{pt}, T_n, U_1 \cap U_2))$ coincide. There is a projection map $\mathbb{M}[\theta_1, \dots, \theta_n] \to (\ker f^*)^{(2n,n)}$, and plainly the polynomial $p_0 = \prod_{i=1}^{n} \theta_i$ is in the kernel of this projection. Since $p_0$ is of degree $n$, and since the projection map factors

$$\mathbb{M}[\theta_1, \dots, \theta_n] \to \frac{\mathbb{Z}[\theta_1, \dots, \theta_n]^{(n)}}{(p(\theta_1, \dots, \theta_n))} \to (\ker f^*)^{(2n,n)}$$

it follows that $p = a p_0$, where $a \in \mathbb{M}^{0,0}$. By reference to (3.2), we see that $a = 1$. $\square$

**Proposition 50.** *Suppose $n \geq 1$. Let $T_n = (\mathbb{G}_m)^n$ act on $\mathrm{GL}(n)$ via the action of left multiplication by diagonal matrices. Then the $E_2$-page of the associated spectral sequence takes the form*

$$E_2^{*,*} = \frac{\Lambda_{\mathbb{M}}(\rho_n, \rho_{n-1}, \ldots, \rho_2, \rho_1)[\theta_1, \ldots, \theta_n]}{(\rho_1, \sum_{i=1}^n \theta_i)} \qquad |\rho_i| = (0, 2n-1, n), \quad |\theta_i| = (1,1,1) \quad (3.4)$$

*and the differentials are generated by $d_i(\rho_i) = \sigma_i(\theta_i)$ for $n \geq i \geq 2$, where $\sigma_i(\theta_i)$ is the i-th elementary symmetric function in the $\theta_i$.*

Note: this $E_2$ page can be thought of as coming from a fictitious $E_1$-page

$$\Lambda_{\mathbb{M}}(\rho_n, \ldots, \rho_1)[\theta_1, \ldots, \theta_n]$$

with nonzero differential $d_1(\rho_1) = \sum \theta_i$. This explains the rather awkward presentation given in (3.4). In classical algebraic topology this $E_2$-page is isomorphic to the $E_3$-page of the Serre spectral sequence of the fibration $T_n \to ET_n \times_{T_n} \mathrm{GL}(n) \to B\,\mathrm{GL}(n)$, with $E_2$-page isomorphic to our fictitious $E_1$-page.

*Proof.* The proof is by induction on $n$. When $n = 1$, $T_1 \cong \mathbb{G}_m \cong \mathrm{GL}(1)$, so there is nothing to prove.

In the case $n > 1$, we know that the $E_2$-page of the spectral sequence takes the anticipated form from proposition 46.

Given $\pi \in \Sigma_n$, the symmetric group on $n$ letters, there are maps $f_\alpha : \mathrm{GL}(n) \to \mathrm{GL}(n)$ and $g_\alpha : T_n \to T_n$ given by permuting the columns in the first case and permuting the multiplicands in the second. It is apparent that

$$\begin{array}{ccc} T_n \times \mathrm{GL}(n) & \xrightarrow{g_\alpha, f_\alpha} & T_n \times \mathrm{GL}(n) \\ \downarrow & & \downarrow \\ \mathrm{GL}(n) & \xrightarrow{f_\alpha} & \mathrm{GL}(n) \end{array}$$

commutes, and as a result we obtain a $\Sigma_n$-action on the spectral sequence under consideration. The action of $g_\alpha^*$ on $H^{*,*}(T_n; R)$ is to permute the generators, whereas the action of

$f_\alpha^*$ is trivial, by proposition 11. In particular this implies that the action on the $E_2$-page of the spectral sequence is to permute the classes of the form $\theta_i$s, but to fix those of the form $\hat{\rho}_j$.

We fix canonical maps $\mathrm{GL}(n-1) \to \mathrm{GL}(n)$ and $T_n \to T_{n-1}$, the former being the inclusion of proposition 9, the latter being projection onto the first $n-1$ terms. The following diagram of group actions commutes

$$\begin{array}{ccccc} T_{n-1} \times \mathrm{GL}(n-1) & \longrightarrow & T_{n-1} \times \mathrm{GL}(n) & \longleftarrow & T_n \times \mathrm{GL}(n) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{GL}(n-1) & \longrightarrow & \mathrm{GL}(n) & \longleftarrow & \mathrm{GL}(n) \end{array}$$

and as a result we obtain three spectral sequences, the pages of which we will denote by ${}_IE_*$, ${}_{II}E_*$ and ${}_{III}E_*$ respectively. We have maps $\phi^* : {}_{II}E_* \to {}_IE_*$ and $\pi^* : {}_{II}E_* \to {}_IE_*$.

We fix notation

$$\begin{aligned} {}_IE_2^{*,*} &= \frac{\Lambda_{\mathbb{M}}(\hat{\rho}_{n-1}, \ldots, \hat{\rho}_2)[\theta_{n-1}, \ldots, \theta_1]}{(\sum \theta_i)} \\ {}_{II}E_2^{*,*} &= \frac{\Lambda_{\mathbb{M}}(\hat{\rho}''_n, \ldots, \hat{\rho}''_2)[\theta''_{n-1}, \ldots, \theta''_1]}{(\sum \theta''_i)} \\ {}_{III}E_2^{*,*} &= \frac{\Lambda_{\mathbb{M}}(\hat{\rho}'_n, \ldots, \hat{\rho}'_2)[\theta'_n, \ldots, \theta'_1]}{(\sum \theta_i)} \end{aligned}$$

We have $\phi^*(\hat{\rho}''_i) = \hat{\rho}_i$ for $i < n$, and $\phi^*(\theta_i) = \theta''_i$ for all $i$, moreover $\pi^*(\hat{\rho}''_i) = \hat{\rho}'_i$ and $\pi^*(\theta''_i) = \theta_i$ for all $i$.

Let $A_i \subset Z[\theta'_1, \ldots, \theta'_{n-1}]$ denote the $i$-th graded part (each $\theta'$ being assigned grading 1). Let $I_i$ be the ideal of $\mathbb{Z}[\theta'_1, \ldots, \theta'_n]$ generated by the elementary symmetric polynomials $\sigma_1(\theta'_i), \sigma_2(\theta'_i), \ldots, \sigma_i(\theta'_i)$, and let $B_i = A_i/(I_{i-1} \cap A_i)$.

The usual argument by total Chow height rules out all differentials supported by $\theta'_i$ and $\theta''_i$, and reduces the possible differentials supported by $\hat{\rho}'_i$ to

$$d_i : \mathbb{Z}\hat{\rho}'_i \to {}_IE_i^{i,i,i} \cong B_i$$

Since the terms $\sigma_i(\theta''_1, \ldots, \theta''_{n-1})$ for $1 \le i \le n-1$ generate the subring of symmetric polynomials in $\mathbb{Z}[\theta''_1, \ldots, \theta''_{n-1}]$, it follows that $\hat{\rho}''_n$ can support no nonzero differential, since the only such differential would be symmetric because of the $\Sigma_n$-action.

Suppose $i < n$, then by comparison, we have

$$\pi^* d_i(\hat{\rho}'_i) = d_i \hat{\rho}''_i = \phi^* d_i(\hat{\rho}_i) = \phi^*(\sigma_i(\theta_1, \ldots, \theta_n)) = \sigma_i(\theta''_1, \ldots, \theta''_n)$$

so that $d_i \hat{\rho}'_i = p(\theta_1, \ldots, \theta_n)$ is represented by a symmetric polynomial of degree $n$ whose evaluation at $(\theta_1, \theta_2, \ldots, \theta_{n-1}, 0)$ is $\sigma_i(\theta_1, \ldots, \theta_{n-1})$. We can write such a polynomial as

$$p(\theta_1, \ldots, \theta_n) = \sigma_i(\theta_1, \ldots, \theta_{n-1}) + \theta_n q(\theta_1, \ldots, \theta_{n-1}) + \theta_n^2 r(\theta_1, \ldots, \theta_n)$$

Since this polynomial is symmetric, and there are no terms involving $\theta_i^s$ for $i < n$ and $s > 2$ that do not also involve $\theta_n$, it follows that there are no forms of the form $\theta_n^s$ for $s > 2$ that do not involve at $\theta_i$ for all $i$. The monomial of lowest degree that satisfies this criterion is $\theta_1 \ldots \theta_{n-1} \theta_n^2$, but this is of degree $n+1$, which is too high. It follows that $r = 0$, and then it is a trivial matter to observe that $q = \sigma_{i-1}(\theta_1, \ldots, \theta_{n-1})$, so that $p(\theta_1, \ldots, \theta_{n-1}) = \sigma_i(\theta_1, \ldots, \theta_n)$, as claimed.

As for the case of $d_n(\hat{\rho}_n)$, the above-stated comparisons tell us nothing. On the other hand, there is a diagram of group actions

$$\begin{array}{ccc} T_n \times \mathrm{GL}(n) & \longrightarrow & T_n \times \mathbb{A}^n \setminus \{0\} \\ \downarrow & & \downarrow \\ \mathrm{GL}(n) & \longrightarrow & \mathbb{A}^n \setminus \{0\} \end{array}$$

where the underlying map of $T_n$-spaces is simply projection on the first column, the effect of which on cohomology was computed in proposition8. which implies there is a map of spectral sequences from our ${}_{II}E_*^{*,*}$ to the spectral sequence of proposition 49.

On the $E_2$-page, we can calculate this map by straightforward homological algebra

$$\Lambda_{\mathbb{M}}(\hat{\rho}_n)[\theta_1, \dots, \theta_n] \to \frac{\Lambda_{\mathbb{M}}(\hat{\rho}_n, \dots, \hat{\rho}_2)[\theta'_1, \dots, \theta'_n]}{(\sum \theta'_i)}$$

$$\hat{\rho}_n \mapsto \hat{\rho}_n,\ \theta_i \mapsto \theta_i$$

and it follows immediately from the comparison and proposition 49 that $d_n(\hat{\rho}_n) = \sigma_n(\theta'_i)$ in ${}_{III}E$ as required. □

Let $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{Z}^n$ be a set of $n$ *weights*[1]. We consider the action of $\mathbb{G}_m$ on $\mathrm{GL}(n)$ given on the level of $R$-points by

$$\phi : \mathbb{G}_m(R) \times \mathrm{GL}(n, R) \to \mathrm{GL}(n, R)), \quad z \cdot A = \begin{pmatrix} z^{w_1} & & & \\ & z^{w_2} & & \\ & & \ddots & \\ & & & z^{w_n} \end{pmatrix} A$$

**Proposition 51.** *For the given action of $\mathbb{G}_m$ on $\mathrm{GL}(n)$, there is a spectral sequence computing the equivariant motivic cohomology. The $E_2$-page is given as*

$$E_2^{p,q} = \mathrm{Ext}^{p,q}_{\hat{H}^{*,*}(\mathbb{G}_m)}(\hat{H}^{*,*}(\mathrm{GL}(n), \mathbb{M})) = \frac{\Lambda_{\mathbb{M}}(\hat{\rho}_n, \dots, \hat{\rho}_1)[\theta]}{(\hat{\rho}_1, \sum_{i=1}^n w_i \theta)}$$

$$|\hat{\rho}_i| = (0, 2i-1, i) \qquad |\theta| = (1, 1, 1)$$

*with differentials $d_i(\hat{\rho}_i) = \sigma_i(w_1, \dots, w_n)\theta^i$. All other differentials are determined by these and dimensional vanishing.*

[1]We shall distinguish these from the weight-filtration of motivic cohomology by referring to the latter always in full

*Proof.* There is a map of group actions

$$\begin{array}{ccc} \mathbb{G}_m \times \mathrm{GL}(n) & \longrightarrow & T_n \times \mathrm{GL}(n) \\ \downarrow & & \downarrow \\ \mathrm{GL}(n) & = & \mathrm{GL}(n) \end{array}$$

Where the map of groups is that which on $R$-points gives $z \mapsto (z^{w_1}, z^{w_2}, \ldots, z^{w_n})$. The map on cohomology induced by this map is

$$H^{*,*}(T_n) \longrightarrow H^{*,*}(\mathbb{G}_m)$$
$$\Lambda_{\mathbb{M}}(\tau_1, \ldots, \tau_n) \longrightarrow \Lambda_{\mathbb{M}}(\tau)$$
$$\tau \longmapsto w_i \tau$$

It is straightforward to calculate that this induces a map of $E_2$-pages of spectral sequences

$$\mathrm{Ext}_{\hat{H}^{*,*}(T_n)}(\hat{H}^{*,*}(\mathrm{GL}(n)), \mathbb{M}) \to \mathrm{Ext}_{\hat{H}^{*,*}(\mathbb{G}_m)}(\hat{H}^{*,*}(\mathrm{GL}(n), \mathbb{M}))$$
$$\frac{\Lambda_{\mathbb{M}}(\hat{\rho}_n, \ldots, \hat{\rho}_1)[\theta_1, \ldots, \theta_n]}{(\hat{\rho}_1, \sum \theta_i)} \to \frac{\Lambda_{\mathbb{M}}(\hat{\rho}_n, \ldots, \hat{\rho}_1)[\theta]}{(\hat{\rho}_1, \sum_{i=1}^{n} w_i \theta)}$$
$$\hat{\rho}_i \mapsto \hat{\rho}_i \qquad |\hat{\rho}_i| = (0, 2i-1, i)$$
$$\theta_i \mapsto w_i \theta \qquad |\theta| = (1,1,1)$$

By comparison with proposition 50, the differentials are already known. As usual, the only classes capable of supporting a differential are the classes $\hat{\rho}_i$, and by comparison we must have $d_i(\hat{\rho}_i) = \sigma_i(w_i)\theta^i$. The assertion that this determines all differentials is trivially verified, since the $\hat{\rho}_i$s, and $\theta$ generate the $E_2$-page as a ring. □

Our final result on the equivariant cohomology of GL$(n)$ deals with the equivariant cohomology for the following $\mathbb{G}_m$ action on the left and on the right. If $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^n$ are two

$n$-tuples of integers, one defines an action of $\mathbb{G}_m^1$ on GL($n$) as (on $R$-points)

$$z \cdot A = \begin{pmatrix} z^{u_1} & & & \\ & z^{u_2} & & \\ & & \ddots & \\ & & & z^{u_n} \end{pmatrix} A \begin{pmatrix} z^{-v_1} & & & \\ & z^{-v_2} & & \\ & & \ddots & \\ & & & z^{-v_n} \end{pmatrix}$$

**Proposition 52.** *For the $\mathbb{G}_m^1$ action given above, the equivariant cohomology spectral sequence has $E_2$-page*

$$\mathrm{Ext}_{\hat{H}^{*,*}(\mathbb{G}_m)}(\hat{H}^{*,*}(\mathrm{GL}(n), \mathbb{M})) = \frac{\Lambda_{\mathbb{M}}(\hat{\rho}_n, \ldots, \hat{\rho}_1)[\theta]}{(\epsilon\hat{\rho}_1, [\sum u_i - \sum v_i]\,\theta)}$$

*Where $\epsilon$ is $0$ if $\sum u_i - \sum v_i = 0$ and $1$ otherwise. The differentials in this sequence are described summarily by*

$$d_i(\hat{\rho}_i) = [\sigma_i(\mathbf{u}) - \sigma_i(\mathbf{v})]\,\theta^i \pmod{\sigma_1(\mathbf{u}) - \sigma_1(\mathbf{v}), \ldots, \sigma_{i-1}(\mathbf{u}) - \sigma_{i-1}(\mathbf{v})}$$

*where $i$ is any integer between $1$ and $n$; the differentials can be deduced on all other elements by means of the product structure.*

The slightly awkward phrasing, involving $\epsilon$, is of course exactly what one would expect from a hypothetical $E_1$-page that involved only the generators enumerated above, and the $\epsilon$-term is in mimicry of the result of taking homology with respect to a putative $d_1$ differential.

We shall also need the following ring-theoretic lemma

**Lemma 53.** *Let $R$ be a ring, and let $S = R[c_1, \ldots, c_i, c'_1, \ldots, c'_i]$ be a polynomial ring. Let $\phi : S \to S$ be the involution that exchanges $c_j$ and $c'_j$ for all $j$. If $f \in S$ and $f + \phi(f) = 0$, then $f \in (c_1 - c'_1, c_2 - c'_2, \ldots, c_i - c'_i)$*

*Proof.* We use induction on $i$. If $i = 1$, the problem is trivial. Suppose therefore that the theorem holds in $R[c_1, \ldots, c_{i-1}, c'_1, \ldots, c'_{i-1}]$.

Suppose $f$ satisfies $f + \phi(f) = 0$, so we can write it as

$$f = \sum_{j=0}^{\infty} a_j \left[ c_i^j s_j - (c_i')^j \phi(s_j) \right] + S$$

where $S, s_j \in R[c_1, \ldots, c_{i-1}, c_1', \ldots, c_{i-1}']$, $a_j \in R$, almost all of the $a_j$s are 0 and $S + \phi(S) = 0$. We have the identity

$$c_i^j s_j - (c_i')\phi(s_j) - c_i^j \phi(s_j) + c_i^j \phi(s_j) = c_i^j [s_j - \phi(s_j)] + [c_i^j - (c_i')]^j \phi(s_j)$$

but here $s_j - \phi(s_j)$ satisfies the condition, and depends on fewer variables, so $s_j - \phi(s_j)$ lies in the stated ideal by induction. Consequently, all the leading terms in our decomposition of $f$ lie in the ideal. By induction, so does $S$, and as a result $f$ does too. □

We now return to the proof of the proposition

*Proof.* Let $T_{2n} = (\mathbb{G}_m)^n \times (\mathbb{G}_m)^n$ act on $\mathrm{GL}(n)$ by (on $R$-points)

$$(a_1, \ldots, a_n, b_1, \ldots, b_n) \cdot A = \mathrm{diag}\,(a_1, \ldots, a_n)\, A\, \mathrm{diag}\left(b_1^{-1}, \ldots, b_n^{-1}\right)$$

There is an evident group homomorphism $\mathbb{G}_m \to T_{2n}$ given by

$$z \mapsto (z^{u_1}, \ldots, z^{u_n}, z^{v_1}, \ldots, z^{v_n})$$

and for this group homomorphism, we have a commutative map of group actions

$$\begin{array}{ccc} \mathbb{G}_m \times \mathrm{GL}(n) & \longrightarrow & T_{2n} \times \mathrm{GL}(n) \\ \downarrow & & \downarrow \\ \mathrm{GL}(n) & = & \mathrm{GL}(n) \end{array}$$

Suppose $\alpha, \beta \in \Sigma_n$ are each permutations on $n$-letters. Then the pair $(\alpha, \beta)$ acts on the group action $T_{2n} \times \mathrm{GL}(n) \to \mathrm{GL}(n)$, where $\alpha$ permutes the first $n$ terms of $T_n$ and the columns of $\mathrm{GL}(n)$, and $\beta$ permutes the last $n$ terms of $T_n$, and the rows of $\mathrm{GL}(n)$. We denote this action by $f_{\alpha,\beta}$. There is also an involution, which we denote $\gamma$, which acts by interchanging the first and last $n$ terms of $T_n$ and is the map $A \mapsto A^{-1}$ on $\mathrm{GL}(n)$. The identity

$$\left[\operatorname{diag}(a_1,\ldots,a_n)\,A\,\operatorname{diag}\left(b_1^{-1},\ldots,b_n^{-1}\right)\right]^{-1} = \operatorname{diag}(b_1,\ldots,b_n)\,A\,\operatorname{diag}\left(a_1^{-1},\ldots,a_n^{-1}\right)$$

ensures that this involution is compatible with the group action.

The action of $T_{2n}$ on $\mathrm{GL}(n)$ yields a coaction on cohomology. Write

$$H^{*,*}(T_{2n}) = \Lambda_{\mathbb{M}}(\tau_1,\ldots,\tau_n,\tau_1',\ldots,\tau_n'), \quad |\tau_i| = |\tau_i'| = (1,1)$$
$$H^{*,*}(\mathrm{GL}(n)) = \Lambda_{\mathbb{M}}(\rho_n,\ldots,\rho_1), \quad |\rho_i| = (2i-1,i)$$

For dimensional reasons, the coaction must be $\tau_i \mapsto 1\otimes\tau_i$ for $i \geq 2$, but the case of $\tau_1$ is more involved. Again, by dimensional reasons we have

$$\rho_1 \mapsto 1\otimes\rho_1 + p_1(\tau_1,\ldots,\tau_n,\tau_1',\ldots,\tau_n')\otimes 1$$

where $p_1$ is a homogeneous linear polynomial.

For e.g. the inclusion $\mathbb{G}_m \to T_{2n}$ of the first factor, the map

$$\begin{array}{ccc} \mathbb{G}_m \times \mathrm{GL}(n) & \longrightarrow & T_{2n}\times\mathrm{GL}(n) \\ \downarrow & & \downarrow \\ \mathrm{GL}(n) & = & \mathrm{GL}(n) \end{array}$$

is a map of group actions. On cohomology, this map is evaluation of $(\tau_1,\ldots,\tau_n,\tau_1',\ldots,\tau_n')$ at $(\tau,0,\ldots,0,0,\ldots,0)$. For the $\mathbb{G}_m$-action, the coaction on cohomology is $\rho_1 \mapsto 1\otimes\rho_1 + \tau\otimes 1$, by proposition 21. In particular, by naturality, we must have $p_1(\tau,0,\ldots,0,0,\ldots,0) = \tau$.

On cohomology, we have $f_{\alpha,\beta}^*(\tau_i) = \tau_{\alpha^{-1}(i)}$, $f_{\alpha,\beta}^*(\tau_i') = \tau_{\beta^{-1}(i)}'$ and $f_{\alpha,\beta}(\rho_i) = \rho_i$, the last by reference to proposition 11. It follows that $p_1$ must be symmetric in $(\tau_1,\ldots,\tau_n)$ and

$(\tau'_1, \ldots, \tau'_n)$ , and

$$p_1(\tau'_1, \ldots, \tau'_n, \tau_1, \ldots, \tau_n) = -p(\tau_1, \ldots, \tau_n, \tau'_1, \ldots, \tau'_n)$$

It follows that $p_1 = \sum \tau_i - \sum \tau'_i$.

By reference to proposition 47, we can write down the $E_2$-page of the spectral sequence for the $T_{2n}$ action on GL$(n)$. We denote this by ${}_IE_*$. The $E_2$-page is

$$\mathrm{Ext}_{\hat{H}^{*,*}(T_{2n})}(\hat{H}^{*,*}(\mathrm{GL}(n)), \mathbb{M}) = \frac{\Lambda_{\mathbb{M}}(\hat{\rho}_n, \ldots, \hat{\rho}_2)[\theta_1, \ldots, \theta_n, \theta'_1, \ldots, \theta'_n]}{(\sum \theta_i - \sum \theta'_i)}$$

The symmetric group actions give $\hat{f}^*_{\alpha,\beta}(\hat{\rho}_i) = \hat{\rho}_i$, $\hat{f}^*_{\alpha,\beta}(\theta_i) = \theta_{\alpha(i)}$ and $\hat{f}^*_{\alpha,\beta}(\theta'_i) = \theta_{\beta(i)}$. The involution acts as $\hat{\rho}_i \mapsto -\hat{\rho}_i$, $\hat{\theta}_i = \theta'_i$ and $\theta'_i \mapsto \theta_i$.

The usual arguments mean that the only differentials we need consider are $d_i(\hat{\rho}_i)$. We deduce from the symmetric group actions and the involution that

$$d_i(\rho_i) \equiv p_i(\theta_1, \ldots, \theta_n, \theta'_1, \ldots, \theta'_n) \pmod{(p_1, p_2, \ldots, p_n)}$$

where $p_i$ is symmetric in the $\theta_i$, $\theta'_i$ individually, and antisymmetric in the interchange of the two. In particular, writing $c_i$ for $\sigma_i(\theta_i)$ and $c'_i$ for $\sigma_i(\theta'_i)$, it follows from standard results on symmetric polynomials that $p_i$ is a function of the $c_i$, $c'_i$

There is map of group action $T_n \times \mathrm{GL}(n)$, being the action of the previous proposition, to the $T_{2n} \times \mathrm{GL}(n)$ action at hand. By comparison of the spectral sequences, it follows that $p(\theta_1, \ldots, \theta_n, 0, \ldots, 0) \equiv \sigma_n(\theta_i)$. By antisymmetry we have

$$p_i(\theta_1, \ldots, \theta_n, \theta'_1, \ldots, \theta'_n) = \sigma_i(\theta_i) - \sigma_i(\theta'_i) + q_i(\theta_1, \ldots, \theta_n, \theta'_1, \theta'_n)$$

where $q_i$ is of degree $i$, symmetric in $\theta_1, \ldots, \theta_n$ and $\theta'_1, \ldots, \theta'_n$, antisymmetric in the interchange of the $\theta_i$ and $\theta'_i$, and $q_i$ lies in the product ideal $(\theta_1, \ldots, \theta_n)(\theta'_1, \ldots, \theta'_n)$. In terms of the $c_i$, we have

$$p_i = c_i - c'_i + r_i(c_1, \ldots, c_{i-1}, c'_1, \ldots, c'_{i-1})$$

Where $r(c'_1, \ldots, c'_{i-1}, c_i, \ldots, c_{i-1}) = -r(c_1, \ldots, c_{i-1}, c'_i, \ldots, c'_{i-1})$. By the lemma, $r$ lies in the ideal generated by $(c_1 - c'_1, \ldots, c_{i-1} - c'_{i-1})$. We have recursively described the differentials in the spectral sequence, they are

$$d_i(\rho_i) \equiv c_i - c'_i = \sigma_i(\theta_i) - \sigma_i(\theta'_i) \pmod{\sigma_1(\theta_i) - \sigma_1(\theta'_i), \ldots, \sigma_{i-1}(\theta_i) - \sigma_{i-1}(\theta'_i)}$$

It is now a matter of no great difficulty to use our original group homomorphism $\mathbb{G}_m \to T_{2n}$ to describe in full the spectral sequence for $\mathbb{G}_m$ acting on $\mathrm{GL}(n)$. We write $H^{*,*}(\mathbb{G}_m) = \Lambda_{\mathbb{M}}(\tau)$. It follows by naturality that the coaction of $H^{*,*}(\mathbb{G}_m)$ on $H^{*,*}(\mathrm{GL}(n))$ is given by $\rho_i \mapsto 1 \otimes \rho_i$ for $i \geq 2$ and $\rho_1 \mapsto 1 \otimes \rho_1 + [\sum u_i - \sum v_i]\, \tau \otimes 1$. By application of proposition 47, the $E_2$-page of the spectral sequence has the form asserted in the proposition:

$$\mathrm{Ext}_{\hat{H}^{*,*}(\mathbb{G}_m)}(\hat{H}^{*,*}(\mathrm{GL}(n), \mathbb{M})) = \frac{\Lambda_{\mathbb{M}}(\hat{\rho}_n, \ldots, \hat{\rho}_1)[\theta]}{(\epsilon \hat{\rho}_1, [\sum u_i - \sum v_i]\, \theta)}$$

Where $\epsilon$ is 0 if $\sum u_i - \sum v_i = 0$ and 1 otherwise. We denote this spectral sequence by ${}_{II}E_*$. There is a comparison map of spectral sequences ${}_{I}E_* \to {}_{II}E_*$, sending $\hat{\rho}_i$ to $\hat{\rho}_i$ for $i \geq 2$, sending $\theta_i$ to $u_i\theta$ and $\theta'_i$ to $v_i\theta$. It follows from the comparison that in the spectral sequence ${}_{II}E_*$, the differentials satisfy $d_i(\hat{\rho}_i) = [\sigma_i(u_i) - \sigma_i(v_i)]\, \theta^i$, as claimed. □

## 3.4 The Equivariant Cohomology of Stiefel Varieties

**Theorem 54.** *Let* $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{Z}^n$ *and* $\mathbf{v} = (v_1, v_2, \ldots, v_m) \in \mathbb{Z}^m$ *be two sequences of weights. We assume (since the other case has already been dealt with) that* $n > m$. *Consider the* $\mathbb{G}_m$*-action on* $W(n, m)$ *given by (or R-points)*

$$z \cdot A = \mathrm{diag}\,(u_1, \ldots, u_n)\, A\, \mathrm{diag}\left(v_1^{-1}, \ldots, v_m^{-1}\right)$$

*Let $s_1, \ldots, s_{n-m}$ be integers defined (recursively) by the relations*

$$\sigma_i(\mathbf{u}) = \sum_{j=1}^{i} \sigma_j(\mathbf{v}) s_{i-j}.$$

*and let $s_i = 0$ for $i > n - m$.*

*The spectral sequence associated with the $\mathbb{G}_m$-action has $E_2$-page*

$${}_I E_2 = \operatorname{Ext}_{\hat{H}^{*,*}(\mathbb{G}_m)}(\hat{H}^{*,*}(\mathrm{GL}(n)), \mathbb{M}) = \Lambda_{\mathbb{M}}(\hat{\rho}_{n-m+1}, \ldots, \hat{\rho}_n)[\theta]$$

*Let $n - m + 1 \leq k \leq n$ and suppose that*

$$d_j(\rho_j) = 0 \quad \textit{for } n - m + 1 \leq i \leq k$$

*then we have, in the given spectral sequence*

$$d_k(\rho_k) = \left[\sigma_k(\mathbf{u}) - \sum_{j=1}^{k} \sigma_j(\mathbf{v}) s_{k-j}\right] \theta^k \tag{3.5}$$

*Proof.* Strictly speaking, the proof proceeds by induction on $\ell = k - (n - m + 1)$, although most of the difficulty is already evident in the case $\ell = 0$. The arguments for the case $\ell = 0$ and for the induction step are very similar; we shall give both in parallel as much as possible.

Given the cohomology of Stiefel varieties, it follows immediately from 46 that the $E_2$-page is as claimed.

As usual, arguments from Chow height eliminate all differentials not generated by $d_i(\rho_i)$. The argument in the proof will be to take $\mathbb{Z}/p$-coefficients, then verify equation (3.5) for infinitely many primes $p$. The case of $\mathbb{Z}\left[\frac{1}{2}\right]$ coefficients then follows.

Consider the integer polynomial

$$f(x) = x^{n-m+1} - s_1 x^{n-m} + s_2 x^{n-m-1} + \cdots + (-1)^{n-m-1} s_{n-m-1} x + (-1)^{n-m} s_{n-m}$$

By the a corollary of the Frobenius density theorem, [Sur03], this polynomial splits over $\mathbb{Z}/p$ for infinitely many odd primes $p$. Let $\mathcal{P}$ denote the set of all such primes. We will first establish equation (3.5) for all $p \in \mathcal{P}$.

To show the problem reduces well to prime coefficients, we need only observe that the natural map on spectral sequences induced by $\mathbb{Z}\left[\frac{1}{2}\right] \to \mathbb{Z}/p$ commutes with differentials, and the description of the $E_2$-page is good for $\mathbb{Z}\left[\frac{1}{2}\right]$ and $\mathbb{Z}/p$ coefficients.

Working modulo a particular prime, $p \in \mathcal{P}$, we write $d_k(\hat{\rho}_k) = C\theta^k$, by abuse of notation, where all the terms are understood as the reduction mod $p$ of their $\mathbb{Z}\left[\frac{1}{2}\right]$ analogues. By the choice of the prime $p$, we can find $\mathbf{v}' = (v'_1, \ldots, v'_{n-m})$, roots of the polynomial $f(x)$ in $Z/p$. We chose particular integer representatives for the $v'_i$, denoting them by $v'_i$, again by abuse of notation. We note that this choice of $v'_i$ has been made so that $\sigma_i(\mathbf{v}') \equiv s_i \pmod{p}$. We work exclusively with $\mathbb{Z}/p$ coefficients from now on. We therefore have $\sigma_i(\mathbf{v}') = s_i$, which is key to the whole argument.

Consider the $\mathbb{G}_m$-action on GL$(n)$ given by weights $(u_1, \ldots, u_n)$ on the left and weights $(v_1, \ldots, v_m, v'_1, \ldots, v'_{n-m})$ on the right. Let $\pi$ denote the projection map $\pi : \mathrm{GL}(n) \to W(n,m)$ we obtain by considering only the first $m$ columns. This map is $\mathbb{G}_m$-equivariant for the given actions.

For the purposes of computing with the $\mathbb{G}_m$-action on GL$(n)$, it will be convenient to define $\mathbf{v}''$ as the concatenation of $\mathbf{v}$ and $\mathbf{v}'$, that is $(v_1, \ldots, v_m, v'_1, \ldots, v'_{n-m})$. The spectral sequence arising from the $\mathbb{G}_m$ action on GL$(n)$ has $E_2$ page, by 52,

$$ {}_{II}E_2 = \Lambda_{\mathbb{M}_p}(\hat{\rho}_n, \ldots, \hat{\rho}_1)[\theta] $$

since in particular, with $\mathbb{Z}/p$-coefficients, one has $\sum u_i - (\sum v_i + \sum v'_i) = 0$. In fact, the first nonvanishing differential is given by $d_j(\hat{\rho}_j) = [\sigma_j(\mathbf{u}) - \sigma_j(\mathbf{v}'')]\theta^j$, where $j$ is the least positive integer such that $\sigma_j(\mathbf{u}) - \sigma_j(\mathbf{v}'') \neq 0$.

Suppose for the sake of contradiction that this $j \leq n - m$.

$$ d_j(\hat{\rho}_j) = \left[\sigma_j(\mathbf{u}) - \sigma_j(\mathbf{v}'')\right]\theta^j = \left[\sigma_j(\mathbf{u}) - \sum_{i=1}^{j} \sigma_i(\mathbf{v})\sigma_{j-i}(\mathbf{v}')\right]\theta^j = \left[\sigma_j(\mathbf{u}) - \sum_{i=1}^{j} \sigma_i(\mathbf{v}) s_{j-i}\right]\theta^j $$

The term on the RHS is 0, by the definition of $s_{j-i}$, a contradiction.

If we are in the case where $k > n - m + 1$, that is, not the base case for the purpose of induction, we suppose for the sake of contradiction that $j < k$. Again we have

$$d_j(\hat{\rho}_j) = \left[\sigma_j(\mathbf{u}) - \sigma_j(\mathbf{v}'')\right]\theta^j = \left[\sigma_j(\mathbf{u}) - \sum_{i=1}^{j} \sigma_i(\mathbf{v})\sigma_{j-i}(\mathbf{v}')\right]\theta^j = \left[\sigma_j(\mathbf{u}) - \sum_{i=1}^{j} \sigma_i(\mathbf{v})s_{j-i}\right]\theta^j$$

but here we know that, by applying the result to compute $d_j(\hat{\rho}_j)$, that we have $d_j(\hat{\rho}_j) = \left[\sigma_j(\mathbf{u}) - \sum_{i=1}^{j} \sigma_i(\mathbf{v})s_{j-i}\right]\theta^j$. Since $d_j(\hat{\rho}_j)$ was assumed to be 0 for $j$ in the range $n - m < j < k$, we have a contradiction.

We return to considering both cases. The purpose of the inductive argument was to be able to assert that $d_j(\hat{\rho}_j) = 0$ for $j < k$ in the spectral sequence for $\mathrm{GL}(n)$. This allows us to compute the differential $d_k(\hat{\rho}_k)$. With $\mathbb{Z}/p$-coefficients we obtain,

$$d_k(\hat{\rho}_k) = [\sigma_k(\mathbf{u}) - \sigma_k(\mathbf{v}'')]\theta^k = \left[\sigma_k(\mathbf{u}) - \sum_{j=1}^{k} \sigma_j(\mathbf{v})\sigma_{k-j}(\mathbf{v}')\right]\theta^k = \left[\sigma_k(\mathbf{u}) - \sum_{j=1}^{k} \sigma_j(\mathbf{v})s_{k-j}\right]\theta^k$$

We again consider the equivariant cohomology of $W(n, m)$. From the comparison map

$$\begin{array}{ccc}
\mathrm{Ext}_{\hat{H}^{*,*}(\mathbb{G}_m;\mathbb{Z}/p)}(\hat{H}^{*,*}(W(n,m);\mathbb{Z}/p), \mathbb{M}_p) & \Longrightarrow & H^{*,*}(B(\mathrm{pt}, \mathbb{G}_m, W(n,m)); \mathbb{Z}/p) \\
\downarrow{\scriptstyle \mathrm{id}\otimes\pi^*} & & \downarrow \\
\mathrm{Ext}_{\hat{H}^{*,*}(\mathbb{G}_m;\mathbb{Z}/p)}(\hat{H}^{*,*}(\mathrm{GL}(n);\mathbb{Z}/p), \mathbb{M}_p) & \Longrightarrow & H^{*,*}(B(\mathrm{pt}, \mathbb{G}_m, \mathrm{GL}(n)); \mathbb{Z}/p)
\end{array}$$

we obtain a commutative square

$$\begin{array}{ccc}
\hat{\rho}_k & \xmapsto{d_{2k}} & C\theta^k \\
\downarrow & & \downarrow \\
\hat{\rho}_k & \xmapsto{d_{2k}} & \left[\sigma_k(\mathbf{u}) - \sum_{j=1}^{k} \sigma_j(\mathbf{v})s_{k-j}\right]\theta^k
\end{array}$$

which proves that $C = \left[\sigma_k(\mathbf{u}) - \sum_{j=1}^{k} \sigma_j(\mathbf{v}) s_{k-j}\right]$ in $\mathbb{Z}/p$, or equivalently that we have

$$p \mid \left(C - \left[\sigma_k(\mathbf{u}) - \sum_{j=1}^{k} \sigma_j(\mathbf{v}) s_{k-j}\right]\right)$$

Since there are infinitely many $p \in \mathcal{P}$, and this relation holds for them all, it follows that

$$C = \left[\sigma_k(\mathbf{u}) - \sum_{j=1}^{k} \sigma_j(\mathbf{v}) s_{k-j}\right]$$

as required. □

Unfortunately this method of proof establishes only the first non-zero differential of the form $d_k(\hat{\rho}_k) = C\theta^k$, we cannot push it further to describe the subsequent differentials. We conjecture that the pattern established in the theorem continues, that the differential takes the form

$$d_k(\hat{\rho}_k) = \left[\sigma_k(\mathbf{u}) - \sum_{j=1}^{k} \sigma_j(\mathbf{v}) s_{k-j}\right] \theta^k$$

modulo the appropriate indeterminacy for all $k$.

# Chapter 4

# Varieties of Long Exact Sequences

## 4.1 The Variety of Long Exact Sequences

We continue to fix a perfect field $k$ throughout this chapter. We continue to use the notation $H^{*,*}(X)$ for motivic cohomology taken with coefficients in an unspecified commutative ring $A$ with $1/2 \in A$. All notational conventions from the previous chapter are still in effect.

We consider long exact sequences of graded $k$-vector spaces, for instance:

$$0 \longrightarrow A_1 \xrightarrow{d_1} B_1 \xrightarrow{e_1} A_2 \xrightarrow{d_2} \cdots \xrightarrow{e_{n-1}} A_n \xrightarrow{d_n} B_n \longrightarrow 0 \tag{4.1}$$

since there are $2n$ terms to this sequence, we designate this as the *even* case, the odd case will be that where the last term is $A_n$. We shall treat mainly the even case in what follows, the odd case is generally much the same and we shall try to spare the reader by proving each result only once.

Our initial treatment does not involve the grading, and is equally true of the ungraded case. Later, the grading will play a meaningful role.

We write $a_i$ for $\dim_k A_i$ and $b_i$ for $\dim_k B_i$. We will construct a variety that parametrizes such sequences. We will denote this variety by $X(a_1, \ldots, a_n; b_1, \ldots b_n)$ or by $X$ when brevity is important.

$X$ is really the variety parametrizing all possible matrices representing $(d_1, \dots, d_n)$ and $(e_1, \dots, e_{n-1})$. We begin therefore with the large affine space

$$Y = \prod \mathbb{A}^{a_i \times b_i} \times \prod \mathbb{A}^{b_i \times a_{i+1}}$$

If $R$ is a $k$-algebra, then $Y(R)$ is the set of pairs of finite sequences $(d_1, \dots, d_n)$, $(e_1, \dots, e_{n-1})$ of matrices with entries in $R$ of the appropriate dimensions. The conditions $d_i e_i = 0$ and $e_{i-1} d_i = 0$ are polynomial conditions, and so there exists a closed subvariety $Z \rightarrowtail Y$ whose $R$-points are sequences of matrices whose products are 0, viz. chain complexes of the form

$$0 \longrightarrow R^{b_n} \xrightarrow{d_n} R^{a_n} \xrightarrow{e_{n-1}} R^{b_{n-1}} \xrightarrow{d_{n-1}} \cdots \xrightarrow{e_1} R^{b_1} \xrightarrow{d_1} R^{a_1} \longrightarrow 0$$

but we should like to have a variety of exact sequences. The variety $Z$ is interesting in its own right, but it is unfortunately singular, and therefore less tractable by the methods we have at our disposal.

We define the rank of a matrix, $d$, naïvely. If the matrix is defined over a field, it is the largest integer $\ell$ for which the matrix $d$ has some $\ell$-th minor which is a unit. Over a more complicated ring, the rank is of course less easy to define, and we shall try to avoid working with rank conditions over arbitrary $k$-algebras. On the other hand, however, the condition of a particular minor's vanishing is a polynomial one in the entries of the matrix, so we can define open subschemes as complements of loci where certain minors vanish.

The schemes we work with are in fact locally closed varieties in $\mathbb{A}^N_k$ for some sufficiently large $N$. We shall at least once have occasion to exploit this. If we wish to define a map $X \to Y$ of schemes, and if $Y$ is locally closed in $\mathbb{A}^N$, we need only define a map $X \to \mathbb{A}^N$ and then verify that $X(\bar{k}) \to \mathbb{A}^N(\bar{k})$ factors through $Y(\bar{k}) \to \mathbb{A}^N(\bar{k})$. To see this, consider the two cases of $Y$ being either closed or open in an ambient variety $A$ to which

$X$ maps, and construct the pull-back

$$\begin{array}{ccc} X_0 & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & A \end{array}$$

Note that $X_0$ is (depending on the case) either closed or open in $X$. Moreover, if, by hypothesis every geometric point of $X$ maps to a geometric point of $Y$ in $A$, then every geometric point of $X$ is the image of some geometric point of $X_0$. It follows that $X_0 = X$. By factoring an arbitrary locally-open $Y \to \mathbb{A}^N$ into a closed subscheme map followed by an open immersion, the result follows.

This allows the following style of argument, as presaged in lemmas 1 and 2. To define a map $X \to Y$, we first use Yoneda's lemma to construct a natural map $X(R) \to \mathbb{A}^N(R)$ and then verify that if $R = \bar{k}$ than the resulting map actually factors $X(\bar{k}) \to Y(\bar{k}) \to \mathbb{A}^N(\bar{k})$, which in practice allows us to consider linear algebra only over an (algebraically closed) field.

Note that if $\bar{k}^a \xrightarrow{d} \bar{k}^b \xrightarrow{e} \bar{k}^c$ is a sequence of spaces with $ed = 0$, we have $\text{rank}(d) + \text{rank}(e) \leq d$.

What we mean by exactness is that the matrix direct-sum $\bigoplus d_i \oplus \bigoplus e_i$ should have the largest possible rank compatible with the condition $e_{i-1}d_i = 0$ and $d_i e_i = 0$ respectively, which is to say $\text{rank}\,(d_i) + \text{rank}\,(e_i) = b_i$ and $\text{rank}\,(e_{i-1}) + \text{rank}\,(d_i) = a_i$. This, combined with the convention that $e_0 = 0$ has rank 0, gives a recursive formula for the ranks of $d_i$, $e_i$, writing $r_i$ for this maximal rank for $d_i$, and $s_i$ for the maximal rank of $e_i$ it is

$$r_i = a_i - s_{i-1}$$
$$s_i = b_i - r_i$$

To demand that an $a \times b$ matrix have rank at least $r$ on closed points is an open condition, since demanding that any particular $r$-th minor be a unit is an open condition. Let $I$ index the set of $r$-th minors of an $a \times b$-matrix, that is to say, an element of $I$ is a choice of $r$ columns and $r$ rows. Let $U_i$ denote the subvariety determined by the condition that the $i$-th minor is a unit. We write $M(a,b;r)$ for the subvariety of $\mathbb{A}^{a\times b}$ covered by the open subvarieties $U_i$. It is clear that $M(a,b;r)$ is an open subvariety of $\mathbb{A}^{a\times b}$.

It follows that $M = \prod M(a_i, b_i; r_i) \times \prod M(b_{i-1}, a_i; s_i)$ is an open subscheme of $Y$, and that $M \cap Z$ gives a locally closed subscheme of $Y$, and open subscheme of $Z$. We write $X$ for $M \cap Z$; it is the scheme whose $R$ points are complexes

$$0 \longrightarrow R^{b_n} \xrightarrow{d_n} R^{a_n} \xrightarrow{e_{n-1}} R^{b_{n-1}} \xrightarrow{d_{n-1}} \cdots \xrightarrow{e_1} R^{b_1} \xrightarrow{d_1} R^{a_1} \longrightarrow 0$$

where the matrix direct-sum $\bigoplus d_i \oplus \bigoplus e_i$ has the maximal possible rank. If $F$ is a field, then this condition is obviously equivalent to the condition that the $X(F)$-points be long exact sequences.

In order to attain exactness (i.e. so that $X(k)$ is nonempty) we demand that the Euler characteristic of the sequence vanishes, which is to say $\sum a_i = \sum b_i$. We also do not allow the degenerate case where $s_i = 0$ for any $i$, or where $r_i = 0$ for $i < n$. As a technical convenience we allow $r_n = 0$ and $b_n = 0$.

## 4.2 Presenting $X$ as a Homogeneous Space

In this section, we write $G = \mathrm{GL}(b_1, a_2, \ldots, b_{n-1}, a_n, b_n)$ as a shorthand for

$$G = \mathrm{GL}(a_1, b_1, \ldots, b_{n-1}, a_n) = \mathrm{GL}(b_1) \times \mathrm{GL}(a_2) \times \cdots \times \mathrm{GL}(a_n) \times \mathrm{GL}(b_n)$$

We consider the space of $M(a,b;r)$, of rank $r$ $a \times b$ matrices, and $\mathrm{GL}(a,b)$. There is an action

$$\mathrm{GL}(a,b) \times M(a,b;r) \longrightarrow M(a,b;r)$$

which on $R$-points is given by

$$(x,y) \cdot d \mapsto xdy^{-1}$$

and we can extend this action to an action of $G$ on $X(a_1, \ldots, a_n, b_1 \ldots, b_n)$. On $R$-points this action is given by

$$(y_1, x_2, \ldots, y_n) \cdot (d_1, e_1, \ldots, e_{n-1}, d_n) = (d_1 y_1^{-1}, y_1 e_1 x_2^{-1}, \ldots, y_{n-1} e_{n-1} x_n^{-1}, x_n d_n y_n^{-1})$$

(note that the $x_1$ -term one might expect is omitted, equivalently one can consider a $x_1$-term to exist but always to be the identity matrix). We shall prove that $G \times X \to X$ is a transitive action and gives rise to a presentation of $G$ as a Nisnevich sheaf quotient of $G$ by a closed subgroup $H$.

We define $u_i$ and $v_i$ as the $a_i \times b_i$ and $b_i \times a_{i+1}$ matrices with entries in $k$

$$u_i = \begin{pmatrix} 0 & 0 \\ I_{r_i} & 0 \end{pmatrix}, \quad v_i = \begin{pmatrix} 0 & 0 \\ I_{s_i} & 0 \end{pmatrix}$$

Plainly $(u_1, v_1, \ldots, v_{n-1}, u_n)$ is a $k$-point of $X(a_1, \ldots, a_n; b_1, \ldots, b_n)$, and we denote this by $u : \mathrm{pt} \to X$. We therefore have

$$G \cong G \times \mathrm{pt} \longrightarrow G \times X \longrightarrow X \tag{4.2}$$

which on $R$-points sends $g \in G$ to $g \cdot u$.

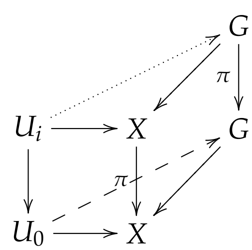
**Proposition 55.** *With notation as above, there is a Zariski cover $c : \mathcal{U} \to X$ such that the pull-back of $G \to X$, denoted $c^{-1}(G) \to \mathcal{U}$, splits.*

*Proof.* Recall that $X$ represents $2n$-tuples of matrices $(d_1, e_1, \ldots, e_{n-1}, d_n)$ satisfying $d_i e_i = e_i d_{i+1} = 0$ and with a rank condition on the $d_i, e_i$, By construction $r_i + s_i = b_i$ and $s_i + r_{i+1} = a_{i+1}$. For each $i$, choose $r_i$ integers in $\{1, \ldots, a_i\}$ and denote this set by $P_i$ and choose $s_i$ integers in $\{1, \ldots, b_i\}$ and denote this set by $Q_i$. We denote by $U$ the open subvariety of $X$ whose $R$-points are matrices $(d_1, e_1, \ldots, e_{n-1}, d_n)$ where the submatrix of $d_i$ given by the columns indexed by $P_i$ and the rows indexed by $\{1, \ldots, b_i\} \setminus Q_i$ has full rank, and the submatrix of $e_i$ given by the columns indexed by $Q_i$ and the rows indexed by $\{1, \ldots, a_{i+1}\} \setminus P_{i+1}$ has full rank. The variety $U$ is plainly open in $X$, and the union of all such $U$ cover $X$. We write $\mathcal{U}$ for the disjoint union of all such $U$.

We choose a distinguished open set, $U_0$, where $P_i = \{1, \ldots, r_i\}$ and $Q_i = \{1, \ldots, s_i\}$. The basepoint $u : \mathrm{pt} \to X$ factors through $u : \mathrm{pt} \to U \to X$. All of the open sets $U_i$ in our cover differ from this one by consistent permutation of rows and columns in the matrices $d_i$, $e_i$, and any such consistent permutation of rows and columns can be effected by the action of a rational point $\pi : \mathrm{pt}_k \to G$ given by permutation matrices. The action is via $\mathrm{pt} \times X \to G \times X \to X$, and therefore is an isomorphism $X \to X$ which we denote by $\pi$. Similarly, $\pi : \mathrm{pt} \times G \to G \times G \to G$ gives an automorphism of $G$. The square

$$\begin{array}{ccc} G & \xrightarrow{\pi} & G \\ \downarrow & & \downarrow \\ X & \xrightarrow{\pi} & X \end{array}$$

commutes. For any $U_i$ in our open cover, therefore, there is a map $\pi : \mathrm{pt} \to G$ for which $U_i \to X$ is the pull-back of the square below




It suffices to construct the section denoted by the dashed arrow above, which is to say, a map $U_0 \to G$ such that the triangle containing $U_0, X$ and $G$ commutes. The existence of the dotted arrow then follows by abstract nonsense.

We describe the map $U_0 \to G$. Let $R$ be a $k$-algebra and let $(d_1, e_1, \ldots, e_{n-1}, d_n)$ be a sequence of matrices in $U_0(R)$. We decompose

$$d_i = \begin{pmatrix} D_{i,1} & D_{i,2} \\ D_{i,3} & D_{i,4} \end{pmatrix}, \quad e_i = \begin{pmatrix} E_{i,1} & E_{i,2} \\ E_{i,3} & E_{i,4} \end{pmatrix}$$

where $D_{i,3}$ (resp. $E_{i,3}$) is an $r_i \times r_i$- (resp. $s_i \times s_i$-) matrix satisfying a rank condition. Let

$x_1 = I_{a_1}$. Let

$$\bar{y}_i = \begin{pmatrix} D_{i,3} & D_{i,4} \\ 0 & I_{b_i - r_i} \end{pmatrix}, \quad \bar{x}_i = \begin{pmatrix} E_{i,3} & E_{i,4} \\ 0 & I_{a_{i+1} - s_i} \end{pmatrix}$$

*A priori* we have constructed a map $U_0 \to \mathbb{A}^N$ where $N$ is some colossal affine space in which $G$ is dense. If we consider the situation now with $R = \overline{k}$, the rank condition simply means that $D_{i,3}$ and $E_{i,3}$ are invertible matrices, and so $\bar{x}_i$, $\bar{y}_i$ are invertible matrices. We define $x_i = \bar{x}_i^{-1}$ and $y_i = \bar{y}_i^{-1}$, and this is a valid definition of a map $U_0 \to G$, by lemma 2.

We also observe that for reasons of exactness, $d_1$, which is an $a_1 \times b_1$-matrix has rank $r_1 = a_1$. It follows that it decomposes in particular as

$$d_1 = \begin{pmatrix} D_{1,3} & D_{1,4} \end{pmatrix}$$

where $D_{1,3}$ is a full-rank $r_1 \times r_1$-matrix. We have

$$\begin{pmatrix} I_{r_1} & 0 \end{pmatrix} y_1^{-1} = d_1$$

as required, and now, starting with the case $i = 1$ and proceeding from $d_1$ to $e_1$ to $d_2$ and so on, by elementary matrix-multiplication arguments which we suppress, we obtain

$$\begin{pmatrix} 0 & 0 \\ I_{r_i} & 0 \end{pmatrix} \begin{pmatrix} D_3 & D_4 \\ 0 & I_{b_i - r_i} \end{pmatrix} = \begin{pmatrix} E_3 & E_4 \\ 0 & I_{a_{i+1} - s_i} \end{pmatrix} d_i \tag{4.3}$$

which is to say $\begin{pmatrix} 0 & 0 \\ I_{r_i} & 0 \end{pmatrix} y_i^{-1} = x_i^{-1} d_i$, and similarly we deduce $y_i \begin{pmatrix} 0 & 0 \\ I_{s_i} & 0 \end{pmatrix} x_{i+1}^{-1} = e_i$.

What we have shown is that for any point $(d_1, \ldots, e_{n-1}, d_n)$ in $U_0(R)$, we can construct $(y_1, x_2 \ldots, y_{n-1}, x_n, y_n) \in G(R)$ such that $(y_1, \ldots, x_n, y_n) \mapsto (d_1, \ldots, e_{n-1}, d_n)$ . The construction

$$(d_1, \ldots, e_{n-1}, d_n) \longmapsto (y_1, x_2 \ldots, y_{n-1}, x_n, y_n)$$

is plainly natural in $R$, and so is given by a map of representing objects, viz. a scheme map

$U_0 \to G$. Equation (4.3) is the assertion that the triangle

$$\begin{array}{ccc} & & G \\ & \nearrow & \downarrow \\ U_0 & \longrightarrow & X \end{array}$$

commutes, or in other words, that this is a local splitting. This completes the proof. □

We remind the reader that we wish to present $X$ as $G/K$ for some closed subgroup $K$. To this end, define $K$ as the pullback

$$\begin{array}{ccc} K & \longrightarrow & G \\ \downarrow & & \downarrow \\ \mathrm{pt} & \xrightarrow{u} & X \end{array}$$

which is to say, $K$ is the stabilizer subgroup of the point $u$.

**Proposition 56.** *With notation as above, $K$ is $\mathbb{A}^1$-equivalent to $\mathrm{GL}(s_1, \ldots, s_{n-1}, r_n, s_n)$ and the inclusion $K \to G$ is homotopy equivalent to the map*

$$\mathrm{GL}(s_1, r_2 \ldots, s_{n-1}, r_n, s_n) \to \mathrm{GL}(b_1, a_2, \ldots, b_{n-1}, a_n, b_n) = G$$

*that sends*

$$(g_1, f_2, \ldots, g_{n-1}, f_n, g_n) \mapsto \left( \begin{pmatrix} I_{r_1} & 0 \\ 0 & g_1 \end{pmatrix}, \begin{pmatrix} g_1 & 0 \\ 0 & f_2 \end{pmatrix}, \ldots, \begin{pmatrix} g_{n-1} & 0 \\ 0 & f_n \end{pmatrix}, \begin{pmatrix} f_n & 0 \\ 0 & g_n \end{pmatrix} \right)$$

*on points*

*Proof.* It is not difficult to calculate $K$ explicitly. Again, we work with $R$-points. Suppose

$(y_1, x_2, \dots, y_{n-1}, x_n, y_n) \in G(R)$ fixes the point $u \in X(R)$. We have

$$x_i \begin{pmatrix} 0 & 0 \\ I_{r_i} & 0 \end{pmatrix} y_i^{-1} = \begin{pmatrix} 0 & 0 \\ I_{r_i} & 0 \end{pmatrix}$$
$$y_i \begin{pmatrix} 0 & 0 \\ I_{s_i} & 0 \end{pmatrix} x_{i+1}^{-1} = \begin{pmatrix} 0 & 0 \\ I_{s_i} & 0 \end{pmatrix}$$

and it follows from this that $x_i$ and $y_i$ decompose as

$$x_i = \begin{pmatrix} f_i & 0 \\ * & g_i \end{pmatrix}, \quad y_i = \begin{pmatrix} g_i & 0 \\ * & f_{i+1} \end{pmatrix}$$

where $f_i, g_i$ are invertible matrices of size $r_i \times r_i$ and $s_i \times s_i$ respectively and $*$ indicates the entries in question can take on any value in $R$. We have therefore described $K$ as the variety representing sequences of matrices

$$(g_1, f_2, \dots, g_{n-1}, f_n, g_n) \mapsto \left( \begin{pmatrix} I_{a_1} & 0 \\ * & g_1 \end{pmatrix}, \begin{pmatrix} g_1 & 0 \\ * & f_2 \end{pmatrix}, \dots, \begin{pmatrix} g_{n-1} & 0 \\ * & f_n \end{pmatrix}, \begin{pmatrix} f_n & 0 \\ * & g_n \end{pmatrix} \right)$$

(the case at the beginning being a degenerate instance of the general rule). Plainly the map which on $R$-points is

$$(g_1, f_2, \dots, g_{n-1}, f_n, g_n) \mapsto \left( \begin{pmatrix} I_{a_1} & 0 \\ 0 & g_1 \end{pmatrix}, \begin{pmatrix} g_1 & 0 \\ 0 & f_2 \end{pmatrix}, \dots, \begin{pmatrix} g_{n-1} & 0 \\ 0 & f_n \end{pmatrix}, \begin{pmatrix} f_n & 0 \\ 0 & g_n \end{pmatrix} \right)$$

induces an $\mathbb{A}^1$-equivalence $\mathrm{GL}(s_1 \dots, s_{n-1}, r_n, s_n) \to K$, since its image is a deformation retract of $K$. In composition $\mathrm{GL}(s_1 \dots, s_{n-1}, r_n, s_n) \to K \to G$, this map is exactly as described. □

The transitivity of the $G$-action implies that the variety $X$ is in fact smooth over pt, a fact we have not yet proved, but we shall take for granted. It is a basic result in the geometry of varieties that $X$ is generically smooth, which is to say that there exists a dense open $U \subset X$

such that $U$ is smooth over pt, see [Har77]. On the other hand, because of the transitivity of the $G(\overset{k}{})$-action, if $X$ fails to be regular at a geometric point $x_0 \to X \setminus U$, then $X$ fails to be regular at all geometric points, contradicting the density of $U$.

## 4.3 The Non-Equivariant Cohomology

We are now in a position to apply some of the machinery we developed in previous chapters. We observe that $K$ as defined above is the stabilizer of a point under an action by a group scheme $G$ on $X$, and in the terminology of section 2.5, $G \times K \to G \to X$ is a pseudobundle. The existence of local sections of the map $G \to X$ is the other ingredient necessary for the identification of $X$ and $G/K$ as Nisnevich sheaves, which will allow us to compute the cohomology of the former.

We remind the reader that the schemes $\mathrm{GL}(n)$ and their products are cellular in the sense of [DI05], and their cohomology is $H^{*,*}(\mathrm{GL}(n); A) = \Lambda_{\mathbb{M}}(\alpha_n, \alpha_{n-1}, \ldots, \alpha_1)$. In particular, it is free as an $\mathbb{M}$-module. Therefore a particularly simple Künneth formula applies and we can write the cohomology of $G = \mathrm{GL}(a_1, b_1, \ldots, a_n, b_n)$ as

$$H^{*,*}(\mathrm{GL}(a_1, b_1, \ldots, a_n) = \prod_{i=2}^{n} H^{*,*}(\mathrm{GL}(a_i)) \times \prod_{i=1}^{n} H^{*,*}(\mathrm{GL}(b_i)) \tag{4.4}$$

which is itself an exterior algebra, on classes $\alpha_{i,j}, \beta_{\ell,k}$ where $2 \leq i \leq n$, $1 \leq \ell \leq n$ and $1 \leq j \leq a_i$ and $1 \leq k \leq b_i$. The projection maps $G \to \mathrm{GL}(a_i)$ and the inclusion maps $\mathrm{GL}(a_i) \to G$ both behave entirely as expected on cohomology. Moreover, these maps being group homomorphisms, they respect the Hopf-algebra structure on cohomology. By comparison with proposition 19 we immediately deduce the Hopf algebra structure on $H^{*,*}(G)$, which is very simple

$$\alpha_{i,j} \mapsto 1 \otimes \alpha_{i,j} + \alpha_{i,j} \otimes 1, \qquad \beta_{i,j} \mapsto 1 \otimes \beta_{i,j} + \beta_{i,j} \otimes 1$$

At times it is more convenient to distinguish the odd and even terms in the long exact sequences, and at other times it seems more convenient to view them as being of a kind.

We will occasionally therefore use the notation

$$\alpha_{i,j} = \gamma_{2i-1,j}, \quad \beta_{i,j} = \gamma_{2i,j}, \quad a_i = c_{2i-1}, \quad b_i = c_{2j}.$$

A similar argument to that used to compute $H^{*,*}(G)$ computes

$$H^{*,*}(K;R) = \prod_{i=2}^{n} \Lambda_R(\hat{\rho}_{i,1}, \ldots, \hat{\rho}_{i,r_i}) \times \prod_{i=1}^{n} \Lambda_R(\hat{\sigma}_{i,1}, \hat{\sigma}_{i,2}, \ldots, \hat{\sigma}_{i,s_i})$$

with again the expected Hopf algebra structure

$$\rho_{i,j} \mapsto 1 \otimes \rho_{i,j} + \rho_{i,j} \otimes 1, \qquad \sigma_{i,j} \mapsto 1 \otimes \sigma_{i,j} + \sigma_{i,j} \otimes 1$$

Again we may occasionally write

$$\rho_{i,j} = \tau_{2i-1,j}, \quad \sigma_{i,j} = \tau_{2i,j}, \quad r_i = t_{2i-1}, \quad s_i = t_{2i}$$

we shall also use the convention $t_0 = 0$, so that for all $c_i$ there is a corresponding $t_{i-1}$.

The Hopf-algebra structures on $H^{*,*}(G)$ and $H^{*,*}(K)$ being very simple, it is easy to compute the corresponding ring structures on $\hat{H}^{*,*}(G) = \mathrm{Hom}_{\mathbb{M}}(H^{*,*}(G), \mathbb{M})$ and also on $\hat{H}^{*,*}(K) = \mathrm{Hom}_{\mathbb{M}}(H^{*,*}(K), \mathbb{M})$. Both are again exterior algebras on generators we denote $\hat{\alpha}_{i,j}, \hat{\beta}_{i,j}$ and $\hat{\rho}_{i,j}, \hat{\sigma}_{i,j}$ respectively, which are the duals of their un-hatted namesakes with respect to the obvious bases of monomials in $H^{*,*}(G)$ and in $H^{*,*}(K)$. We denote these duals of cohomology classes also by $\hat{\gamma}_{i,j}$ and $\hat{\tau}_{i,j}$ whenever it seems more convenient.

The inclusion of $K$ in $G$ induces the following map on the duals of cohomology

$$\begin{array}{ccc} \hat{H}^{*,*}(K;R) & \xrightarrow{\iota_*} & \hat{H}^{*,*}(G;R) \\ \| & & \| \\ \prod_{i=1}^{2n-1} \Lambda_R(\hat{\tau}_{i,1}, \ldots, \hat{\tau}_{i,t_i}) & \xrightarrow{\iota_*} & \prod_{i=1}^{2n-1} \Lambda_R(\hat{\gamma}_{i,1}, \ldots, \hat{\gamma}_{i,c_i}) \end{array} \tag{4.5}$$

$$\iota_*(\hat{\tau}_{i,j}) = \hat{\gamma}_{i-1,j} + \hat{\gamma}_{i,j}$$

which is a map of Hopf algebras, in particular of rings. By a change of coordinates, replacing $\hat{\gamma}_{i,j}$ by $\iota_*(\hat{\tau}_{i,j})$ in our presentation of $\hat{H}^{*,*}(G;R)$, equation (4.4) and subsequent remarks, we see that $\hat{H}^{*,*}(G;R)$ is itself an exterior algebra over $\hat{H}^{*,*}(K;R)$. We find a set $\hat{N}$ so that $\hat{H}^{*,*}(G;R) = \Lambda_{\hat{H}^{*,*}(K;R)}(\hat{N})$.

Let $i$ be an integer satisfying $1 \le i \le 2n$. Suppose $j$ is an integer satisfying $t_{i-1} < j \le c_i$. We define $\ell$ to be the least integer $i \le \ell$ such that $t_\ell < j$, then define

$$\hat{\kappa}_{i,j} = \sum_{k=i}^{\ell} (-1)^{k-1} \hat{\gamma}_{k,j}. \tag{4.6}$$

We denote the set of all $\hat{\kappa}_{i,j}$ by $\hat{N}(X)$, or by $\hat{N}$ when the dependence on $X$ is clear. We also denote by $\kappa_{i,j}$ the duals of the $\hat{\kappa}_{i,j}$, with respect to the evident basis already given to $H^{*,*}(G)$ and $\hat{H}^{*,*}(G)$. We write the set of all $\kappa_{i,j}$ as $N$.

A pictorial description of the $\kappa_{i,j}$ is perhaps of use. Take for example the case where $(c_1, c_2, c_3, c_4, c_5, c_6) = (1,4,5,4,3,1)$ and $(t_1, t_2, t_3, t_4, t_5) = (1,3,2,2,1)$. Pictorially we denote this as

$$\begin{array}{ccccccccc}
 & & \gamma_{3,5} & & & & & & \\
\gamma_{2,4} & & \gamma_{3,4} & & \gamma_{4,4} & & & & \\
\gamma_{2,3} & — & \gamma_{3,3} & & \gamma_{4,3} & & \gamma_{5,3} & & \\
\gamma_{2,2} & — & \gamma_{3,2} & — & \gamma_{4,2} & — & \gamma_{5,2} & & \\
\gamma_{2,1} & — & \gamma_{3,1} & — & \gamma_{4,1} & — & \gamma_{5,1} & — & \gamma_{6,1}
\end{array} \tag{4.7}$$

the columns arising from all but the first of the $c_i$ and the horizontal lines being numbered by all but the first of the $t_i$. In this case one has

$$\begin{aligned}
\kappa_{1,1} &= -\gamma_{2,1} + \gamma_{3,1} - \gamma_{4,1} + \gamma_{5,1} - \gamma_{6,1} \\
\kappa_{2,2} &= -\gamma_{2,2} + \gamma_{3,2} - \gamma_{4,2} + \gamma_{5,2} \\
\kappa_{2,3} &= -\gamma_{2,3} + \gamma_{3,3} \\
\kappa_{i,j} &= (-1)^{i+1}\gamma_{i,j} \quad \text{if } (i,j) \text{ is any of } (4,3), (5,3), (2,4), (3,4), (4,4) \text{ or } (5,3)
\end{aligned}$$

In practice, the question of signs is nugatory.

**Theorem 57.** *In the notation of this section, the cohomology of $X(a_1, \ldots, a_n, b_1, \ldots, b_n)$ (abbreviated $X$) is $\mathrm{Hom}_{\hat{H}^{*,*}(K;R)}(\hat{H}^{*,*}(G), \mathbb{M})$ and the map $G \to G/K \simeq X$ induces the evident injective map*

$$H^{*,*}(X) \cong \mathrm{Hom}_{\hat{H}^{*,*}(K)}(\hat{H}^{*,*}(G), \mathbb{M}) \hookrightarrow \mathrm{Hom}_{\mathbb{M}}(\hat{H}^{*,*}(G), \mathbb{M}) \cong H^*(G;R)$$

*and we have an identification*

$$\mathrm{Hom}_{\hat{H}^{*,*}(K;R)}(\hat{H}^{*,*}(G), \mathbb{M}) = \Lambda_{\mathbb{M}}(N) \subset H^{*,*}(G)$$

*Proof.* Since $X$ is the homogeneous space of cosets of $G$ by $K$, we can apply propositions 41 and 42, to obtain a spectral sequence which on the $E_2$-page is $\mathrm{Ext}^{p,q}_{\hat{H}^{*,*}(K)}(\hat{H}^{*,*}(G), \mathbb{M})$. The $\hat{H}^{*,*}(K)$-module structure of $\hat{H}^{*,*}(G)$ is given by (4.5), $H^{*,*}(G)$ is an exterior algebra over $\hat{H}^{*,*}(K)$, in particular it is a free $\hat{H}^{*,*}(K)$-module. The $E_2$-page is concentrated in the column $p = 0$, and it collapses thereafter. We therefore have

$$H^*(X;R) = \mathrm{Hom}_{\hat{H}^{*,*}(K)}(\hat{H}^{*,*}(G), \mathbb{M})$$

which is the first assertion.

We can calculate the comparison map $H^{*,*}(X) \to H^{*,*}(G)$ by taking $G$ to be $G/\{e\}$, the trivial quotient in the latter case, and using the naturality of the Rothenberg-Steenrod spectral sequence. For algebraic reasons the map is an inclusion of subrhomotopicalings, since $H^{*,*}(X)$ can be viewed as a restricted set of maps $\hat{H}^{*,*}(G) \to \mathbb{M}$. Since

$$H^{*,*}(X) = \mathrm{Hom}_{\hat{H}^{*,*}(K)}(\hat{H}^{*,*}(G), \mathbb{M})$$

and the latter is a direct sum of $\mathbb{M}$-modules isomorphic to $H^{*,*}(K)$, $H^{*,*}(X)$ is *a fortiori* a free $\mathbb{M}$-module.

To prove the last assertion, that $H^{*,*}(X) = \Lambda_{\mathbb{M}}(N) \subset H^{*,*}(G; R)$, we first remark that since the $\kappa_{i,j}$ are $\mathbb{M}$-independent and satisfy $\kappa_{i.j}^2 = 0$, the subalgebra of $H^{*,*}(G; R)$ they generate is exactly $\Lambda_{\mathbb{M}}(N)$.

Secondly, it is easily verified that the $\kappa_{i,j} \in \mathrm{Hom}_{\mathbb{M}}(\hat{H}^{*,*}(G), \mathbb{M})$ are in fact $\hat{H}^{*,*}(K; R)$-linear, so we certainly have an inclusion $\Lambda_{\mathbb{M}}(N) \subset H^{*,*}(X)$. We prove that we have an equality by reduction to the case of Chow height 1, i.e. those groups $H^{p,q}(G)$ with $2q - p = 1$. The algebra $\hat{H}^{*,*}(G)$ is generated in Chow height 1 by the $\hat{\gamma}_{i,j}$s as an $\mathbb{M}$-algebra, and consequently as an $\hat{H}^{*,*}(K)$- algebra. In fact, it is easily seen by explicit computation that $\hat{H}^{*,*}(G)$ is generated by $\hat{N}$ as an $\hat{H}^{*,*}(K)$-algebra. We also know that $\hat{\kappa}_{i,j}$ are linearly independent and satisfy $\hat{\kappa}_{i,j}^2 = 0$, so we have a description $\hat{H}^{*,*}(G) = \Lambda_{\hat{H}^{*,*}(K)}(\hat{N})$, and it follows from proposition 46 that $\mathrm{Ext}_{\hat{H}^{*,*}(K)}(\hat{H}^{*,*}(G), \mathbb{M}) = \Lambda_{\mathbb{M}}(N)$, as asserted. □

## 4.4 The Equivariant Cohomology

We suppose given the following data: for each $a_i$, an $a_i$-tuple of integers $\mathbf{v}_i = (v_{i,1}, \ldots, v_{i,a_i})$ and for each $b_i$ a similar $b_i$-tuple $\mathbf{w}_i = (w_{i,1}, w_{i,2}, \ldots, w_{i,b_i})$. We shall call these $v_{i,j}, w_{i,j}$s *weights*. We can put a $\mathbb{G}_m$-action on $X$ by means of these weights.

Explicitly, we consider a point of $X(R)$ as a sequence of matrices $(d_1, e_1, \ldots, d_n)$ and then the action is given by e.g.

$$z \cdot d_1 = \begin{pmatrix} z^{v_{1,1}} & & & \\ & z^{v_{1,2}} & & \\ & & \ddots & \\ & & & z^{v_{1,a_1}} \end{pmatrix} d_1 \begin{pmatrix} z^{-w_{1,1}} & & & \\ & z^{-w_{1,2}} & & \\ & & \ddots & \\ & & & z^{-w_{1,a_1}} \end{pmatrix} \tag{4.8}$$

and

$$z \cdot e_1 = \begin{pmatrix} z^{w_{1,1}} & & & \\ & z^{w_{1,2}} & & \\ & & \ddots & \\ & & & z^{w_{1,b_1}} \end{pmatrix} e_1 \begin{pmatrix} z^{-v_{2,1}} & & & \\ & z^{-v_{2,2}} & & \\ & & \ddots & \\ & & & z^{-v_{2,a_2}} \end{pmatrix}$$

and similarly for the other $d_i, e_i$, viz. it is an action on the left & right similar to that described in proposition 52.

When $X$ is thought of as coming equipped with this $\mathbb{G}_m$-action as it shall be for much of the rest of the chapter, and when we wish to make this explicit, we shall write

$$X = X(\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{w}_1, \ldots, \mathbf{w}_n) \tag{4.9}$$

Since the cohomology of $X$ is known, $H^{*,*}(X) = \Lambda_{\mathbb{M}}(N)$, we should like to be able to compute the equivariant cohomology $H^{*,*}(B(\mathrm{pt}, \mathbb{G}_m, X))$, at least so far as being able to write down a spectral sequence, which is almost within our grasp, and then to compute the differentials. The following two comparison results together allow us to do this.

The first is a form of splitting principle, since in its essentials it measures the difference in cohomology between sequences $0 \to A \to B \to C \to 0$ and $0 \to B \to A \oplus C \to 0$, by effecting the change one 1-dimensional summand at a time.

**Proposition 58.** *Suppose given*

$$X = X(a_1, \ldots, a_n, b_1, \ldots, b_n)$$
$$X' = X(a_1 - 1, a_2 + 1 \ldots, a_n, b_1, \ldots, b_n)$$

*Let $G, G'$ denote $\mathrm{GL}(b_1, a_2 \ldots, a_{n-1}, b_{n-1}, a_n, b_n)$ and $\mathrm{GL}(b_1, a_2 + 1, b_2 \ldots, b_{n-1}, a_n)$ respectively. Present $H^{*,*}(G)$ (resp. $H^{*,*}(G')$) in the way we have done heretofore, as $\Lambda_{\mathbb{M}}(\{\gamma_{i,j}\})$ for classes*

*$\gamma_{i,j}$ (resp. $\Lambda_{\mathbb{M}}(\{\gamma'_{i,j}\})$). Present $H^{*,*}(X)$ (resp. $H^{*,*}(X')$) as in the notation of theorem 57*

$$H^{*,*}(X) = \Lambda_{\mathbb{M}}(N) \subset H^{*,*}(G)$$
$$H^{*,*}(X') = \Lambda_{\mathbb{M}}(N') \subset H^{*,*}(G')$$

*There exists a zig-zag diagram of schemes*

$$X \xleftarrow{\simeq} W \longrightarrow X' \tag{4.10}$$

*In which the map $W \to X$ is an $\mathbb{A}^1$-weak equivalence. These maps fit in a diagram*

$$\begin{array}{ccccc} G & = & G & \longrightarrow & G' \\ \downarrow & & \downarrow & & \downarrow \\ X & \xleftarrow{\simeq} & W & \longrightarrow & X' \end{array}$$

*where the vertical maps on the right & left are the maps of diagram* (4.2)*, and the map $G \to G'$ is a standard inclusion. This allows the computation of the map $H^{*,*}(X) \to H^{*,*}(X')$.*

*Suppose further that weights $\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{w}_1, \dots, \mathbf{w}_n$ have been given, so there is a $\mathbb{G}_m$-action on X. Let*

$$\mathbf{w}'_i = \mathbf{w}_i$$
$$\mathbf{v}'_i = \mathbf{v}_i, \quad \textit{for } 2 < i$$
$$\mathbf{v}'_2 = (v_{2,1}, \dots, v_{2,a_2}, v_{1,a_1})$$
$$\mathbf{v}'_1 = (v_{1,1}, \dots, v_{1,a_1-1})$$

*so that the integer sequences $\mathbf{v}'_i$ and $\mathbf{w}'_i$ impose a $\mathbb{G}_m$-action on $X'$. Then there is a $\mathbb{G}_m$-action on W with respect to which the maps $X \longleftarrow W \longrightarrow X'$ are $\mathbb{G}_m$-equivariant.*

*Proof.* The proof will again use the artifice of constructing the maps using a Yoneda-type argument on $R$-points and then check the linear algebra conditions on $\bar{k}$-points. We establish the diagram

$$\begin{array}{ccccc} G(R) & = & G(R) & \longrightarrow & G'(R) \\ \downarrow & & \downarrow & & \downarrow \\ X(R) & \xleftarrow{\simeq} & W(R) & \longrightarrow & X'(R) \end{array}$$

and the attendant properties explicitly. Let $W$ be the scheme whose $R$-points consist of sequence of matrices $(d_1, \ldots, d_n)$ and $(e_1, \ldots, e_{n-1})$

$$0 \longrightarrow R^{b_n} \xrightarrow{d_n} R^{a_n} \xrightarrow{e_{n-1}} R^{b_{n-1}} \xrightarrow{d_{n-1}} \cdots \xrightarrow{e_1} R^{b_1} \xrightarrow{d_1} R^{a_1} \longrightarrow 0$$

where the matrix direct-sum $\bigoplus d_i \oplus \bigoplus e_i$ has the maximal possible rank (viz. an $R$-point of $X$) along with the following additional data: Writing

$$d_1 = \begin{pmatrix} \delta_{1,1} & \delta_{1,2} & \ldots & \delta_{1,b_1} \\ \delta_{2,1} & \delta_{2,2} & \ldots & \delta_{2,b_1} \\ \vdots & \vdots & & \vdots \\ \delta_{a_1,1} & \delta_{a_1,2} & \ldots & \delta_{a_1,b_1} \end{pmatrix}$$

we demand an additional $b_1$-tuple of elements of $R$, $\eta = (\eta_1, \ldots, \eta_{b_1})$ such that

$$\sum_j \delta_{i,j}\eta_j = \begin{cases} 0, & \text{if } i < a_1 \\ 1, & \text{if } i = a_1. \end{cases} \tag{4.11}$$

This is the same as demanding $d_1\eta = (0, \ldots, 0, 1)^t$. Since these are polynomial conditions, one can easily show that $W$ is a closed subscheme of $X \times \mathbb{A}^{a_n}$. By simply forgetting the data $\eta$, one obtains a map $W \to X$. In fact, by restricting to e.g. the open cover $\mathcal{U}$ of $X$ considered in proposition 55, it is easily seen that $W \to X$ is a Zariski-locally trivial affine $\mathbb{A}^T$-bundle for some $T$ (in fact $T = b_1 - a_1$). Perforce $W \to X$ is an $\mathbb{A}^1$-equivalence as claimed.

We demonstrate the map $W \to X'$ on $R$-points, its being evident that it is indeed natural in $R$ and therefore a map of schemes. Given an $R$-point of $W$, written as

$$(d_1, e_1, \ldots, d_{n-1}, e_{n-1}, d_n, \jmath) \in G(R)$$

the $R$-point of $X'$ it maps to is that given by $(d'_1, e'_1, \ldots, d'_{n-1}, e'_{n-1}, d'_n)$ with $d'_i = d_i$ for $i > 2$ and $e_i = e_i$ for $i \geq 2$. Writing, in addition to our notation for $d_1$ above

$$e_1 = \begin{pmatrix} \varepsilon_{1,1} & \varepsilon_{1,2} & \ldots & \varepsilon_{1,a_2} \\ \varepsilon_{2,1} & \varepsilon_{2,2} & \ldots & \varepsilon_{2,a_2} \\ \vdots & \vdots & & \vdots \\ \varepsilon_{b_1,1} & \varepsilon_{b_1,2} & \ldots & \varepsilon_{b_1,a_2} \end{pmatrix}$$

we put

$$d'_1 = \begin{pmatrix} \delta_{1,1} & \delta_{1,2} & \ldots & \delta_{1,b_1} \\ \delta_{2,1} & \delta_{2,2} & \ldots & \delta_{2,b_1} \\ \vdots & \vdots & & \vdots \\ \delta_{a_1-1,1} & \delta_{a_1-1,2} & \ldots & \delta_{a_1-1,b_1} \end{pmatrix}, \quad e'_n = \begin{pmatrix} \varepsilon_{1,1} & \varepsilon_{1,2} & \ldots & \varepsilon_{1,a_2} & \eta_1 \\ \varepsilon_{2,1} & \varepsilon_{2,2} & \ldots & \varepsilon_{2,a_2} & \eta_2 \\ \vdots & \vdots & & \vdots & \vdots \\ \varepsilon_{b_1,1} & \varepsilon_{b_1,2} & \ldots & \varepsilon_{b_1,a_2} & \eta_{b_1} \end{pmatrix}$$

We also augment $d_2$ with a trailing row of 0s as $d'_2 = \begin{pmatrix} d_2 \\ 0 \end{pmatrix}$. At worst, this establishes a map $W \to \mathbb{A}^N$, where the latter is a very large affine space in which $X'$ is locally closed. We now consider the construction on the level of geometric points, and shall prove a factorization $W(\bar{k}) \to X'(\bar{k}) \to \mathbb{A}^N(\bar{k})$

One verifies immediately $e'_1 d'_2 = d'_1 e'_{11} = 0$. Since the rank of $d_1$ was $a_1$ (as large as possible), the rank of $d'_1$ is $a_1 - 1$. The rank of $e'_1$ is easily seen to be one greater than the rank of $e_1$ since $\jmath$ is linearly independent of the other rows, being unique in not annihilating $(\delta_{a_1,1}, \delta_{a_1,2}, \ldots, \delta_{a_1,b_1})$. The rank of $d'_2$ is unchanged from that of $d_2$. By use of lemmas 1, and 2, we have a map $W \to X'$.

One allows $G$ to act on $W$ by (on $R$-points) defining the action of $g = (b_1 \ldots, b_{n-1}, a_n, b_n)$

on $(d_1, e_1, \ldots, e_{n-1}, d_n)$ in the same way as for $X$, and setting $g \cdot \jmath = y_2 \jmath$. Since the action of $g$ on $d_1$ is via $d_1 \mapsto d_n y_2^{-1}$, it follows that the $g$-action preserves the product $d_1 \eta$. So we really have an action of $G$ on $W$. We pick a particular $k$-point of $W$, taking the point of diagram (4.2) and adding the additional data that $\eta_0 = (0, 0, \ldots, 0, 1, 0, \ldots 0)^t$, the 1 being in the $a_1$-th position. This gives a commutative diagram

$$\begin{array}{ccc} G & = & G \\ \downarrow & & \downarrow \\ X & \longleftarrow & W \end{array}$$

Using the standard inclusion $\mathrm{GL}(b_{n-1}) \to \mathrm{GL}(b_{n-1}+1)$ to obtain a map $G \to G'$, we observe directly that the map $W \to X'$ commutes with the $G$-action, and so the diagram extends to

$$\begin{array}{ccccc} G & = & G & \longrightarrow & G' \\ \downarrow & & \downarrow & & \downarrow \\ X & \overset{\simeq}{\longleftarrow} & W & \longrightarrow & X' \end{array}$$

as required.

Finally we must prove that the maps $X \longleftarrow \overline{W} \longrightarrow X'$ are $\mathbb{G}_m$-equivariant. The action of $\mathbb{G}_m$ on $\overline{W}$ extends that of $\mathbb{G}_m$ on $X$, and on $R$-points we demand

$$z \cdot (\eta_1, \ldots, \eta_{b_1})^t = (z^{w_{1,1} - v_{1,a_1}} \eta_1, z^{w_{1,2} - v_{1,a_1}} \eta_2, \ldots, z^{w_{1,b_1} - v_{1,a_1}} \eta_{b_1})^t$$

The verification that this preserves the defining condition is routine, since the action of $z$ on $\delta_{i,j}$ is $z \cdot \delta_{i,j} = z^{v_{1,i} - w_{1,j}} \delta_{i,j}$ so the equations of (4.11) are satisfied. It follows that $W \to X$ is $\mathbb{G}_m$-equivariant. Equivariance of $W \to X'$ is just as easily shown. The only thing that really needs to be verified is the action of $z$ on $e_1'$, which is by

$$z \cdot e_1' = \operatorname{diag}(z^{w_{1,1}}, \ldots, z^{w_{1,b_1}})\, e_n' \operatorname{diag}\left(z^{-v_{2,1}}, \ldots, z^{-v_{2,a_2}}, z^{-v_{1,a_n}}\right)$$

and which we can see by immediate inspection is gives the same action as we have defined

$z \cdot \jmath$, where $\jmath$ is now the last row of $e'_n$.

This concludes the proof. □

The importance of this proposition is to understand the map $H^{*,*}(X') \to H^{*,*}(X)$. Each is a subring, of $H^{*,*}(G)$ and of $H^{*,*}(G')$ respectively, and three out of the four maps in

$$\begin{array}{ccc} H^{*,*}(G') & \longrightarrow & H^{*,*}(G) \\ \uparrow & & \uparrow \\ H^{*,*}(X') & \longrightarrow & H^{*,*}(X) \end{array}$$

are already known. We obtain in this way a map $H^{*,*}(X') \to H^{*,*}(X)$. Although the map is easily computed in practice, writing it down in full generality without reference to $H^{*,*}(G)$ obscures more than it reveals. Returning to our example, diagram (4.7), the case where $X$ is the variety of sequences of vector spaces of dimensions $(c_1, c_2, c_3, c_4, c_5, c_6) = (1, 4, 5, 4, 3, 1)$, and $X'$ is the variety of vector spaces of dimensions $(0, 4, 6, 4, 3, 1)$ and maps of rank $(0, 4, 2, 2, 1)$. Here $H^{*,*}(G)$ is generated by classes $\gamma_{i,j}$ as we have already enumerated. The rign $H^{*,*}(G')$ is generated by classes $\gamma_{i,j}$ which correspond under the map to those in $H^{*,*}(G)$, with the addition of a single class $\gamma'_{3,6}$.

Pictorially we denote this as

$$\begin{array}{ccccccccc}
 & & \gamma'_{3,6} & & & & & & \\
 & & \gamma_{3,5} & & & & & & \\
\gamma_{2,4} & - - - & \gamma_{3,4} & & \gamma_{4,4} & & & & \\
\gamma_{2,3} & \text{———} & \gamma_{3,3} & & \gamma_{4,3} & & \gamma_{5,3} & & \\
\gamma_{2,2} & \text{———} & \gamma_{3,2} & \text{———} & \gamma_{4,2} & \text{———} & \gamma_{5,2} & & \\
\gamma_{2,1} & \text{———} & \gamma_{3,1} & \text{———} & \gamma_{4,1} & \text{———} & \gamma_{5,1} & \text{———} & \gamma_{6,1}
\end{array}$$

Where the class $\gamma'_{3,6}$ and the dotted arrow are different from the case of $H^{*,*}(X)$. As such $H^{*,*}(X')$ is an exterior algebra on classes $\kappa'_{i,j}$ that satisfy

$$\begin{aligned}
\kappa'_{1,1} &= -\gamma_{2,1} + \gamma_{3,1} - \gamma_{4,1} + \gamma_{5,1} - \gamma_{6,1} \\
\kappa'_{2,2} &= -\gamma_{2,2} + \gamma_{3,2} - \gamma_{4,2} + \gamma_{5,2} \\
\kappa'_{2,3} &= -\gamma_{2,3} + \gamma_{3,3} \\
\kappa'_{4,3} &= -\gamma_{4,3} + \gamma_{5,3} \\
\kappa'_{2,4} &= -\gamma_{2,4} + \gamma_{3,4} \\
\kappa_{i,j} &= (-1)^{i+1}\gamma_{i,j} \quad \text{if } (i,j) \text{ is any of } (4,3), (5,3), (4,4), (3,5), \text{ or } (3,6)
\end{aligned}$$

and the map $H^{*,*}(G') \to H^{*,*}(G)$ induces an identification $\kappa_{i,j} = \kappa'_{i,j}$ except for $\kappa'_{3,6}$ which, being new maps to 0, and $\kappa'_{2,4} \mapsto -\kappa_{2,4} + \kappa_{3,4}$.

There is a next step in this 'folding-up' procedure. We can present $X'$ as being the variety representing sequences of dimensions $(4,6,4,3,1)$, and we can present the cohomology of such a variety as

$$\begin{array}{ccccccc}
\gamma''_{2,6} & & & & & & \\
\gamma''_{2,5} & & & & & & \\
\gamma''_{2,4} & & \gamma''_{3,4} & & & & \\
\gamma''_{2,3} & & \gamma''_{3,3} & & \gamma''_{4,3} & & \\
\gamma''_{2,2} & \text{---} & \gamma''_{3,2} & \text{---} & \gamma''_{4,2} & & \\
\gamma''_{2,1} & \text{---} & \gamma''_{3,1} & \text{---} & \gamma''_{4,1} & \text{---} & \gamma_{5,1}
\end{array}$$

Which is the same as the previous diagram excepting the omission of the first column and a consequent re-indexing. We can then apply to comparison procedure to obtain

$$\begin{array}{ccccccc}
\gamma''_{2,6} & & & & & & \\
\gamma''_{2,5} & & \gamma'''_{3,5} & & & & \\
\gamma''_{2,4} & & \gamma''_{3,4} & & & & \\
\gamma''_{2,3} & --- & \gamma''_{3,3} & & \gamma''_{4,3} & & \\
\gamma''_{2,2} & \text{——} & \gamma''_{3,2} & \text{——} & \gamma''_{4,2} & & \\
\gamma''_{2,1} & \text{——} & \gamma''_{3,1} & \text{——} & \gamma''_{4,1} & \text{——} & \gamma_{5,1}
\end{array}$$

Where $\gamma'''_{3,5}$ and the dashed line are new. These amount to a comparison of the variety $X' = X(4,4,1,6,3)$ with $X'' = X(3,5,1,6,3)$. The effect of such a comparison is to take a class $\kappa''_{2,3} \in H^{*,*}(X'')$ to the sum $-\gamma''_{2,3} + \gamma''_{3,3} - \gamma''_{4,3} = -\kappa'_{3,3} - \kappa'_{4,3} \in H^{*,*}(X')$— bear in mind that $\gamma''_{i,j}$ are reindehomotopicalxings of the $\gamma'_{i,j}$, which explains the slightly anomalous indexing of the $\gamma''_{i,j}$s versus the $\kappa'_{i,j}$s.

There is associated with the $\mathbb{G}_m$-action on $X$ a Rothenberg-Steenrod spectral sequence computing the cohomology of $H^{*,*}(B(\mathrm{pt}, \mathbb{G}_m, X))$, and we should like to compute the differentials in it. To do so, we need a comparison result with some simpler space, which is precisely what the following proposition provides.

**Proposition 59.** *Let $X = X(a_1, \ldots a_n, b_1, \ldots, b_n)$ and $G = \mathrm{GL}(a_1, \ldots, b_{n-1}, a_n)$ be as above, and suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{w}_1, \ldots, \mathbf{w}_n$ have been given, so there is a $\mathbb{G}_m$-action on $X$. There is a* *commutative diagram of schemes*

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{GL}(a_n) \\ \downarrow & & \downarrow \\ X & \longrightarrow & W(b_1, a_1) \end{array}$$

*where* $G \to X$ *is the map of diagram* (4.2) *and* $\mathrm{GL}(b_1) \to W(b_1, a_1)$ *is a standard projection map. The map* $G \to \mathrm{GL}(b_1)$ *is projection of a product group on the first factor. If we give* $W(b_1, a_1)$ *the* $\mathbb{G}_m$*-action coming from action on the left & right as in theorem 54 with weights* $\mathbf{v}_1, \mathbf{w}_1$*, then the map* $X \to W(b_1, a_1)$ *is* $\mathbb{G}_m$*-equivariant.*

*Proof.* On $R$-points, the map $X(R) \to W(a_n, b_n)(R)$ is the map that reads off the last matrix $d_n$. All the assertions are immediately verified, except perhaps the commutativity of the square. Recall that in the presentation of $X$ as a quotient of $G$, we had $d_n = x_n \begin{pmatrix} 0 & \\ & I_{b_n} \end{pmatrix}$, which is to say that $d_n$ is the result of reading off the last $b_n$-columns of $x_n$, which is among the many projections of $\mathrm{GL}(a_n) \to W(a_n, b_n)$. □

We observe that as a special case, when the long exact sequence is very short, $X(a, a)$, we have an equivariant isomorphism $X(a, a) \cong \mathrm{GL}(a)$, where the latter has $\mathbb{G}_m$-action on the left & right.

Consider a particular $X = X(a_1, \ldots, a_n, b_1, \ldots, b_n)$. Recall that we assumed $\sum a_i = \sum b_i$, and denote this integer by $M$. By repeated use of proposition 58, we have a map $X \to \mathrm{GL}(M)$. Under the nondegenercy assumption that none of the ranks in the exact sequences represented by $X$ is 0, there is only one $\kappa_{1,j}$ in the presentation of $H^{*,*}(X)$, to wit $\kappa_{1,1}$. Writing $H^{*,*}(\mathrm{GL}(M)) = \Lambda_{\mathbb{M}}(\rho_1, \ldots, \rho_M)$, under the map $H^{*,*}(\mathrm{GL}(N)) \to H^{*,*}(X)$, we have $\rho_1 \mapsto \kappa_{1,1}$.

If we have weights $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and $\mathbf{w}_1, \ldots, \mathbf{w}_n$, and so an action of $\mathbb{G}_m$ on $X$, then the equivariant nature of the comparison in proposition 58 means, after repeated application, that there is a succession of equivariant zig-zags

$$X \xleftarrow{\simeq} W_1 \longrightarrow X_1 \longleftarrow \cdots \xleftarrow{\simeq} W_\omega \longrightarrow \mathrm{GL}(M)$$

which we privately think of as a map $X \to \mathrm{GL}(M)$, since it behaves as such on cohomology.

The target, $\mathrm{GL}(M)$, is given the $\mathbb{G}_m$-action on the left & right as in proposition 52 by means of the weights $\mathbf{v}$, $\mathbf{w}$ which are respectively the concatenations of the weights $\mathbf{v}_i$ and $\mathbf{w}_i$, as can be seen from proposition 58.

Write $H^{*,*}(\mathbb{G}_m) = \mathbb{M}[\tau]/(\tau^2)$

From this comparison, and the proof of proposition 52, we see that the coaction on cohomology of $\mathbb{G}_m \times X \to X$ satisfies

$$\kappa_{1,1} \mapsto 1 \otimes \kappa_{1,1} + [\sigma_1(\mathbf{v}) - \sigma_1(\mathbf{w})]\tau \otimes 1$$

This calculation allows us to apply proposition 47 to write down explicitly

**Theorem 60.** *Let $X = X(a_1, \ldots, a_n, b_1, \ldots, b_n)$ be the variety representing long exact sequences. Let $\mathbf{v}_i, \mathbf{w}_i$ be weights, giving an action of $\mathbb{G}_m$ on $X$ as in equation 4.8 Let $G$ and $N \subset H^{*,*}(G)$ be as in theorem 60.*

*There is a spectral sequence computing $H^{*,*}(B(\mathrm{pt}, \mathbb{G}_m, X)$ which on the $E_2$ page takes the following form*

$$E_2^{*,*} = \frac{\Lambda_{\mathbb{M}}(N \setminus \{\kappa_{1,1}\})[\theta]}{\left(\left[\sum v_{i,j} - \sum w_{ij}\right] \theta\right)} \qquad \text{if } \sum v_{i,j} - \sum w_{ij} \neq 0$$

$$E_2^{*,*} = \Lambda_{\mathbb{M}}(N)[\theta] \qquad \text{if } \sum v_{i,j} - \sum w_{ij} = 0$$

*in each case $|\kappa_{i,j}| = (0, 2i - i, i)$ and $|\theta| = (1, 1, 1)$*

As before, the two cases can be thought of as arising from a hypothetical $E_1$-page, the first case being that when $\kappa_{1,1}$ supports a nonzero differential, the second when it does not.

## 4.5 Generalized Herzog-Kühl Equations

It behooves us to give at least one application of the machinery established above. Fix an integer $n$ and a perfect field $k$. Let $S = k[x_1, \ldots, x_m]$ be the polynomial ring, graded by placing 1 in degree 0 and $x_1, \ldots, x_m$ in degree $-1$.

Consider first the case of an artinian $S$-module, $M$, and a free resolution

$$0 \longrightarrow F_a \longrightarrow F_{a-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M$$

Decomposing the free modules into their graded parts: $F_i = \bigoplus_{j=0}^{\infty} S(j)^{\beta_{i,j}}$, we obtain a matrix $\beta_{i,j}$ of integers, referred to as the *Betti table* of the resolution. The standard Herzog-Kühl equations, as presented in [BS08], [ES09] for example, give relations among the $\beta_{i,j}$,

$$\sum_{i=1}^{a} \sum_{j \in \mathbb{Z}} (-1)^j \beta_{i,j} j^s = 0$$

where $s$ ranges over the integers $\{0, \ldots, n\}$.

We generalize this slightly. Instead of a resolution, we begin instead with a chain complex of f.g. graded free $S$-modules (which is to say both that the modules are graded and that the maps between them respect the grading)

$$\Theta: \; 0 \longrightarrow F_{c_m} \longrightarrow F_{c_{m-1}} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow 0$$

having finite length homology (the homology is no longer concentrated at one end). For the sake of concreteness we work with the case of even $m$, the case of odd $m$ being the same in all essentials. We have therefore a chain complex of graded free $S$-modules

$$\Theta: \; 0 \longrightarrow F_n \xrightarrow{D_n} G_n \xrightarrow{E_{n-1}} \cdots \xrightarrow{E_1} G_1 \xrightarrow{D_1} F_1 \longrightarrow 0$$

having artinian homology. We decompose $F_i$ and $G_i$ into homogeneous parts,

$$F_i = \bigoplus_{j=1}^{a_i} S(v_{i,j}), \quad G_i = \bigoplus_{j=1}^{b_i} S(w_{i,j})$$

where $S(d)$ denotes $S$ placed in degree $d$. We define two further sets of integers $s_i, r_i$ recursively, starting with $s_0 = 0$ and proceeding

$$r_i = a_i - s_{i-1}$$
$$s_i = b_i - r_i$$

It will transpire that $r_i, s_i$ are 'ranks' of $D_i$, $E_i$ in the sense that $r_i = \sup_{\mathfrak{m} \subset S} \operatorname{rank} D_i \otimes S/\mathfrak{m}$ where $\mathfrak{m}$ ranges over all maximal ideals of $S$. A similar statement applies to $s_i$, $E_i$.

The main result of this section is the following.

**Theorem 61** (Herzog-Kühl Equations)**.** *Let $\Theta$, $r_i$, $s_i$, $v_{j,k}$ and $w_{j,k}$ be as defined above (a similar result holds for a complex of odd length). Let $q \leq n$. Let $\mathbf{v}_q$, $\mathbf{w}_q$ denote the vectors of integers $(v_{j,k})$, $(w_{j,k})$ for $j \leq q$. Let $u_1, \ldots, u_q$ be integers defined recursively by*

$$\sigma_i(\mathbf{w}_q) = \sum_{j=1}^{i} \sigma_j(\mathbf{v}_q) u_{i-j}$$

*for $1 \leq i \leq r_q$. Let $u_j = 0$ for $j > q$. Then we have*

$$\sigma_i(\mathbf{w}_q) = \sum_{j=1}^{i} \sigma_j(\mathbf{v}_q) u_{i-j}$$

*for $r_q + 1 \leq i \leq m - 1$.*

*Proof.* It follows from the exactness result of [Car86] that evaluation at all maximal ideals of $S$ other than the irrelevant ideal yields a long exact sequence of graded vector spaces

$$0 \longrightarrow K^{a_n} \longrightarrow K^{b_{n-1}} \longrightarrow \cdots \qquad \longrightarrow K^{a_1} \longrightarrow 0$$

where $K$ is some algebraic extension of $k$.

The matrices representing the differentials in $\Theta$ are matrices of polynomials in $m$ variables, or, to put it another way, when taken all together they give rise to a map $\mathbb{A}_k^m \to \mathbb{A}_k^N$ for some very large $N$. If $(\xi_1, \ldots, \xi_m) : \bar{k} \to \mathbb{A}_k^m$ is a geometric point of $\mathbb{A}_k^m$, then the composite $\overline{k} \to \mathbb{A}^m_k \to \mathbb{A}^N_k$ is evaluation of the matrices $d_i, e_i$ at the $m$-tuple $(\xi_1, \ldots, \xi_m)$.

The exactness result we cited above implies that on the $\overline{k}$-points of $\mathbb{A}^m_k \setminus \{0\}$, the image of this evaluation map is an exact sequence. In other words there is a factorization $(\mathbb{A}^m_k \setminus \{0\})(\overline{k}) \to X(\overline{k}) \to \mathbb{A}^N_k(\overline{k})$, from which it follows that there exists a map

$$f : \mathbb{A}^m_k \setminus \{0\} \to X$$

which on $\overline{k}$-points is given by evaluation.

The graded nature of the complex $\Theta$ means that the differentials are given by homogeneous polynomial functions. The homogeneity of the polynomials implies that this map $f$ is $\mathbb{G}_m$ equivariant, where $\mathbb{G}_m$ acts on $\mathbb{A}^m \setminus \{0\}$ in the obvious way

$$z \cdot (\xi_1, \ldots, \xi_m) = (z\xi_1, \ldots, z\xi_m)$$

and on $X$ according to the grading given by the integers $\mathbf{v}_i$, $\mathbf{w}_i$ on the various $F_i \cong S^{a_i}$, $G_i \cong S^{b_i}$. Properly $X$ is to be understood as

$$X = X(\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{w}_1, \ldots, \mathbf{w}_n)$$

as in equation 4.9.

Since we understand the equivariant cohomology of $X$ reasonably well, we shall try to obtain obstructions to $\mathbb{G}_m$-equivariant maps $\mathbb{A}^m_k \setminus \{0\} \to X$.

We consider the equivariant composition:

$$\mathbb{A}^m \setminus \{0\} \longrightarrow X \longrightarrow \mathrm{pt}$$

There is a composition of maps of Rothenberg-Steenrod spectral sequences arising from this composition. The spectral sequence for $\mathbb{A}^m_k \setminus \{0\}$ has $E_2$-page as described in proposition 48 $\mathbb{M}[\rho_n, \theta]/(\rho_n^2)$, with $|\rho_n| = (0, 2n-1, n)$ and

$$\frac{\mathbb{M}[\rho_m, \theta]}{(\rho_n^2)} \quad |\rho_n| = (0, 2n-1, n), \quad |\theta| = (1,1,1)$$

The sole nontrivial differential is $d_m(\rho_m) = \theta^m$. The spectral sequence computing the equivariant cohomology of pt is even more straightforward, the $E_2$-page being

$$\mathrm{Ext}_{\hat{H}^{*,*}(\mathbb{G}_m)}(\mathbb{M}, \mathbb{M}) = \mathbb{M}[\theta]$$

and the sequence being in an immediate state of collapse.

The point is that the $\mathbb{G}_m$-equivariant map $\mathbb{A}^m \setminus \{0\} \to \mathrm{pt}$ induces comparison maps of spectral sequences. In particular, using the naturality of proposition 46 we obtain, on every page subsequent to the $E_2$-page, for $j < m$ and all $\ell > 1$

$$\mathbb{M}^{0,0}\theta^j \cong E_\ell^{j,j,j}(B(\mathrm{pt}, \mathbb{G}_m, \mathrm{pt})) \xrightarrow{\cong} E_\ell^{j,j,j} B(\mathrm{pt}, \mathbb{G}_m, \mathbb{A}^m \setminus \{0\} \cong \mathbb{M}^{0,0}\theta^j$$

Since this map must factor through $\mathbb{A}^m \setminus \{0\}$, taking $R = \mathbb{Z}[\frac{1}{2}]$, we see that the spectral sequence computing $H^{*,*}(B(\mathrm{pt}, \mathbb{G}_m, X); R)$, derived in theorem 60 must also have $E_\ell^{j,j,j} \cong \mathbb{M}^{0,0}\theta^j$. By referring to that theorem, we deduce first that

$$\sigma_1(\mathbf{v}_i) - \sigma_1(\mathbf{w}_1) = \sum v_{i,j} - \sum w_{i,j} = 0$$

and secondly that the differentials $d_j(\kappa_{i,j})$ must all vanish for $j < m$. We now apply comparison theorems to deduce the nature of these differentials.

We first apply the comparison-by-folding map, proposition 58, repeatedly to obtain the comparison map in homotopy

$$X(a_1, \ldots, a_n; b_1, b_2, \ldots, b_n) \to X(a_1 + a_2 + \cdots + a_q, a_{q+1}, \ldots, a_n; \\ b_1 + b_2 + \cdots + b_q, b_{q+1}, \ldots, b_n)$$

To this we apply a comparison with a Stiefel variety

$$X(a_1 + a_2 + \cdots + a_q, a_{q+1}, \ldots, a_n; \\ b_1 + b_2 + \cdots + b_q, b_{q+1}, \ldots, b_n) \to W(b_1 + \cdots + b_q, a_1 + \cdots + a_q)$$

We write $b = \sum_{i=1}^{q} b_i$, $a = \sum_{i=1}^{q} a_i$ and write $W = W(b, a)$. We point out that by an exactness argument, we have $b - a = r_q$. We have

$$H^{*,*}(W) = \Lambda_{\mathbb{M}}(\rho_{b-a+1}, \ldots, \rho_b)$$

and the map induced on cohomology by the comparisons take $\rho_k$ to $\sum_{i=1}^{2q} \kappa_{i,k}$ for $1 \leq k \leq b - a + 1$. Since the differentials in the spectral sequence supported by the classes $\kappa_{i,k}$ must vanish when $k < m$, and since the comparison maps are $\mathbb{G}_m$-equivariant, it follows that the differentials supported by the classes $\rho_k$, where $k < m$, in the cohomology of $W(b, a)$ similarly must vanish. The $\mathbb{G}_m$-action on $W(b, a)$ which makes the comparison maps equivariant is given by the weights $(v_{1,1}, v_{1,2}, \ldots, v_{q,a_q}; w_{1,1}, w_{1,2}, \ldots, w_{q,b_q})$. The numerical formulas of the proposition now follow by considering theorem 54. □

The numerical conditions of this theorem may be restated in the following form. Given $(v_{j,k})$ and $(w_{j,k})$ as in the theorem, there exist complex numbers $v'_{q+1,1}, \ldots, v_{q+1,r_q}$ (unique up to permutation) so that, taking $\mathbf{v}'_q$ to be the vector of all $v_{j,k}$ and $v'_{q+1,k}$, we have

$$\sigma_i(\mathbf{v}'_q) = \sigma_i(\mathbf{w}_q)$$

The $u_j$ of the theorem are the elementary symmetric functions $\sigma_j(\{v'_{q+1,1}, \ldots, v'_{q+1,r_q}\})$. The theory of symmetric polynomials now allow us to restate the theorem as follows

**Corollary 61.1.** *Let $\Theta$, $v_{j,k}$ and $w_{j,k}$ be as defined above (a similar result holds for a complex of odd length). Let $q \leq n$ and consider $r_q = \operatorname{rank} D_q$ (a similar result holds with the roles of $F_j$, $G_j$ reversed). Let $\mathbf{v}_q$, $\mathbf{w}_q$ denote the vectors of integers $(v_{j,k})$, $(w_{j,k})$ for $j \leq q$. Let $v'_{q+1,1}, \ldots, v'_{q+1,r_q}$*

*be complex numbers defined by the system of equations*

$$\sum_{j=1}^{q}\sum_{k=1}^{b_j} w_{j,k}^i = \sum_{j=1}^{q}\sum_{k=1}^{a_j} v_{j,k}^i + \sum_{k=1}^{r_q} (v'_{q+1,k})^i$$

*for* $1 \leq i \leq r_q$*. Then we have*

$$\sum_{j=1}^{q}\sum_{k=1}^{b_j} w_{j,k}^i = \sum_{j=1}^{q}\sum_{k=1}^{a_j} v_{j,k}^i + \sum_{k=1}^{r_q} (v'_{q+1,k})^i$$

*for* $r_q + 1 \leq i \leq m - 1$*.*

The classical Herzog-Kühl equations correspond to the case where $q = n$, in that case $r_q = 0$ and there are no integers $u_j$ (and therefore no $v'_{q+1,j}$) to be considered. In this case we simply have the following

**Corollary 61.2.** *Let* $\Theta$*,* $v_{j,k}$ *and* $w_{j,k}$ *be as defined above (a similar result holds for a complex of odd length). We have*

$$\sum_{j=1}^{n}\sum_{k=1}^{b_j} w_{j,k}^i = \sum_{j=1}^{n}\sum_{k=1}^{a_j} v_{j,k}^i$$

*for* $0 \leq i \leq n$*.*

In cases where $r_q \geq m$, of course, all the above statements are vacuous. When the complex under consideration is a resolution, so there is homology only at one end, the strong Buchsbaum-Eisenbud conjecture (see for instance [CE92], we do not know where this conjecture first appears) says that the 'ranks' $r_q \geq \binom{m-1}{2q-1}$ and $s_q \geq \binom{m-1}{2q}$. Since this generally exceeds $m - 1$, the more intricate relations above are conjectured not to matter for resolutions. For arbitrary complexes, however, the ranks of the syzygies may in fact be quite small, as the following example, which I learned from [Geo09], shows.

Let $k$ be any field, let $S = k[x, y, z, w, t]$. Let $A$, $B$ be the matrices

$$A = \begin{pmatrix} y & z & w & t & 0 & 0 & 0 \\ -x & 0 & z & w & t & 0 & 0 \\ 0 & -x & -y & 0 & w & t & 0 \\ 0 & 0 & -x & -y & -z & 0 & t \\ 0 & 0 & 0 & -x & -y & -z & -w \end{pmatrix}, \quad B = \begin{pmatrix} x & y & z & w & t \end{pmatrix}$$

We observe that $BA = \mathbf{0}$, so that

$$0 \longrightarrow \ker A \longrightarrow S^7 \xrightarrow{A} S^5 \xrightarrow{B} S \longrightarrow 0$$

is a complex. By direct calculation one sees it has only artinian homology. Let $\Psi_*$ be a graded finite free resolution of the $S$-module $\ker A$. Then the composite $\Psi_* \to S^7 \to S^5 \to S \to 0$ is a graded complex having artinian homology. The 'rank' of the map $S^7 \to S^5$ is $s_1 = 5 - 1 = 4$, whereas the strong Buchsbaum-Eisenbud conjecture calls for $\binom{4}{2} = 6$.

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