THE PRIME DIVISORS OF THE PERIOD AND INDEX OF A BRAUER CLASS

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ABSTRACT. We show that in locally-ringed connected topoi the primes dividing the period and index of a Brauer class coincide. The result applies in particular to Brauer classes on connected schemes, algebraic stacks, topological spaces and to the projective representation theory of profinite groups.

This preprint is an extended version of a paper of the same title. Material appearing here but not in the short version, which is intended for publication, appears in blue. The bulk of the additional material is by way of background on topoi.

1. INTRODUCTION

If k is a field, then the Brauer group, $\mathrm{Br}(k)$, is the group of equivalence classes of central simple k–algebras modulo Morita equivalence. A theorem of Wedderburn's states that every central simple k–algebra A is isomorphic to a matrix algebra $\mathrm{Mat}_n(D)$, where D is a finite-dimensional central k–division algebra. Since the rings D and $\mathrm{Mat}_n(D)$ are Morita equivalent, the Brauer group is identified with the set of isomorphism classes of finite-dimensional k–division-algebras. The Brauer group was introduced by Brauer in the 1930s, and has been studied extensively since—the monograph of Gille and Szamuely [9] is a good reference.

For a class $\alpha \in \mathrm{Br}(k)$, one defines the period $\mathrm{per}(\alpha)$ to be its order as a group element. If A is a central simple k–algebra, write $\alpha = [A]$ for the associated class in the Brauer group; the integer $\mathrm{per}(\alpha)$ is the smallest positive integer such that $A^{\otimes_k \mathrm{per}(\alpha)} \cong \mathrm{Mat}_n(k)$ for some integer n. The index, $\mathrm{ind}(\alpha)$, is the greatest common divisor of the degrees—the square-roots of the dimensions over k—of the central simple algebras in the class α.

It is not hard to show that $\mathrm{per}(\alpha) | \mathrm{ind}(\alpha)$. Three additional facts about $\mathrm{per}(\alpha)$ and $\mathrm{ind}(\alpha)$ concern us in this paper, all of which are classical and can be found in [9]:

(1) $\mathrm{ind}(\alpha)$ is the degree of the lowest-dimensional element of α, namely the unique division algebra D with $[D] = \alpha$;
(2) one may use Galois splitting fields and Sylow subgroups of Galois groups to prove that $\mathrm{per}(\alpha)$ and $\mathrm{ind}(\alpha)$ have the same prime divisors;
(3) as a consequence, there exists a central simple algebra A, specifically the unique division algebra D in the class α, with class α such that $\deg(A)$ has the same prime divisors as $\mathrm{per}(\alpha)$.

Work of Azumaya [7] and then of Auslander and Goldman [6] established the notion of an *Azumaya algebra* over a commutative ring R, and defined $\mathrm{Br}(R)$ as a group of equivalence classes of Azumaya algebras, generalizing the Brauer group of a field. These Azumaya algebras are flat families of central simple algebras. The idea was extended by Grothendieck [11] to the case of a locally-ringed topos, $(\mathbf{X}, R)$, although the emphasis in that work was on the specific case where the topos is $\tilde{X}_{\acute{e}t}$, the étale topos of a scheme X, and where the local ring is $\mathcal{O}_X$, the structure sheaf of X. The definition of Azumaya algebras and $\mathrm{Br}(\mathbf{X}, R)$, when applied to the étale topos of $\mathrm{Spec}\, R$, locally ringed by the structure sheaf, specializes to the definitions of Auslander and Goldman.

In the generality of a locally ringed topos, it is possible to define the period and the index of a class $\alpha \in \mathrm{Br}(\mathbf{X}, R)$, although one must allow for pathologies if $\mathbf{X}$ is badly disconnected. For instance $\mathrm{Br}(\mathbf{X}, R)$ need not be a torsion group in general, so $\mathrm{per}(\alpha)$ may be infinite. Unless otherwise stated, we assume $\mathbf{X}$ is connected, an assumption which greatly simplifies the theory and costs very little applicability. We define $\mathrm{per}(\alpha)$ as the order of α, which is finite under this assumption, and we may define

$$\mathrm{ind}(\alpha) = \gcd\{\deg(A) : [A] = \alpha\}.$$

We wish to determine whether the statement $\text{per}(\alpha) | \text{ind}(\alpha)$ and the analogues of (1)–(3) above hold in general.

Of these, $\text{per}(\alpha) | \text{ind}(\alpha)$ is generally seen to be true, whereas we have already proved in [1] that (1) does not always hold. Namely, we showed that there exists a smooth affine complex 6-fold X and a Brauer class $\alpha \in \text{Br}(X)$ with $\text{per}(\alpha) = 2$ and $\text{ind}(\alpha) = 2$, but where there is no degree 2 Azumaya algebra defined on X with class α. In more recent work, [3], we extended the arguments of [1] to give examples of this failure with $\text{per}(\alpha) = p$ for every prime p. Our smallest example is 6-dimensional. In a positive direction, it is known that if X is a regular noetherian 2-dimensional scheme, then (1) holds: there exists an Azumaya algebra of degree equal to $\text{per}(\alpha)$. In the affine case, this follows from the proof of [6, Proposition 7.4], and the general case is similar.

Property (3) was known to hold in the following situations: the classical case of the étale sites of fields, and the étale sites of regular noetherian 2–dimensional schemes, [6], and the étale sites of schemes X that are unions of two affine schemes along an affine intersection, where it was deduced by [8, Chapter II]. To our knowledge, no other results along the lines of (3) were known for schemes.

If property (3) obtains, then property (2) must obtain as well.We asked in [2, Problem 1.8] when property (2) holds and we are aware of some additional cases where it was known to hold where (3) was not known. By [2, Theorem 3.1], (2) holds for finite CW complexes. The proof employs the Hurewicz isomorphism theorem and twisted topological K–theory, and is peculiar to the case of CW complexes. If X is a regular noetherian scheme, then we showed in [3, Proposition 6.5] that the period and index have the same prime divisors. This time, the proof was by using the inclusion $\text{Br}(X) \subseteq \text{Br}(K)$, where K is the field of fractions of X, and an argument of Saltman to show that the index of $\alpha \in \text{Br}(X)$ is the same when computed over X or over K.

In this paper, we consider the common generalization of Azumaya algebras in the étale topology on a scheme and in the ordinary topology on a CW complex: the theory of Azumaya algebras in a locally ringed topos.

Azumaya algebras in locally ringed topoi also generalize the special case of projective representations of finite groups. Given a finite group, G, we may form the topos $\mathbf{B}G$ of discrete G–sets, and endow it with a local ring R. In this topos, an Azumaya algebra of degree n is tantamount to a representation $\rho : G \to \text{PGL}_n(R)$. When $R = \mathbb{C}$, the Brauer group of the locally ringed topos $\text{Br}(\mathbf{B}G, R)$ is the Schur multiplier $H^2(G, \mathbb{C}^\times)$ of G. In this setting, (1) is known not to hold. Higgs communicated to us that $\text{PSL}_2(\mathbb{F}_7)$ has Brauer group $\mathbb{Z}/2$, where the non-zero class, α, is represented by irreducible projective representations of degrees $4, 4, 6, 6, 8$, so that $\text{per}(\alpha) = \text{ind}(\alpha) = 2$, but there is no degree-2 Azumaya algebra with class α. On the other hand, (2) is known to hold for all finite G, and (3) is known to hold when G is a finite p–group, [13].

In this paper, we establish (3), and therefore (2), under mild hypotheses on the topos. This solves Problem 1.8 of [2], where to our knowledge the question of whether (2) holds in full generality was first posed.

Theorem (Theorem 6)**.** *Let $(\mathbf{X}, R)$ be a connected locally ringed topos, and let $\alpha \in \text{Br}(\mathbf{X}, R)$. There exists an Azumaya algebra over $(\mathbf{X}, R)$ the degree of which is divisible only by those primes dividing $\text{per}(\alpha)$. In particular, $\text{per}(\alpha)$ and $\text{ind}(\alpha)$ have the same prime divisors.*

While the result is stated in a general and abstract language, the proof when it comes is simple, being little more than the construction of homomorphisms between projective linear groups. It would be possible to give a different, but conceptually identical, proof in the language of twisted sheaves.

The paper therefore provides, in the first place, a unified proof of a statement that had previously been known only by different arguments in different contexts. In the second place, it covers the cases of the étale site on singular or non-noetherian schemes and the case of infinite CW complexes. In the third, it strengthens the result of Saltman for regular noetherian schemes by producing an Azumaya algebra the degree of which is divisible only by primes dividing the period.

The main theorem, Theorem 6, is not stated in maximum generality; it holds for instance if the hypothesis that $\mathbf{X}$ be connected is weakened to the hypothesis that $\pi_0(\mathbf{X})$ be compact. In [8], a more general definition of Azumaya algebra than that of [11] is given, appertaining to the case of a ringed topos. The two definitions coincide in the cases of étale sites of schemes and in the case of CW complexes locally ringed by the sheaf of continuous complex-valued functions. We do not explore an expansion of Theorem 6 to the generality of the Azumaya algebras of [8].

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2. Azumaya Algebras in Grothendieck Topoi

In this section and the next, we present the theories of Azumaya algebras and of PGL_n–bundles in a locally-ringed connected Grothendieck topos, and show that they are equivalent.We claim no originality for this material. We assure the reader that the abstractions of these two sections are embodied in several down-to-earth examples: a short list of applications is given in section 3.1, after the exposition of the theory. Our reference for the theory of topoi is [4], and we adopt the theory of universes of [4, Exposé I, Appendice]. We refer the reader also to [10], especially Chapter 0 for the set- and topos-theoretic preliminaries, to Chapter III for the theory of torsors in a site, and to Chapter V.4 for the theory of the Brauer group in a site. We assume the existence of an uncountable universe, $\mathcal{U}$, the elements of which will be called *small* sets. A category $\mathbf{C}$ is a *locally small* category if the sets of morphisms $\mathbf{C}(X, Y)$ are small. A locally small category, $\mathbf{C}$, equipped with a Grothendieck topology τ is a locally small site if there is a small set $\mathcal{X} = \{X_i\}_{i\in I}$ of objects in $\mathbf{C}$ such that every object of $\mathbf{C}$ has a covering family consisting of morphisms the sources of which are in $\mathcal{X}$, see [4, Exposé II]. A small sheaf or presheaf is a sheaf or presheaf of small sets. A locally small Grothendieck *topos*, [4, Exposé IV], is a category equivalent to a category of small sheaves of on some locally small site. We shall not have occasion to discuss non-small sets or sheaves or presheaves, or non-locally-small categories, sites or topoi. We drop these modifiers throughout and write **Set** etc. for the category of small sets. A small category, site or topos is a (locally small) category, site or topos the objects of which form a set. We omit the modifier 'Grothendieck' in 'Grothendieck topos'.

Topoi admit several characterizations, one of which we use freely: a topos, $\mathbf{X}$, is a category such that the canonical topology endows $\mathbf{X}$ with the structure of a site for which every sheaf is representable, [4, Exposé IV, Théorème 1.2]. This implies that the Yoneda map η which sends an object Y of $\mathbf{X}$ to the presheaf $\mathbf{X}(\cdot, Y)$ is full, faithful and essentially surjective onto the subcategory of sheaves for the canonical topology $\mathbf{Sh}_{\mathrm{can}}(\mathbf{X})$ of $\mathbf{Pre}(\mathbf{X})$. By [16, Chapter IV, Theorem 4.1], it is an equivalence of categories $\mathbf{X} \to \mathbf{Sh}_{\mathrm{can}}(\mathbf{X})$. There is an adjunction

$$\tilde{\cdot} : \mathbf{Pre}(\mathbf{X}) \rightleftarrows \mathbf{X} : \eta$$

where η is the Yoneda embedding , and the left adjoint functor $Y \mapsto \tilde{Y}$, which we call *sheafification* in an abuse of terminology, commutes with finite limits.

All topoi are closed under taking small limits and small colimits, and for any two objects $Y_1, Y_2 \in \mathbf{X}$, the mapping-presheaf $U \mapsto \mathbf{Set}(\eta_{Y_1}(U), \eta_{Y_2}(U))$ is a sheaf, denoted $(Y_2)^{Y_1}$. We write $\emptyset$ for a colimit of the empty diagram and $*$ for a limit of the empty diagram.

If $\{U_i \to V\}_{i\in I}$ is a set of maps in a topos $\mathbf{X}$, we say that the U_i cover V if the induced map $\coprod_{i\in I} U_i \to V$ is an epimorphism.

A *geometric morphism* of topoi $f : \mathbf{X} \to \mathbf{Y}$ is an adjoint pair of functors

$$f^* : \mathbf{Y} \rightleftarrows \mathbf{X} : f_*$$

such that f^* commutes with finite limits. A *point* p of a topos $\mathbf{X}$ is a geometric morphism $p : \mathbf{Set} \to \mathbf{X}$. A topos $\mathbf{X}$ is said to have *enough points* if there exists a set of points $\{p_i\}_{i\in I}$ such that a map $g : Y \to Y'$ in $\mathbf{X}$ is an isomorphism if and only if $p_i^*(g)$ is an isomorphism for all $i \in I$. The set $p_i^*(Y)$ is the *stalk* of Y at p_i. In general, the topoi we encounter shall all have enough points, although this is inessential to the argument.

We abuse notation and write Y for both an object in $\mathbf{X}$ and for the presheaf it represents under the fully faithful Yoneda embedding, η, so that if A and Y are objects of $\mathbf{X}$, the notation $A(Y)$ means $\mathbf{X}(Y, A)$. In order to define a morphism $f : X \to Y$ in $\mathbf{X}$, it suffices to define a morphism of presheaves $f : X(\cdot) \to Y(\cdot)$, and to define this it suffices to define maps of sets $f(U) : X(U) \to Y(U)$ as U ranges over the objects in $\mathbf{X}$, and to show that the definition of f is functorial in U. We refer to this as arguing with elements.

A group object in $\mathbf{X}$ is an object G of $\mathbf{X}$ equipped with a multiplication $\mu : G \times G \to G$, an inverse $\iota : G \to G$ and a unit $e : * \to G$ making the usual diagrams of group theory commute. Equivalently, up to unique isomorphism in $\mathbf{Pre}(\mathbf{X})$, a group object G of $\mathbf{X}$ is a representable presheaf $G : \mathbf{X}^{op} \to \mathbf{Grp} \to \mathbf{Set}$. In order to specify a homomorphism of group objects $\psi : G \to H$ in $\mathbf{X}$ it suffices to specify a natural transformation of group-valued contravariant functors $\psi(\cdot) : G(\cdot) \to H(\cdot)$ on $\mathbf{X}$. We define abelian group objects, ring objects etc. similarly. In the sequel we shall write 'group' for 'group object', and 'abelian group' for 'abelian group object' and so on when no confusion is likely to occur.

Given a ring object R, assumed throughout to be unital and associative, we may form the group $R^\times$ of units in Ras the limit

$$\begin{array}{ccc} R^\times & \longrightarrow & * \\ \downarrow & & \downarrow 1 \\ R \times R & \xrightarrow{\mu} & R. \end{array}$$

There is a composite morphism $u : R^\times \to R \times R \xrightarrow{\pi_1} R$ given by projection on the first factor. Since the definition of $R^\times$ is as a limit, and the formation of limits commutes with the Yoneda embedding, it follows that for all objects U of $\mathbf{X}$, the map of sets $u : R^\times(U) \to R(U)$ is an injection with image $R(U)^\times$. In particular, $u : R^\times \to R$ is a submonoid of the multiplicative monoid structure on R, and arguing with elements of $R^\times(U)$ we see that $R^\times$ is a group.

If U is an object of $\mathbf{X}$ and $f \in R(U)$, we define U_f to be the largest subobject of U in which the image of f is invertible; it is the pull-back:

$$\begin{array}{ccc} U_f & \longrightarrow & R^\times \\ \downarrow & & \downarrow u \\ U & \xrightarrow{f} & R. \end{array}$$

A ringed topos, $(\mathbf{X}, R)$, is *locally-ringed* if the ring object R is commutative for any object U in $\mathbf{X}$, and for any $f \in R(U)$, the objects U_f and U_{1-f} cover U—see [4, Exposé IV, Exercice 13.9] or [11, Section 2] and [17, Chapter VIII]. If p is a point of X and R is a local ring object, then the ring $p^*(R)$ has the property that for all elements $f \in p^*(R)$, either f or $(1-f)$ is a unit. Consequently, either p^*R is empty or p^*R has a unique maximal ideal. In the presence of enough points of the topos, a ring object is a local ring object if and only if p^*R is either local or empty at all points.

If S is a set, then S extends in an obvious way to a constant presheaf on $\mathbf{X}$. The sheafification, $\tilde{S}$, will be called the *constant sheaf* on S; this is a misnomer in that $\tilde{S}(Y)$ is not necessarily constant as Y varies. When we say two objects Y, Z in $\mathbf{X}$ are 'locally isomorphic', we mean that there is an epimorphism $U \to *$ onto the terminal object of $\mathbf{X}$ and an isomorphism $f : Y \times U \to Z \times U$ over U. The functor $\mathbf{X} \to \mathbf{Sets}$ given on objects by $Y \mapsto \mathrm{Hom}(*, Y)$ is known as the *global section functor*, and is left-adjoint to the constant-sheaf functor.

The topos $\mathbf{X}$ is *connected* if the constant-sheaf functor is fully faithful. The exposition is greatly simplified if we assume $\mathbf{X}$ is connected, which is also the most applicable case, so all topoi we consider are assumed connectedunless the contrary is stated. There is an abelian category of R–modules, in which one may form free- and locally-free-modules, tensor products, and homomorphism objects.and among the objects in this category are the free modules of finite rank. These are isomorphic to R^n where $n \geq 0$ is an integer.

Suppose V, W are two R–modules, then $V \otimes_R W$ and $\mathrm{Hom}_R(V, W)$ may again be defined as R–modules. The tensor product $V \otimes_R W$ is the sheafification of the presheaf $U \mapsto V(U) \otimes_{R(U)} W(U)$. The case of $\mathrm{Hom}_R(\cdot, \cdot)$ is similar. If both arguments are free R–modules of finite rank, then $\mathrm{Hom}_R(R^n, R^m) \cong R^{nm}$.

One may define an R–algebra to be an R–module equipped with a multiplication map $A \otimes_R A \to A$ and a structure map $R \to A$ making the usual diagrams commute. We do not require R–algebras to be commutative, but we do require the action of R on A to be central. The R–algebra $\mathrm{Hom}_R(V, V)$ will be written as $\mathrm{End}_R(V)$, and $\mathrm{End}_R(R^n)$ will be identified with the algebra of $n \times n$ matrices over R, denoted $\mathrm{Mat}_n(R)$.

We recall from [11] that an *Azumaya algebra*, $\mathcal{A}$, on $(\mathbf{X}, R)$ is an R-algebra in $\mathbf{X}$ which locally is isomorphic to an algebra of the form $\mathrm{Mat}_n(R)$; the integer n is called the *degree* of $\mathcal{A}$. The tensor product $\mathcal{A} \otimes_R \mathcal{A}'$ is an Azumaya R-algebra formed by means of the Kronecker product $\mathrm{Mat}_n(R) \otimes \mathrm{Mat}_m(R)$, applied locally.

A *locally free module of finite rank* on $(\mathbf{X}, R)$ is an R–module that locally is isomorphic to R^n where n is an integer called the *rank*.

The *Brauer group* $\mathrm{Br}(\mathbf{X}, R)$ of $(\mathbf{X}, R)$ is the set of Azumaya algebras under the equivalence relation that says $\mathcal{A} \simeq \mathcal{A}'$ if there exist locally free R–modules E and E' of finite rank such that

$$\mathcal{A} \otimes_R \mathrm{End}_R(E) \cong \mathcal{A}' \otimes_R \mathrm{End}_R(E').$$

The Brauer group is indeed a group under tensor product, with the inverse of $\mathcal{A}$ being given by the opposite algebra, since $\mathcal{A} \otimes_R \mathcal{A}^{op} \cong \mathrm{End}_R(\mathcal{A})$.

If V is a free R–module then the exterior power $V^{\wedge d}$ is defined as the presheaf

$$Y \mapsto V(Y)^{\wedge d}$$

on $\mathbf{X}$. It is in fact a sheaf, since V is free, and is an R–module. If R^n is given the basis $\{e_1, \dots, e_n\}$, then $(R^n)^{\wedge d}$ is a free R–module of rank $\binom{n}{d}$ having basis consisting of wedge-products of the form

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_d}$$

where $1 \le i_1 < i_2 < \cdots < i_d \le n$. The construction of $V^{\wedge d}$ is functorial.

2.1. Aside on Topoi that are not Connected.

If $\mathbf{X}$ is not connected, then the nature of free and locally free R–modules becomes more intricate, since the constant sheaf $\tilde{\mathbb{Z}}_{\ge 0}$, the object in which the rank is defined, may have nonconstant sections.

In the case of ordinary integers, there is an order relation $T \subset \mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\ge 0}$, to wit $(a, b) \in T$ if $a \le b$. For any constant function $c : * \to \mathbb{Z}_{\ge 0}$, we have a map $\mathbb{Z}_{\ge 0} \cong \mathbb{Z}_{\ge 0} \times * \to \mathbb{N} \times \mathbb{Z}_{\ge 0}$, given by $a \mapsto (a, c)$. The pullback of this map along T defines the initial-segment subset $I_c = \{1, 2, \dots, c\} \subset \mathbb{N}$. For a commutative ring R, one may then define R^c as the free R–module on I_c; in this way there are standard inclusions $R^0 \subset R^1 \subset \cdots \subset R^c \subset \cdots \subset R^{\mathbb{N}}$.

This can all be generalized to the case of a locally ringed topos, $(\mathbf{X}, R)$, associating to a global section $c : * \to \tilde{\mathbb{Z}}_{\ge 0}$ an initial-segment object I_c, and, by applying the ordinary free-module functor throughout, the free R–module R^c. Locally free R–modules are R–modules that are locally isomorphic to free modules. The rank of a locally free R–module is not an integer in general, but a global section of $\tilde{\mathbb{Z}}_{\ge 0}$. Azumaya algebras are R–algebras that are locally isomorphic to endomorphism algebras of free R–modules, $\mathrm{Mat}_c(R)$, and their degrees are again global sections of $\tilde{\mathbb{Z}}_{\ge 0}$.

We return to our standing assumption that $\mathbf{X}$ is connected.

The object $GL_n(R) = GL(R^n)$ which takes Y to $GL(R^n(Y))$ is the group of units in $\mathrm{Mat}_n(R)$. We define $\mathbb{G}_m = GL_1 = R^\times$. The objects GL_n are groups, and $\mathbb{G}_m$ is an abelian group.

Given an element $f \in GL(R^n(Y))$, we may form corresponding elements $f \otimes \ldots \otimes f \in GL(R^n(Y)^{\otimes d})$ and $f \wedge \cdots \wedge f \in GL(R^n(Y)^{\wedge d})$. These give rise to homomorphisms of groups $GL_n(R) \to GL_{dn}(R)$ and $GL_n(R) \to GL_{\binom{n}{d}}(R)$, which we refer to in the sequel as *diagonal* homomorphisms; on the level of elements of $GL_n(R)(U)$, $GL_{dn}(R)$ and $GL_{\binom{n}{d}}(R)$:

$$\mathrm{diag}(f)(v_1 \otimes \ldots \otimes v_d) = f(v_1) \otimes \ldots \otimes f(v_d)$$

and

$$\mathrm{diag}(f)(v_1 \wedge \cdots \wedge v_d) = f(v_1) \wedge \cdots \wedge f(v_d).$$

3. The Projective General Linear Group

This section is an enlargement of [11, §2], and much of the same material appears in [10, Chapitre V, §4].

Define SL_n as the kernel of the determinant homomorphism $\mathbb{G}_m \to GL_n$. There is a diagonal inclusion $\mathbb{G}_m \to GL_n$; it is central and the quotient group is denoted PGL_n. The composite map $\mathbb{G}_m \to GL_n \to \mathbb{G}_m$ is the n–th power map, denoted

$$\epsilon_n : \mathbb{G}_m \to \mathbb{G}_m.$$

We define μ_n, the group of n–th roots of unity, to be the kernel of ϵ_n. Denote the cokernel of ϵ_n by ν_n.

There is a commutative diagram in which both rows and columns are short exact sequences of groups in $\mathbf{X}$, and where those on the left and the bottom are abelian:

(1)
$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \mu_n & \to & SL_n & \to & PSL_n & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \mathbb{G}_m & \to & GL_n & \to & PGL_n & \to & 1 \\
 & & \downarrow & \overset{\epsilon_n}{\searrow} & \downarrow \scriptstyle{\det} & & \downarrow & & \\
1 & \to & \mathbb{G}_m/\mu_n & \to & \mathbb{G}_m & \to & \nu_n & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1. & &
\end{array}$$

Observe that if $\nu_n \cong 1$, as often happens in cases of interest, the canonical map $PSL_n \to PGL_n$ is an isomorphism.

In a ringed topos, it is possible to define the group objects GL_n, $\mathbb{G}_m \cong GL_1$, SL_n, $PGL_n \cong GL_n / \mathbb{G}_m$and $PSL_n \cong SL_n / \mu_n$. If the n–th power map $\mathbb{G}_m \to \mathbb{G}_m$ is an epimorphism, as often happens, then $PGL_n \cong PSL_n$.

If $\mathcal{A}$ is an Azumaya algebra, then it is possible to form $\mathrm{Aut}(\mathcal{A})$ as a group in $\mathbf{X}$; locally this group is isomorphic to a group of the form $\mathrm{Aut}(\mathrm{Mat}_n(R))$. The conjugation action of GL_n on Mat_n means that there is a homomorphism $\phi : PGL_n \to \mathrm{Aut}(\mathrm{Mat}_n(R))$.

The following proposition is asserted in [11].

Proposition 1. *If* $(\mathbf{X}, R)$ *is a locally-ringed topos, then* $\phi : PGL_n \to \mathrm{Aut}(\mathrm{Mat}_n(R))$ *is an isomorphism.*

Proof. We refer to [17, Chapter VIII, Theorem 3], which says that there is a universal locally ringed topos. It is $\mathrm{Spec}(\mathbb{Z}, \mathcal{O})$, the ringed topos associated to the Zariski site on $\mathrm{Spec}\,\mathbb{Z}$. Given any locally-ringed topos $(\mathbf{X}, R)$, there is a geometric morphism $r : \mathbf{X} \to \mathrm{Spec}\,\mathbb{Z}$ such that $r^*\mathcal{O} \cong R$. Since r^* preserves finite limits and all colimits, it follows that $R^n \cong (r^*\mathcal{O})^n$, that $\mathrm{End}(R^n) \cong r^* \mathrm{End}(\mathcal{O})$, that $PGL_n(R) \cong r^* PGL_n(\mathcal{O})$, that $\mathrm{Aut}(\mathrm{End}(R^n)) \cong r^* \mathrm{Aut}(\mathrm{End}(\mathcal{O}^n))$, and all these isomorphisms are compatible with the various actions of these objects on themselves and each other.

It suffices, therefore, to prove the proposition in the case of $(\mathrm{Spec}\,\mathbb{Z}, \mathcal{O})$. Since every projective $\mathbb{Z}$–module is free, the result follows from Theorem 3.6 and Proposition 5.1 of [6], the Skolem-Noether theorem. □

The proof in the case where $\mathbf{X}$ has enough points may be carried at stalks, using the same results in [6] as were used for $\mathrm{Spec}\,\mathbb{Z}$. In the absence of enough points, arguing in $\mathbf{X}$ would require some ungainly maneuvering among the objects of $\mathbf{X}$ in order to mimic the properties of a local ring at a stalk. It is to avoid this that we employ the artifice of the universal example.

This proposition is the point at which it becomes necessary for the topos to be locally ringed, rather than merely ringed, which is why we draw attention in the statement to our standing assumption that $(\mathbf{X}, R)$ is locally ringed.

There exist cohomology functors $H^i(G)$ in the topos $\mathbf{X}$, defined for $i = 0, 1$ in the case of a non-abelian group G, but for all $i \geq 0$ in the case of an abelian group A (see [5, Exposé V] and [10]).The set $H^1(G)$ classifies G–torsors in the topos, and since $\mathrm{Aut}(\mathrm{Mat}_n(R)) \cong PGL_n$, there is a natural bijection between isomorphism classes of Azumaya algebras of degree n and $H^1(PGL_n)$. We view elements $\mathcal{A} \in H^1(PGL_n)$ as being Azumaya algebras of degree n. It is a consequence of the properties of Kronecker product and matrix multiplication that the two definitions of $\otimes$ on the classes in $H^1(PGL_*)$, one given by tensor product of R–algebras, the other by Kronecker products of matrices, agree. Associated to a short exact sequence of groups

$$1 \to G \to G'' \to G' \to 1$$

there is a long exact sequence in cohomology, extending to $H^2(G)$ in the case of a central extension by an abelian group G, so that we have, in particular, the portion of a long exact sequence

$$H^1(\mathrm{GL}_n) \to H^1(\mathrm{PGL}_n) \xrightarrow{\delta} H^2(\mathbb{G}_m). \tag{2}$$

The map $H^1(\mathrm{GL}_n) \to H^1(\mathrm{PGL}_n)$ takes a locally free R–module E of rank n to the PGL_n–torsor $\mathrm{End}_R(E)$.

If G and G' are two groups, then there is an isomorphism $H^i(G \times G') \cong H^i(G) \times H^i(G')$, where applicable, by Giraud [10, Remarque 2.4.4]. This endows the cohomology of an abelian group A in $\mathbf{X}$ with an abelian group structure, and does so in such a way that the n-th power map $A \to A$, which on the level of elements is $a \mapsto a^n$ if A is written multiplicatively, induces multiplication by n on the additive abelian groups $H^*(A)$.

Writing $\otimes$ for the Kronecker product $\mathrm{GL}_n \otimes \mathrm{GL}_m \to \mathrm{GL}_{nm}$, and for the induced product $\mathrm{PGL}_n \otimes \mathrm{PGL}_m \to \mathrm{PGL}_{nm}$,we have a commutative diagram of short exact sequences of groups:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_m \times \mathbb{G}_m & \longrightarrow & \mathrm{GL}_n \times \mathrm{GL}_m & \longrightarrow & \mathrm{PGL}_n \times \mathrm{PGL}_m & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{mult}} & & \downarrow{\scriptstyle \otimes} & & \downarrow{\scriptstyle \otimes} & & \\ 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathrm{GL}_{nm} & \longrightarrow & \mathrm{PGL}_{nm} & \longrightarrow & 1. \end{array}$$

In particular, this means that

$$\delta(\mathcal{A} \otimes \mathcal{A}') = \delta(\mathcal{A}) + \delta(\mathcal{A}'). \tag{3}$$

The following proposition holds in general, the proof being the same is in the case of the étale topos of a scheme. It is also implicit in the discussion of [10, Chapitre V.4.4–5].

Proposition 2. *If* $(\mathbf{X}, R)$ *is a connected, nonempty, locally-ringed topos, then* $\mathrm{Br}(\mathbf{X}, R)$ *can be identified with the image of the map*

$$\coprod_{n=1}^{\infty} H^1(\mathrm{PGL}_n) \to H^2(\mathbb{G}_m). \tag{4}$$

Proof. The hypotheses on $\mathbf{X}$ ensure that all Azumaya algebras represent elements in some cohomology group $H^1(\mathrm{PGL}_n)$, and all locally free R–modules have a well-defined rank. There is, therefore, a surjective map of sets

$$\coprod_{n=1}^{\infty} H^1(\mathrm{PGL}_n) \to \mathrm{Br}(\mathbf{X}, R). \tag{5}$$

Diagram (2) and the identity (3) imply that $\delta(\mathcal{A})$ depends only on the class of $\mathcal{A}$ in $\mathrm{Br}(\mathbf{X})$. We therefore have a factorization of (4) as

$$\coprod_{n=1}^{\infty} H^1(\mathrm{PGL}_n) \to \mathrm{Br}(\mathbf{X}, R) \to H^2(\mathbb{G}_m),$$

where the first map is the surjection 5 and the second map is a homomorphism of groups. Finally, since $\delta(\mathcal{A}) = 1$ if and only if $\mathcal{A}$ is of the form $\mathrm{End}_r(E)$ where E is a locally free R–module, it follows that $\mathrm{Br}(\mathbf{X}, R) \to H^2(\mathbb{G}_m)$ is injective. □

3.1. **Examples.**

(1) If X is a scheme, then one may define the étale site of $X_{\acute{e}t}$ as in [5, Exposé VII].The site consists of X–schemes, $X' \to X$, that are étale over X. The topology is that generated by jointly surjective small families of étale maps. The resulting topos, $\tilde{X}_{\acute{e}t}$, is the étale topos of X. It is connected when X is connected for the Zariski topology: subobjects of the terminal object in $\tilde{X}_{\acute{e}t}$ are isomorphic to Zariski open subsets of X, and the topos is connected if and only if the terminal object cannot be decomposed as a disjoint union of subobjects. The geometric points of X endow $\tilde{X}_{\acute{e}t}$ with enough points, [5, Exposé VIII]. The structure sheaf, $\mathcal{O}_X$, is a local ring object in $\tilde{X}_{\acute{e}t}$, the stalks being strictly Hensel local rings. The theory of Azumaya algebras

in the locally ringed topos $(\tilde{X}_{\acute{e}t}, \mathcal{O}_X)$ is the classical theory of Azumaya algebras of [11], and restricts to the theories of [6], [7] over rings and local rings, by taking $X = \operatorname{Spec} R$, and from there to the theory of central simple algebras over a field, by taking R to be a field.

(2) The construction of the étale site of a scheme can be extended to the lisse-étale site of an algebraic stack $\mathcal{X}$. The structure sheaf $\mathcal{O}_{\mathcal{X}}$ is a local ring object in the topos of sheaves on this site. For particulars, see [15, Chapitre 12] and [18], and for discussion of the Brauer group of a stack: [12].

(3) If X is a topological space, then one may define a topos $\mathbf{X}$ where the objects are sheaves on X. This topos is connected if X is connected and the topos has enough points in all cases. If $\mathbb{K}$ is a topological field, then defining $\mathbb{K}(U)$ to be the set of continuous functions $\mathbf{Cont}(U, \mathbb{K})$ makes $\mathbb{K}(\cdot)$ a sheaf on X, and therefore a local ring object in $\mathbf{X}$. The theory of Azumaya algebras on $(\mathbf{X}, \mathbb{C})$ and $(\mathbf{X}, \mathbb{R})$ are the theories of principal $\mathrm{PGL}_n(\mathbb{C})$–bundles on X and principal $\mathrm{PGL}_n(\mathbb{R})$–bundles, respectively. If X is a CW complex, then these coincide with the theory of principal PU_n– and PO_n–bundles; the first of these two theories is the subject of [2], [3].

(4) If G is a profinite group, we can define the topos $\mathbf{B}G$ of right G–sets, that is to say: discrete sets U equipped with a continuous action map $U \times G \to U$ that is compatible in the obvious ways with the group structure on G. The morphisms in $\mathbf{B}G$ are G–equivariant maps.

The constant-sheaf functor $\tilde{\cdot} : \mathbf{Set} \to \mathbf{B}G$ is the functor giving U the trivial G–action, and is fully faithful, so $\mathbf{B}G$ is connected.

The topos $\mathbf{B}G$ has property that every object decomposes as a disjoint union of orbits of G, and further that every orbit may be covered by a principal free G–space. In particular, every cover of the terminal object in $\mathbf{B}G$ has a refinement of the form $\coprod_{i \in I} e_i G$, where the sets $e_i G$ are isomorphic to G as right G–sets. Evaluation $A \mapsto A(eG)$ at such a principal right G–set is the functor that forgets the underlying G–action on A. This functor forms part of a geometric morphism $\mathbf{Set} \to \mathbf{B}G$, having the free G–object functor as a left adjoint. The topos $\mathbf{B}G$ therefore has $\{\nu\}$ as a conservative set of points, and moreover two objects of $\mathbf{B}G$ are locally isomorphic if and only if there is an isomorphism between them after the G–action is forgotten.

For any ring R with a G–action the associated ring object in $\mathbf{B}G$ is a local ring object if and only if the ring R is local.

Two particular cases of locally ringed topoi $(\mathbf{B}G, R)$ are especially noteworthy, and we enumerate them separately.

(5) First, if k^{sep}/k is a separable closure of fields with Galois group G, then the topos $\mathbf{B}G$ equipped with the ring k^{sep}, on which G has a Galois action, is equivalent as a locally ringed topos to $\widetilde{(\operatorname{Spec} k)}_{\acute{e}t}$ ringed by $\mathcal{O}_k$, so the theory of Azumaya algebras in this instance is the theory of central simple k–algebras.

(6) Second, if R is a local ring given trivial right G–action, then a principal PGL_n–bundle on the locally ringed topos $(\mathbf{B}G, R)$ is equivalent to a right G–set structure on the set $\mathrm{PGL}_n(R)$ compatible with the left $\mathrm{PGL}_n(R)$–structure on $\mathrm{PGL}_n(R)$ itself, this amounts to a continuous homomorphism of groups $\phi : G \to \mathrm{PGL}_n(R)$. When $R = \mathbb{C}$ and G is a finite group, the Brauer group $\mathrm{Br}(\mathbf{B}G, \mathbb{C})$ is the Schur multiplier of G. The basic theory of projective representations of finite groups is set out in [14], and a treatment of the period–index problem in this setting is given in [13].

3.2. **Aside on Topoi that are not Connected.** In general, a proposition similar to Proposition 2 holds, but where the objects PGL_n are replaced by objects PGL_c where c is a global section of the constant sheaf $\tilde{\mathbb{Z}}_{\geq 0}$. This allows us to identify $\mathrm{Br}(\mathbf{X}, R)$ with a subgroup of $H^2(\mathbb{G}_m)$ in all cases.

If every locally free R–module E of locally constant rank can be extended to a locally free module $E \oplus R^c$ of constant rank, where c is a global section of the constant sheaf $\tilde{\mathbb{Z}}_{\geq 0}$, and if every Azumaya algebra $\mathcal{A}$ can be extended to an Azumaya algebra $\mathcal{A} \otimes \mathrm{Mat}_{c'}(R)$ of constant degree, again where c' is a global section of the constant sheaf $\tilde{\mathbb{Z}}_{\geq 0}$, then $\mathrm{Br}(\mathbf{X}, R)$ agrees with the image of the map (4) as written. This is the case if all global sections of $\tilde{\mathbb{Z}}_{\geq 0}$, i.e., all maps $* \to \tilde{\mathbb{Z}}_{\geq 0}$, factor through some map $* \to \tilde{n}$, where $\tilde{n}$ denotes the constant sheaf associated to $\{0, 1, \ldots, n\}$. Such a factorization is guaranteed if the pro-set $\pi_0(\mathbf{X})$ is compact.

4. Period & Index

Henceforth we assume our topos locally-ringed and connected.

Suppose α is an element of $\mathrm{Br}(\mathbf{X}, R) \subset H^2(\mathbb{G}_m)$. We define the *period* of α to be the order of α as a group element in $H^2(\mathbb{G}_m)$, assuming it is finite, and we define the *index* of α to be the greatest common divisor of all integers n such that α is in the image of $H^1(\mathrm{PGL}_n) \to H^2(\mathbb{G}_m)$. If $\mathcal{A}$ is an element in $H^1(\mathrm{PGL}_n)$, then we say that the *degree* of $\mathcal{A}$ is n, and we abuse terminology in saying that the period and index of $\mathcal{A}$ are simply the period and index of the image of $\mathcal{A}$ under the map

$$H^1(\mathrm{PGL}_n) \to \mathrm{Br}(\mathbf{X}, R).$$

Writing α for this image, we say $\mathcal{A}$ *represents* α.

Theorem 3. *If $(\mathbf{X}, R)$ is a ringed topos and if $\alpha \in \mathrm{Br}(\mathbf{X}, R)$ is an element in the Brauer group represented by $\mathcal{A}$, then $\mathrm{per}(\alpha)$ divides the degree of $\mathcal{A}$. Consequently, $\mathrm{per}(\alpha) | \mathrm{ind}(\alpha)$, and $\mathrm{Br}(\mathbf{X}, R)$ is torsion.*

Proof. By reference to diagram (1), and noting that ϵ_n induces multiplication by n in cohomology, we see that thereThere is a factorization of the composite map

$$H^1(\mathrm{PGL}_n) \xrightarrow{\iota} H^2(\mathbb{G}_m) \xrightarrow{\times n} H^2(\mathbb{G}_m)$$

as

$$\begin{array}{ccccc} H^1(\mathrm{PGL}_n) & & & & \\ \downarrow & & & & \\ H^1(\nu_n) & \longrightarrow & H^2(\mathbb{G}_m/\mu_n) & \longrightarrow & H^2(\mathbb{G}_m), \end{array}$$

which is necessarily the trivial map. As a consequence, the image of $\iota : H^1(\mathrm{PGL}_n) \to H^2(\mathbb{G}_m)$ is n–torsion, and it follows that $\mathrm{per}(\alpha) | \deg(\mathcal{A})$ if $\mathcal{A}$ is any representative of α. Since $\mathrm{ind}(\alpha)$ is the greatest common divisor of all such $\mathcal{A}$, the result follows. □

Proposition 4. *Suppose $\mathcal{A}$ in $H^1(\mathrm{PGL}_n)$ and $\mathcal{A}'$ in $H^1(\mathrm{PGL}_m)$ each represent the same element, α, in $\mathrm{Br}(\mathbf{X}, R)$. There exists an element $\mathcal{A} \oplus \mathcal{A}'$ in $H^1(\mathrm{PGL}_{n+m})$ representing α.*

Proof. Suppose $\mathcal{A}$ in $H^1(\mathrm{PGL}_n)$ and $\mathcal{B}$ in $H^1(\mathrm{PGL}_m)$ represent α and β in $\mathrm{Br}(\mathbf{X}, R)$, respectively. As remarked above, there is an isomorphism $H^1(\mathrm{PGL}_n \times \mathrm{PGL}_m) \to H^1(\mathrm{PGL}_n) \times H^1(\mathrm{PGL}_m)$, [10]. The data of $\mathcal{A}$ and $\mathcal{B}$ therefore give an element $\mathcal{A} \times \mathcal{B}$ in $H^1(\mathrm{PGL}_n \times \mathrm{PGL}_m)$.

We may include $\mathbb{G}_m \xrightarrow{\Delta} \mathrm{GL}_n \times \mathrm{GL}_m$ as the subgroup of scalar matrices, and we define $\mathrm{PGL}_{n,m}$ as the quotient. We also write $\Delta : \mathbb{G}_m \to \mathbb{G}_m \times \mathbb{G}_m$ for the diagonal inclusion. There is a short exact sequence of group objects

$$1 \longrightarrow \mathbb{G}_m \xrightarrow{\Delta} \mathbb{G}_m \times \mathbb{G}_m \xrightarrow{q} \mathbb{G}_m \longrightarrow 1$$

where the map $q : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ is that given by $(\lambda, \lambda') \mapsto \lambda/\lambda'$; this map is split, and consequently an epimorphism.

The rows and first two columns of the following diagram are short exact sequences of groups:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{G}_m & \xrightarrow{\Delta} & \mathbb{G}_m \times \mathbb{G}_m & \longrightarrow & \mathbb{G}_m & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{G}_m & \xrightarrow{\Delta} & \mathrm{GL}_n \times \mathrm{GL}_m & \longrightarrow & \mathrm{PGL}_{n,m} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & 1 & \longrightarrow & \mathrm{PGL}_n \times \mathrm{PGL}_n & = & \mathrm{PGL}_n \times \mathrm{PGL}_m & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1. & &
\end{array}$$

By the nine-lemma, the third column is also exact. We conclude that the obstruction to lifting $\mathcal{A} \times \mathcal{B}$ from $H^1(\mathrm{PGL}_n \times \mathrm{PGL}_m)$ to $H^1(\mathrm{PGL}_{n,m})$ is the class $\alpha - \beta$ in $H^2(\mathbb{G}_m)$.

If we take $\mathcal{A}$ and $\mathcal{A}'$, both of which represent $\alpha \in \mathrm{Br}(\mathbf{X}, R)$, then this obstruction vanishes, and we may define $\mathcal{A}''$ to be a lift of $\mathcal{A} \times \mathcal{A}'$ to $H^1(\mathrm{PGL}_{n,m})$.

There is a 'direct-summation' map $\sigma : \mathrm{GL}_n \times \mathrm{GL}_m \to \mathrm{GL}_{n+m}$. The diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{G}_m & \xrightarrow{\Delta} & \mathrm{GL}_n \times \mathrm{GL}_m & \longrightarrow & \mathrm{PGL}_{n,m} & \longrightarrow & 1 \\
 & & \| & & \downarrow{\scriptstyle \sigma} & & \downarrow & & \\
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathrm{GL}_{n+m} & \longrightarrow & \mathrm{PGL}_{n+m} & \longrightarrow & 1
\end{array}$$

commutes, from which we deduce that $\mathcal{A}''$ yields an element $\mathcal{A} \boxplus \mathcal{A}'$ in $H^1(\mathrm{PGL}_{n+m})$. This element represents α, since $\mathcal{A}''$ does. □

We write $\mathrm{Supp}\, n$ for the set of prime numbers dividing an integer n.

Lemma 5. *Let m and n be positive integers, with $m \mid n$. Then there exists a set of integers $\{q_1, \dots, q_\ell\}$ with $1 \le q_i < n$ and $(q_i, m) = 1$ for all i and such that*

$$\mathrm{Supp}\,\gcd\left\{\binom{n}{q_1}, \dots, \binom{n}{q_\ell}\right\} = \mathrm{Supp}\, m. \tag{6}$$

If $\{q_1, \dots, q_\ell\}$ is a set meeting the conditions of the lemma, and if $q_{\ell+1}$ is some number such that $1 \le q_{\ell+1} < n$ and $(q_{\ell+1}, m) = 1$, then it follows from the proof that $\{q_1, \dots, q_\ell, q_{\ell+1}\}$ also meets the conditions of the lemma. The lemma could therefore be stated as saying that the maximal set $\{q : 1 \le q < n, (q, m) = 1\}$ satisfies (6).

Proof. Suppose a and b are two positive integers and p is a prime. Then the value of $\binom{a}{b}$ in $\mathbb{Z}/(p\mathbb{Z})$ is the coefficient of x^b in the expansion of $(1+x)^a$ over that ring. If $a = cp^s$ for some integer s, then

$$(1+x)^a = \left((1+x)^{p^s}\right)^c \equiv (1 + x^{p^s})^c \pmod{p}$$

from which we deduce that $\binom{a}{b} \equiv 0 \pmod{p}$ unless p^s divides b as well, in which case $\binom{a}{b} \equiv \binom{c}{b/p^s} \pmod{p}$.

Let $\{p_1, \dots, p_\ell\}$ denote the set of primes dividing n but not dividing m. Let s_i denote the exponent of the largest power of p_i dividing n. The binomial coefficient $\binom{n}{p_i^{s_i}} \equiv \binom{n/p_i^{s_i}}{1} \not\equiv 0 \pmod{p}$, while, for any prime q dividing m, we have $\binom{n}{p_i^{s_i}} \equiv 0 \pmod{q}$. The set $\{1, p_1^{s_1}, \dots, p_\ell^{s_\ell}\}$ therefore satisfies the assertions of the lemma. □

Now we come to our main theorem, where we show that (3) from the introduction, and hence (2), holds for a broad class of locally ringed topoi.

Theorem 6. *Let* (X, R) *be a locally-ringed connected topos and let* $\alpha \in \mathrm{Br}(X, R)$. *There exists a representative* $\mathcal{A}$ *of* α *such that the prime numbers dividing* $\mathrm{per}(\alpha)$ *and* $\deg(\mathcal{A})$ *coincide.*

Proof. Write m for the period of α, which is finite and divides $\mathrm{ind}(\alpha)$ by Theorem 3. By definition of the Brauer group, there exists some positive integer n and some $\mathcal{B} \in H^1(\mathrm{PGL}_n)$ such that $\mathcal{B}$ represents α. By Theorem 3, we know that $m|n$.

Let V denote the standard free R–module of rank n. Let $\{q_1, \ldots, q_\ell\}$ be a set of integers $1 \leq q_i < n$ with the properties that $(q_i, m) = 1$ for all i, and

$$\operatorname{Supp} \gcd \left\{ \binom{n}{q_1}, \ldots, \binom{n}{q_\ell} \right\} = \operatorname{Supp} m.$$

For each i, let r_i be a positive integer such that $q_i r_i \equiv 1 \pmod{m}$. For each i between 1 and ℓ, define

$$W_i = \left(V^{\wedge q_i}\right)^{\otimes r_i}.$$

The dimension of W_i is

$$s_i = \binom{n}{q_i}^{r_i}$$

and in particular

$$\operatorname{Supp} \gcd\{s_1, \ldots, s_\ell\} = \operatorname{Supp} m.$$

The formation of W_i from V means that there is a diagonal homomorphism from $\mathrm{GL}(V) = \mathrm{GL}_n$ to $\mathrm{GL}(W_i) = \mathrm{GL}_{s_i}$, and this fits in the following diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathrm{GL}_n & \longrightarrow & \mathrm{PGL}_n & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle z \mapsto z^{q_i r_i}} & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathrm{GL}_{s_i} & \longrightarrow & \mathrm{PGL}_{s_i} & \longrightarrow & 1 \end{array}$$

In particular, there is an induced map $f_i : H^1(\mathrm{PGL}_n) \to H^1(\mathrm{PGL}_{s_i})$ with the property that $f_i(\mathcal{B})$ represents $q_i r_i \alpha = \alpha$ in $H^2(\mathbb{G}_m)$.

For any sufficiently large integer g divisible by $\gcd\{s_1, \ldots, s_\ell\}$ we can find nonnegative integers $\{c'_1, \ldots, c'_\ell\}$ such that

$$g = \sum_{i=1}^{\ell} c'_i s_i.$$

In particular, we can find some sufficiently large integer N such that $\operatorname{Supp} N = \operatorname{Supp} m$ and

$$N = \sum_{i=1}^{\ell} c_i s_i$$

where the c_i are nonnegative integers.

The elements $f_i(\mathcal{B})$ in $H^1(\mathrm{PGL}_{s_i})$ all represent α, and by the construction of Proposition 4, we can form

$$\mathcal{A} = \bigoplus_{i=1}^{\ell} \Big(\bigoplus_{j=1}^{c_i} f_i(\mathcal{B}) \Big),$$

of degree $\deg(\mathcal{A}) = N$, which represents α as well. It lies in $H^1(\mathrm{PGL}_N)$. Since $\operatorname{Supp} N = \operatorname{Supp} m$, the theorem is proved. ☐

We note that the bound on $\deg(\mathcal{A})$ implicit in the proof does not depend on the topos, and is probably wildly inefficient in many interesting cases. For instance, in the case of an element α of period 3, represented by a class $\mathcal{A}$ of degree 60, we must eliminate the primes 2 and 5. We may take as our set $\{q_1, q_2, q_3, q_4\} = \{1, 4, 55, 58\}$, all of which are congruent to 1 modulo 3, which means that we may take $r_1 = r_2 = r_3 = r_4 = 1$. Setting

$$c_1 = 137400, \quad c_2 = 1, \quad c_3 = 1, \quad c_4 = 88,$$

and using the identity

$$N = 3^{15} = 137400\binom{60}{1} + \binom{60}{4} + \binom{60}{55} + 88\binom{60}{58},$$

we deduce that an element of $\mathrm{Br}(X, R)$ having period 3 which is represented by $\mathcal{B}$ of degree 60 may be represented by an Azumaya algebra $\mathcal{A}$ of degree $3^{15} = 14,348,907$.

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