Centre de Recherches Mathématiques
CRM Proceedings and Lecture Notes
Volume **??**, 2008

Global Questions for Map Evolution Equations

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Abstract. Just as the harmonic map equation is a geometric analogue of the classical Laplace equation for harmonic functions, so the classical linear evolution PDEs, the heat, wave, and Schrödinger equations, have geometric "map" analogues: the harmonic map heat-flow, wave map, and Schrödinger map equations. These equations are nonlinear when the target space geometry is nontrivial. Quite remarkably, these equations are all of physical (as well as mathematical) interest, at least when the target space is a 2-sphere, arising variously in the study of ferromagnets (and anti-ferromagnets), liquid crystals, and general relativity. In this article we review some recent results for map evolution equations (focusing on the Landau – Lifshitz family of equations, which includes as special cases the heat-flow and Schrödinger map) concerning the basic global questions: singularity formation vs. global regularity, and long-time asymptotics.

1. Introduction

Let us begin with the harmonic map equation. From the outset, in order to streamline the presentation and make the analysis more concrete, we fix a specific choice of domain and target manifold for our maps:

$$\mathbf{u}\colon \mathbb{R}^n \to \mathbb{S}^2,$$

mostly $n = 2$. We realize $\mathbb{S}^2$ as the unit sphere in $\mathbb{R}^3$:

$$\mathbb{S}^2 := \{\mathbf{u} = (u_1, u_2, u_2) \in \mathbb{R}^3 \mid |\mathbf{u}| = 1\} \subset \mathbb{R}^3.$$

(*Notation.* 3-vectors will be bold-faced throughout.) Harmonic maps are critical points of the Dirichlet energy functional

$$\mathcal{E}(\mathbf{u}) := \frac{1}{2}\int_{\mathbb{R}^n} |\nabla \mathbf{u}|^2 \,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^n} \sum_{j=1}^{n}\sum_{k=1}^{3} \left|\frac{\partial u_k}{\partial x_j}\right|^2 \mathrm{d}x,$$

and so (if regular) solve the corresponding Euler – Lagrange equation

$$0 = -\mathcal{E}'(u) = P^{\mathbf{u}}\Delta\mathbf{u} = \Delta\mathbf{u} + |\nabla\mathbf{u}|^2 u \tag{HM}$$

where $P^{\mathbf{u}}$ denotes the orthogonal projection from $\mathbb{R}^3$ onto the tangent plane

$$T_{\mathbf{u}}\mathbb{S}^2 := \{\boldsymbol{\xi} \in \mathbb{R}^3 \mid \boldsymbol{\xi} \cdot \mathbf{u} = 0\}$$

2000 *Mathematics Subject Classification.* Primary ??; Secondary ??
This is the final form of the paper.

to $\mathbb{S}^2$ at $\mathbf{u}$. Equation (HM) is the equation for harmonic maps between $\mathbb{R}^n$ and $\mathbb{S}^2$. It generalizes Laplace's equation to maps.

1.1. Map evolution equations. Now we let our maps vary with time as well, so that for each time $t \geq 0$,

$$u(\cdot, t)\colon \mathbb{R}^n \to \mathbb{S}^2,$$

or equivalently

$$\mathbf{u}(x,t) = \big(u_1(x,t), u_2(x,t), u_3(x,t)\big) \in \mathbb{R}^3 \text{ with pointwise constraint } |\mathbf{u}(x,t)| \equiv 1.$$

Harmonic map heat-flow. Harmonic maps (between general Riemannian manifolds) have for many years been of interest to differential geometers, and in order to study them, [8] introduced the gradient-flow equations for the energy $\mathcal{E}$, the *harmonic map heat flow* equations $\partial\mathbf{u}/\partial t = -\mathcal{E}'(\mathbf{u})$, which in our setting read

$$\text{(HMHF)} \qquad \frac{\partial \mathbf{u}}{\partial t} = \Delta \mathbf{u} + |\nabla \mathbf{u}|^2 \mathbf{u}$$

The harmonic map heat flow generalizes the linear heat equation to maps.

Landau – Lifshitz equations. Physically, equation (HMHF) is the special case $b = 0$ of the *Landau – Lifshitz* (sometimes *Landau – Lifshitz – Gilbert*) equations modeling dynamics in ferromagnets:

$$\text{(LL)} \qquad \frac{\partial \mathbf{u}}{\partial t} = a(\Delta \mathbf{u} + |\nabla \mathbf{u}|^2 \mathbf{u}) + b\mathbf{u} \times \Delta \mathbf{u}$$

with $a \geq 0$, $b \in \mathbb{R}$, and where $\times$ denotes the usual cross-product in $\mathbb{R}^3$. In fact equation (LL) itself is a special case of a more general equation incorporating additional physical effects such as anisotropy, and demagnetization (see, e.g., [17, 19]).

Schrödinger maps. The opposite limiting case ($a = 0$, i.e., no dissipation) of (LL), can be written

$$\text{(SM)} \qquad \frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} \times \Delta \mathbf{u} = -J^{\mathbf{u}} \mathcal{E}'(\mathbf{u})$$

where the operator

$$J^{\mathbf{u}} := \mathbf{u} \times \, : T_{\mathbf{u}}\mathbb{S}^2 \to T_{\mathbf{u}}\mathbb{S}^2$$

TABLE 1

Linear $u\colon \mathbb{R}^n \to \mathbb{R}$ $(\mathbb{C})$	**Geometric** $u\colon \mathbb{R}^n \to \mathbb{S}^2 \subset \mathbb{R}^3$	**Main Application**
Laplace $\Delta u = 0$	harmonic map $\Delta u = -\lvert\nabla u\rvert^2 u$	geometry
heat $\frac{\partial u}{\partial t} = \Delta u$	h.m. heat-flow $\frac{\partial u}{\partial t} = \Delta u + \lvert\nabla u\rvert^2 u$	geometry, ferromagnets
wave $\frac{\partial^2 u}{\partial t^2} = \Delta u$	wave map $\frac{\partial^2 u}{\partial t^2} = \Delta u + \left(\lvert\nabla u\rvert^2 - \left\lvert\frac{\partial u}{\partial t}\right\rvert^2\right) u$	relativity
Schrödinger $\frac{\partial u}{\partial t} = \mathrm{i}\Delta u$	Schrödinger map $\frac{\partial u}{\partial t} = u \times \Delta u$	ferromagnets

gives a rotation through $\pi/2$ on the tangent plane $T_{\mathbf{u}}\mathbb{S}^2$, and so endows $\mathbb{S}^2$ with a complex structure. Thus equation (SM) can immediately be written for general maps from Riemannian manifolds into Kähler manifolds (see, e.g., [6, 10, 28]).

Since it generalizes the linear Schrödinger equation to maps, (SM) is known as the *Schrödinger map* (sometimes *Schrödinger flow*) equation.

Wave maps. Finally, the *wave map* equation is $P^{\mathbf{u}}(\partial^2\mathbf{u}/\partial t^2 + \Delta\mathbf{u}) = 0$, which in our setting is

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right)\mathbf{u} + \left(\left|\frac{\partial \mathbf{u}}{\partial t}\right|^2 - |\nabla \mathbf{u}|^2\right)\mathbf{u} = 0, \tag{WM}$$

generalizes the linear wave equation to maps. It is has been studied as a toy model in particle physics ("nonlinear σ-model"), but its main interest, aside from the inherent mathematical one, is in general relativity, where it is studied as a (comparatively simple) model for understanding singularity formation (for some background, see, e.g., [23, 24]).

1.2. The energy landscape and equivariant symmetry. The energy $\mathcal{E}(\mathbf{u})$ plays a central role in all of our analysis. We begin by observing that the energy behaves well under the various dynamics introduced above.

Energy identity. Formally taking the dot product of (LL) with $\mathcal{E}'(\mathbf{u}) = -\Delta\mathbf{u} - |\nabla\mathbf{u}|^2\mathbf{u} \in T_{\mathbf{u}}\mathbb{S}^2$ and integrating in space and time yields the basic energy identity

$$\mathcal{E}\big(\mathbf{u}(t)\big) + a\int_0^t \int_{\mathbb{R}^n} |\Delta\mathbf{u} + |\nabla\mathbf{u}|^2\mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}t = \mathcal{E}\big(\mathbf{u}(0)\big). \tag{1.1}$$

For (SM) ($a = 0$) this means energy conservation, while for $a > 0$ (including the (HMHF) case $b = 0$), energy is nonincreasing. A conserved Hamiltonian functional for (WM) is obtained by adding $\frac{1}{2}\int_{\mathbb{R}^n} |\partial\mathbf{u}/\partial t|^2 \,\mathrm{d}x$ to $\mathcal{E}(\mathbf{u})$.

Two space dimensions is energy critical. The energy scales as

$$\mathcal{E}\big(\mathbf{u}(\cdot)\big) = s^{2-n}\mathcal{E}\big(\mathbf{u}(\cdot/s)\big)$$

for $s > 0$, which makes the space dimension $n = 2$ "energy critical." This has important consequences (see below) and in particular leads to the intuition that $n = 2$ should be a borderline case for the formation of singularities for our map dynamics. So $n = 2$ turns out to be particularly interesting mathematically (and of course $n = 2$ and $n = 3$ are physically the most interesting space dimensions). For these reasons, we specialize to $n = 2$ from here on in.

Equivariant symmetry. Since the analysis of our flow equations is a big challenge, a good starting point is to assume some symmetry. Fix an integer $m \in \mathbb{Z}$. By an *m-equivariant map* $\mathbf{u}\colon \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$, we mean a map of the form

$$\mathbf{u}(r,\theta) = \mathrm{e}^{m\theta R}\,\mathbf{v}(r)$$

where (r,θ) are polar coordinates on $\mathbb{R}^2$, $\mathbf{v}\colon [0,\infty) \to \mathbb{S}^2$, and R is the matrix generating rotations around the u_3-axis:

$$R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathrm{e}^{\alpha R} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Radial maps arise as the case $m = 0$, and we may always assume $m \geq 0$ (a trivial transformation flips the sign of m). m-equivariance is preserved by all of the evolution equations considered above. A *subclass* of m-equivariant maps are those of the form

$$\mathbf{u}(r,\theta) = \big(\cos m\theta \sin\phi(r), \sin m\theta \sin\phi(r), \cos\phi(r)\big) \tag{1.2}$$

which depend on a single radial function $\phi(r)$. This subclass is preserved by (HMHF) and (WM), and is much used in the corresponding literature, since the map equation reduces to a scalar PDE for $\phi(r,t)$ (which allows ready use of the maximum principle in the (HMHF) case, for example). This subclass is notably *not* preserved by (LL) or (SM), just as the wave and heat equations preserve real functions, while the Schrödinger equation (or a heat-Schrödinger mix) does not. We will work in the m-equivariant class for most of what follows.

Topological lower bound on energy. There is a well-known energy lower bound

$$\mathcal{E}(\mathbf{u}) \geq 4\pi|\deg(\mathbf{u})|$$

where $\deg(\mathbf{u})$ is the *degree* of the map $\mathbf{u}$, considered (compactifying the domain $\mathbb{R}^2$ via stereographic projection) as a map from $\mathbb{S}^2$ to itself (defined, for example, by integrating the pullback by $\mathbf{u}$ of the volume form on $\mathbb{S}^2$).

This bound is particularly easy to understand when $\mathbf{u}$ is an m-equivariant map, so that

$$\mathcal{E}(\mathbf{u}) = \pi \int_0^\infty \left(\left|\frac{\partial \mathbf{v}}{\partial r}\right|^2 + \frac{m^2}{r^2}(v_1^2+v_2^2)\right) r\,\mathrm{d}r.$$

If $\mathcal{E}(\mathbf{u}) < \infty$, then $\mathbf{v}(r)$ is continuous, and the limits $\lim_{r\to 0}\mathbf{v}(r)$ and $\lim_{r\to\infty}\mathbf{v}(r)$ exist (see [12]), and so we must have $\mathbf{v}(0), \mathbf{v}(\infty) = \pm\widehat{\mathrm{k}}$, where $\widehat{\mathrm{k}} = (0,0,1)$. Without loss of generality we fix $\mathbf{v}(0) = -\widehat{\mathrm{k}}$. The two cases $\mathbf{v}(\infty) = \pm\widehat{\mathrm{k}}$ then correspond to different topological classes of maps. We denote by Σ_m the class of m-equivariant maps with $\mathbf{v}(\infty) = \widehat{\mathrm{k}}$:

$$\Sigma_m = \left\{\mathbf{u} : \mathbb{R}^2 \to \mathbb{S}^2 \mid \mathbf{u} = \mathrm{e}^{m\theta R}\mathbf{v}(r), \mathcal{E}(\mathbf{u}) < \infty, \mathbf{v}(0) = -\widehat{\mathrm{k}}, \mathbf{v}(\infty) = \widehat{\mathrm{k}}\right\}.$$

For $\mathbf{u} \in \Sigma_m$, the energy $\mathcal{E}(\mathbf{u})$ can be rewritten by "completing the square":

$$\mathcal{E}(\mathbf{u}) = \pi \int_0^\infty \left(\left|\frac{\partial \mathbf{v}}{\partial r}\right|^2 + \frac{m^2}{r^2}|J^{\mathbf{v}}R\mathbf{v}|^2\right) r\,\mathrm{d}r = \pi \int_0^\infty \left|\frac{\partial \mathbf{v}}{\partial r} - \frac{|m|}{r}J^{\mathbf{v}}R\mathbf{v}\right|^2 r\,\mathrm{d}r + \mathcal{E}_{\min}$$

(recall $J^{\mathbf{v}} := \mathbf{v}\times$) with

$$\mathcal{E}_{\min} = 2\pi \int_0^\infty \mathbf{v}_r \cdot \frac{|m|}{r} J^{\mathbf{v}}R\mathbf{v}\, r\,\mathrm{d}r = 2\pi|m| \int_0^\infty (v_3)_r\,\mathrm{d}r = 4\pi|m|.$$

Thus we arrive at

$$\mathbf{u} \in \Sigma_m \implies \mathcal{E}(\mathbf{u}) \geq 4\pi|m|.$$

Harmonic maps. This topological lower bound is clearly saturated if and only if

$$\frac{\partial \mathbf{v}}{\partial r} = \frac{|m|}{r} J^{\mathbf{v}}R\mathbf{v}, \tag{1.3}$$

and the minimal energy is attained precisely an explicit two-parameter family of harmonic maps:

$$\mathcal{O}_m := \left\{\mathrm{e}^{(m\theta+\alpha)R}\mathbf{h}(r/s) \mid s>0,\ \alpha\in\mathbb{R}\right\} \tag{1.4}$$

where

$$\mathbf{h}(r) = \begin{pmatrix} h_1(r) \\ 0 \\ h_3(r) \end{pmatrix}, \quad h_1(r) = \frac{2}{r^{|m|} + r^{-|m|}}, \quad h_3(r) = \frac{r^{|m|} - r^{-|m|}}{r^{|m|} + r^{-|m|}}. \tag{1.5}$$

The rotation parameter α is determined only up to shifts of 2π (i.e., really $\alpha \in \mathbb{S}^1$). Note that $\mathcal{O}_m$ is just the orbit of the harmonic map $e^{m\theta R}\mathbf{h}(r)$ under the symmetries of the energy $\mathcal{E}$ which preserve equivariance: scaling and rotation. Of course, these harmonic maps are each static solutions of all of the map evolution equations introduced above.

This phenomenon — attainment of the minimal energy at solutions of a *first-order* PDE, is not special to equivariant maps, it occurs in general, amounting (after stereographically projecting $\mathbb{S}^2$ to the complex plane $\mathbb{C}$) to the Cauchy – Riemann equations for meromorphic (or anti-meromorphic) functions. This is another reason that $n = 2$ is so interesting: the harmonic map problem has a beautiful structure, and a wealth of explicit solutions.

1.3. A little recent history. Here we describe some of the important results for the various map dynamics described above, continuing to focus on maps from $\mathbb{R}^2$ to $\mathbb{S}^2$. We do not attempt to be exhaustive, but rather to point out a few highlights relevant for our discussion.

Harmonic map heat-flow. Of the map evolution problems we are considering, (HMHF) has been studied the longest, and is certainly the best understood. The energy space theory goes back (at least for compact 2-dimensional domains) to [25], where global weak solutions are constructed, with at most finitely many singular space-time points where nonconstant harmonic maps "separate." The small energy solutions are global. Working in the subclass (1.2) with $m = 1$, and on a disk, [4] showed that, indeed, finite time blow-up *does* occur in some solutions. An interesting question, which was studied via formal asymptotics in [29], and addressed rigorously in [11] and in the next section, is the relation between the possibility of singularity formation, and the degree m.

Landau – Lifshitz equation. Once the Schrödinger-type term ($b \neq 0$) in included in (LL), our understanding diminishes considerably. Though the problem is still dissipative, maximum principle-type arguments are not readily applied, and even partial regularity results become more difficult and weaker (see, e.g., [16] and references therein). Singularity formation is an open question, partly because the class (1.2) is no longer preserved. Indeed, the (HMHF) blow-up may not provide a reliable guide for the (LL) problem.

Schrödinger maps. In the absence of dissipation ($a = 0$), the analysis becomes still more difficult. Even the *local* theory is just beginning to be understood. In fact, despite a great deal of recent work on the local well-posedness problem in two space dimensions ([1, 7, 14, 21, 26]; see also [15, 22] for the "modified Schrödinger map" case), there is no general local result for energy space data. For the class of data we consider in the next section, m-equivariant solutions with energy near the minimal energy $2\pi|m|$, an energy-space local well-posedness result is given in [13]. It is worth remarking that the existence time furnished by this theorem depends not on the energy (reflecting the energy-space critical nature of the equation in dimension $n = 2$), but rather on more refined information about the initial data: the "length scale" of the $\dot{H}^1$-nearest harmonic map (see [13] for details).

Very few global results are known. We single out the result of [5] showing that *small energy*, *equivariant* solutions are globally regular. The global results of [12, 13] we describe in the next section can be thought of as analogues of the [5] result for large energy, where the problem is considerably enriched by the presence of the harmonic map family.

Wave maps. Wave maps have received more attention, for a longer time than have Schrödinger maps. There is a large literature, especially concerning local questions, which we will not attempt to summarize here (see, e.g., [24] for some background). Because of the close connection with the $\mathbb{R}^2 \to \mathbb{S}^2$ Schrödinger map problem we are focusing on, we mention only that the possibility of finite-time blow-up for the energy-space critical ($n = 2$) wave maps was established only quite recently, first in [23] for higher degree equivariant maps, and then in [18] for degree $m = 1$. The former result is quite analogous to the Schrödinger map result of [13] which we will describe in the next section, though this is in fact a *no blow-up* result. The essential difference is that the wave map problem allows specification of initial *momentum*, which makes singularity formation easier to force.

2. Global results for Landau – Lifshitz and Schrödinger maps

Here we state some recent results concerning the question of global regularity vs. singularity formation for the Landau – Lifshitz (LL) family of equations

$$\text{(LL)} \qquad \frac{\partial \mathbf{u}}{\partial t} = a(\Delta \mathbf{u} + |\nabla \mathbf{u}|^2 \mathbf{u}) + b\mathbf{u} \times \Delta \mathbf{u}, \quad \mathbf{u}(x, 0) = \mathbf{u}_0(x)$$

($a \geq 0$, $b \in \mathbb{R}$) which of course includes as special cases the harmonic map heat-flow ($a = 1$, $b = 0$) and the Schrödinger map ($a = 0$, $b = 1$). This is mostly work from the papers [12, 13] which address the Schrödinger case, and the paper [11] which addresses the heat-flow case.

The results concern m-equivariant maps with energy near the minimal energy $\mathcal{E}_{\min} = 4\pi|m|$ (the harmonic map energy), and so the standing assumption on the initial data is

$$\mathbf{u}_0 \in \Sigma_m, \quad \mathcal{E}(\mathbf{u}_0) = 4\pi|m| + \delta_0^2, \quad \delta_0 \ll 1.$$

Let

$$\mathbf{u}(t) \in C([0, T); \Sigma_m)$$

(Σ_m topologized with the energy ($\dot{H}^1$) norm) be the solution of (LL) corresponding to the initial data $\mathbf{u}_0$ (which is a priori just a local-in-time solution — see [13] for local well-posedness for this class of data).

Theorem 2.1 ([12] **"orbital stability" of harmonic maps).** *For δ_0 sufficiently small, there exist $s(t) \in \mathcal{C}\big([0, T); (0, \infty)\big)$ and $\alpha(t) \in \mathcal{C}([0, T); \mathbb{R})$ so that*

$$\big\|\mathbf{u}(x, t) - \mathrm{e}^{(m\theta + \alpha(t))R}\mathbf{h}\big(r/s(t)\big)\big\|_{\dot{H}_1(\mathbb{R}^2)} \lesssim \delta_0, \quad \forall t \in [0, T). \tag{2.1}$$

Moreover, suppose $T < \infty$. Then T is the maximal existence time ($\mathbf{u}(t)$ doesn't extend past T as a solution continuous into Σ_m) if and only if

$$\liminf_{t \to T^-} s(t) = 0. \tag{2.2}$$

This theorem can be viewed, on one hand, as an *orbital stability* result for the family $\mathcal{O}_m$ of harmonic maps (at least up to the possible blow-up time), and on the other hand as a characterization of blow-up for energy near $\mathcal{E}_{\min}$: solutions blow-up if and only if the "length-scale" $s(t)$ goes to zero.

The first statement of the theorem is really a kind of convexity result for the energy functional — the only property of the dynamics that is needed is the nonincrease of the energy, as follows from the energy identity (1.1). The details can be found in [12]

The second statement comes from the energy-space local existence theory — see [13].

Of course this theorem leaves open the question of whether or not finite-time singularities can form. The next result shows that when the degree is sufficiently high, singularities will *not* form, and moreover, solutions converge to specific harmonic maps as $t \to \infty$.

Theorem 2.2 ([11, 13] **global regularity and asymptotic stability for high degree).** *For* (LL) *with* $a > 0$*, assume* $|m| \geq 3$*. For* (SM) $(a = 0)$*, assume* $|m| \geq 4$*. As before,* δ_0 *is sufficiently small. Then*

(1) *there is no finite-time blow-up: the solution can be extended to* $\mathbf{u} \in C([0, \infty); \Sigma_m)$.

(2) *For any* $r \in (2, \infty]$*,* $p \in [2, \infty)$ *with* $1/r + 1/p = \frac{1}{2}$*, we have*

$$\left\| \nabla \left[\mathbf{u}(x,t) - \mathrm{e}^{(m\theta + \alpha(t))R} \mathbf{h}\big(r/s(t)\big) \right] \right\|_{L_t^r L_x^p(\mathbb{R}^2 \times [0,\infty))} \lesssim C_p \delta_0$$

(*if* $a > 0$ *we may include* $(r, p) = (2, \infty)$)

(3) *furthermore, there exist* $s_+ > 0$ *and* α_+ *with*

$$s(t) \to s_+, \quad \alpha(t) \to \alpha_+, \quad \text{as } t \to \infty.$$

The space-time estimates above imply asymptotic *convergence* of the solutions to the family of harmonic maps (in a space-time norm ("dispersive") sense, which is the best we can expect for the Schrödinger case $a = 0$). Hence we say the harmonic maps are *asymptotically stable* under the Landau – Lifshitz flow for $|m| \geq 3$ ($|m| \geq 4$ for (SM)).

The question of blow-up for lower degree maps is unresolved, except in the pure heat-flow case. There we have

Theorem 2.3 ([11] **heatflow blow-up for** $\boldsymbol{m = 1}$**).** *Let* $m = 1$*. For any* $\delta > 0$*, there exists* $u_0 \in \Sigma_1$ *with* $0 < \mathcal{E}(u_0) - 4\pi \leq \delta^2$ *such that the corresponding solution of the harmonic map heat flow blows up in finite time, in the sense that (for example)* $\|\nabla \mathbf{u}(\cdot, t)\|_{L_x^\infty} \to \infty$.

This result is an adaptation of the blow-up proof of [4] for a disk domain, to the case of $\mathbb{R}^2$. In particular, it must be verified that the construction (based on a subsolution argument) can be achieved for data with energy arbitrarily close to the harmonic map energy. The reader is referred to [11] for details.

We will present a few of the central ideas of our approach — and in particular of the proof of Theorem 2.2 — in the next two sections.

3. The approach: two geometric coordinate systems

The framework is to write our maps $\mathbf{u}(x, t)$ using two essentially different "coordinate systems" in the energy space of maps. The first decomposes a map into a "nearby" harmonic map (finite-dimensional part), and a deviation from the harmonic map family (infinite-dimensional part). This system is used to track the time-varying parameters of the "nearest" harmonic map. The second coordinate system — the "generalized Hasimoto transform" of [5] — involves a kind of projection which removes the harmonic map component, and produces an equation for which we can do the hard estimates showing the perturbation away from the harmonic maps "decays" with time.

3.1. Splitting and orthogonality. We split $\mathbf{u}(x,t) = \mathrm{e}^{m\theta R}\mathbf{v}(r,t) \in \Sigma_m$ as a harmonic map with time-varying parameters, plus a perturbation:

$$\mathbf{v}(r,t) = \mathrm{e}^{\alpha(t)R}[\mathbf{h}(\rho) + \boldsymbol{\xi}(\rho,t)], \quad \rho := \frac{r}{s(t)}$$

The choice of the parameter paths $s(t)$ and $\alpha(t)$ is important, and we will address it later.

We further split the perturbation into tangent and normal components (to the sphere at the harmonic map) using the explicit orthonormal basis of $T_{\mathbf{h}(\rho)}\mathbb{S}^2$

$$\widehat{\jmath} := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad J^{\mathbf{h}(\rho)}\widehat{\jmath} = \begin{pmatrix} -h_3(\rho) \\ 0 \\ h_1(\rho) \end{pmatrix},$$

$$\xi(\rho,t) = z_1(\rho,t)\widehat{\jmath} + z_2(\rho,t)J^{\mathbf{h}(\rho)}\widehat{\jmath} + \gamma(\rho,t)\mathbf{h}(\rho)$$

where the pointwise constraint $|u(x,t)| \equiv 1$ forces $\gamma = O(|z|^2)$ for $|\xi|$ small. In this way, the complex function

$$z(\rho,t) = z_1(\rho,t) + \mathrm{i}z_2(\rho,t),$$

together with a choice of the parameters $s(t)$ and $\alpha(t)$, gives a full description of the original solution $\mathbf{u}(x,t)$ (provided $|\boldsymbol{\xi}| \leq 1$).

It can then be shown that if $\mathbf{u}(x,t)$ satisfies (LL), then $z(\rho,t)$ satisfies a nonlinear equation of (in general) mixed heat-Schrödinger type, of the form

$$s^2\frac{\partial z}{\partial t} = -(a+\mathrm{i}b)Hz + (s^2\dot{\alpha} - \mathrm{i}m\, s\dot{s})h_1 + F(\rho,t), \quad H := -\partial_\rho^2 - \frac{1}{\rho}\partial_\rho + \frac{m^2}{\rho^2}(1-2h_1^2),$$

where F denotes terms nonlinear in z, $\dot{s}$, and $\dot{\alpha}$.

We next address the question of how to choose $s(t)$ and $\alpha(t)$. Supposing for a moment that $s(t) \equiv 1$, $\alpha(t) \equiv 0$, the linearized equation for $z(\rho,t)$ is

$$\partial_t z = -(a+\mathrm{i}b)Hz.$$

and

$$H = L_0^* L_0, \quad L_0 := \partial_\rho + \frac{m}{\rho}h_3 = h_1\partial_\rho\frac{1}{h_1}$$

(the adjoint L_0^* is taken in the $L^2(\rho\,\mathrm{d}\rho)$ inner product). So in particular, $\ker H = \mathrm{span}\{h_1\}$, and the linearized equation admits the constant (in time) solution $z(\rho,t) \equiv h_1(\rho)$. Since we would like $z(\rho,t)$ to have some decay in time, we must choose $s(t)$ and $\alpha(t)$ in such a way as to avoid such constant solutions. And since H is self-adjoint in L^2, the natural choice is to work in the subspace of functions z satisfying

$$(z,h_1)_{L^2} = \int_0^\infty z(\rho)h_1(\rho)\rho\,\mathrm{d}\rho \equiv 0, \tag{3.1}$$

which is invariant under the linearized equation. We will see below that this choice is good for several other reasons. Imposition of this condition will determine the dynamics of $s(t)$ and $\alpha(t)$. But there is an important drawback to this approach:

neither z, nor h_1, lies in L^2 in general. The natural (energy) space for z is given by the norm (see [12])

$$\|z\|_X^2 := \int_0^\infty \left\{ |z_\rho|^2 + \frac{|z|^2}{\rho^2} \right\} \rho \, \mathrm{d}\rho$$

and so the best we can do is

$$|(z, h_1)_{L^2}| = \left| \left(\frac{z}{\rho}, \rho h_1 \right)_{L^2} \right| \leq \|z\|_X \|\rho h_1\|_{L^2}.$$

So to make sense of our orthogonality condition, we require

$$\rho h_1(\rho) = \frac{2\rho}{\rho^m + \rho^{-m}} \in L^2(\rho, \mathrm{d}\rho),$$

which only holds if $m \geq 3$. This is the main reason we cannot handle the small $|m|$ cases in Theorem 2.2. (The further restriction $|m| > 3$ for (SM) is needed for different reasons, to be explained shortly.)

Differentiating the orthogonality condition with respect to t, and using the equation for z, we arrive at a system of ODEs for $s(t)$ and $\alpha(t)$, coupled to $z(\rho, t)$. A crucial aspect of our choice of orthogonality condition is that the linear terms drop out, and $\dot{s}$ and $\dot{\alpha}$ are *quadratic* in z:

$$|s\dot{s}| + |s^2 \dot{\alpha}| \lesssim \left| \big(h_1, F(\rho, t)\big) \right| \tag{3.2}$$

The objective then is to obtain estimates on $z(\rho, t)$ which are L^2 in time, which will show $\dot{s}$ and $\dot{\alpha}$ are integrable in time, and so $s(t)$ and $\alpha(t)$ converge to limits as $t \to \infty$.

3.2. Generalized Hasimoto transform. The equation for $z(\rho, t)$, however, is not suitable for obtaining estimates, for at least two reasons: (a) the orthogonality condition has to be imposed to avoid nondecaying solutions, and (b) there are nonlinear terms containing derivatives of z. Fortunately, there is a neat way around these problems: the *generalized Hasimoto transform* of [5].

The Landau – Lifshitz equation (SM), written in terms of $\mathbf{v}(r, t)$, can be factored as

$$\frac{\partial \mathbf{v}}{\partial t} = (a + bJ^{\mathbf{v}})[D_r^{\mathbf{v}} + \frac{1}{r} - \frac{m}{r} v_3] \mathbf{W}$$

where

$$\mathbf{W}(r) := \mathbf{v}_r(r) - \frac{m}{r} J^{\mathbf{v}} R \mathbf{v}(r) \in T_{\mathbf{v}(r)} \mathbb{S}^2$$

and

$$D_r^{\mathbf{v}} := P^{\mathbf{v}(r)} \partial_r$$

denotes the *covariant derivative* (with respect to r, along $\mathbf{v}$).

Let $\mathbf{e}(r) \in T_{\mathbf{v}(r)} \mathbb{S}^2$ be a unit-length tangent field satisfying the "gauge condition" $D_r^{\mathbf{v}} \mathbf{e} \equiv 0$. Expressing $\mathbf{W}$ in the orthonormal frame $\{\mathbf{e}, J^{\mathbf{v}} \mathbf{e}\}$,

$$\mathbf{W} = q_1 \mathbf{e} + q_2 J^{\mathbf{v}} \mathbf{e},$$

it is not difficult to arrive at the following equation for the complex function $q(r, t) := q_1(r, t) + \mathrm{i} q_2(r, t)$:

$$q_t = (a + \mathrm{i}b) \left(\Delta_r - \frac{1}{r^2} \big((1 - m v_3)^2 + m r (v_3)_r \big) \right) q - \mathrm{i} S q \tag{3.3}$$

where the function $S(r,t)$ arises as $D_t^{\mathbf{v}}\mathbf{e} = SJ^{\mathbf{v}}\mathbf{e}$. The curvature relation

$$[D_r, D_t]\mathbf{e} = -\operatorname{Re}\left[\left(\partial_r + \frac{1}{r} - \frac{m}{r}v_3\right)q\overline{\left(q + \frac{m}{r}\nu\right)}\right]J^{\mathbf{v}}\mathbf{e},$$

where $P^{\mathbf{v}(r)}\widehat{\mathrm{k}} = \widehat{\mathrm{k}} - v_3\mathbf{v} = \nu_1\mathbf{e} + \nu_2 J^{\mathbf{v}}\mathbf{e}$, then leads to

$$S(r,t) = -\frac{1}{2}Q(r,t) + \int_r^\infty \frac{1}{\tau}Q(\tau,t)\,\mathrm{d}\tau, \quad Q := |q|^2 + \frac{2m}{r}\operatorname{Re}(\bar{\nu}q).$$

So our equation resembles a cubic nonlinear heat-Schrödinger equation, keeping in mind (a) there are nonlocal nonlinear terms, and (b) it is not self-contained: the unknown map $\mathbf{v}(r,t)$ itself appears in several places (including through ν).

A key point is that

$$1 \gg \mathcal{E}(\mathbf{u}) - 4\pi m = \tfrac{1}{2}\|\mathbf{W}\|_{L^2}^2 = \pi\|q\|_{L^2(r\,\mathrm{d}r)}^2.$$

The transformation has "killed" the harmonic map component, leaving us with a *small L^2-data problem* for for the q equation (3.3), even though the map $\mathbf{u}$ is not a small-energy map). What's more, this equation is amenable to estimates.

3.3. Relating the two coordinate systems. Of course, it is useless (and impossible) to obtain estimates for $q(r,t)$, unless we can control z (and hence $\mathbf{v}$) in terms of q. This is certainly *only* possible if we have a supplementary condition such as the orthogonality condition (3.1), since $q \equiv 0$ just means that $\mathbf{v}(r) = \mathrm{e}^{\alpha R}\mathbf{h}(r/s)$ for some s, α).

We have: provided $|m| \geq 3$ and (3.1) holds, and $\|z\|_X \ll 1$ is sufficiently small,

(1) $\|z_\rho\|_{L^p} + \|z/\rho\|_{L^p} \lesssim s^{1-2/p}\|q\|_{L^p}$ $2 \leq p \leq \infty$
(2) if further $|m| \geq 4$, then $\|z_\rho/\rho\|_{L^2} + \|z/\rho^2\|_{L^2} \lesssim s\|q/r\|_{L^2}$.

The proofs can be found in [13] (the $p = \infty$ case of the first statement was not considered there, but it can be shown also to hold).

The second statement is the source of the extra restriction $|m| \geq 4$ in the (SM) case. The point is that we need $\dot{s}$ and $\dot{\alpha}$ to be L_t^1, and hence by (3.2) we need an estimate on z which is L_t^2. As we will see in the next section, we can estimate q in $L_t^2 L_x^\infty$ if $a > 0$ (and so we may use the $p = \infty$ case of the first statement above), but we don't have this estimate when $a = 0$, and so must rely on $q/|x| \in L_t^2 L_x^2$, forcing us to use the second statement above.

4. Linear evolution estimates with critical-decay potentials

So the remaining task is to obtain estimates for $q(r,t)$ (including L^2-in-time estimates). The key here is to understand the linear part, obtained by substituting $\mathbf{h}(r)$ for $\mathbf{v}(r,t)$ in the q equation (3.3), and dropping nonlinear terms, to arrive at

$$q_t = (a + \mathrm{i}b)\left[q_{rr} - \frac{1}{r}q_r - \frac{1}{r^2}(1 + m^2 - 2mh_3)q\right].$$

Thus we need space-time estimates for the linear evolution operator

$$\mathrm{e}^{-(a+\mathrm{i}b)tH}, \quad H = -\Delta + V(|x|), \quad V(r) = \frac{1 + m^2 - 2mh_3(r)}{r^2}$$

where $a \geq 0$, $b \in \mathbb{R}$. This turns out to be an interesting mathematical problem, precisely because the $1/r^2$ behavior of the potential $V(r)$ at the origin and at infinity places it just beyond the reach (on the borderline, in fact) of the typical perturbative arguments used to obtain estimates for such operators. Indeed, the question of estimates for such operators has quite recently been addressed: see [30] for the purely diffusive case ($b = 0$), and [2, 3] for the purely conservative case ($a = 0$). In both cases, the relevant results hold for space dimensions ≥ 3, because of the lack of Hardy inequality in dimension 2. Our job, then is to recover 2-dimensional versions of these estimates, exploiting our symmetry (equivariance) assumption. We summarize the resulting estimates for L^2 initial data:

Theorem 4.1. *Let the exponent pairs (r, p) and $(\tilde{r}, \tilde{p})$ both satisfy the "admissibility" condition $1/r + 1/p = \frac{1}{2}$. The space-time estimates*

$$\|\mathrm{e}^{-t(a+\mathrm{i}b)H}\varphi\|_{L_t^r L_x^p} + \|\int_0^t \mathrm{e}^{-(t-s)(a+\mathrm{i}b)H} f(\cdot, s)\,\mathrm{d}s\|_{L_t^r L_x^p} \lesssim \|\varphi\|_{L^2} + \|f\|_{L_t^{\tilde{r}'} L_x^{\tilde{p}'}}$$

hold in the following cases:

• **pure heat ($a > 0$, $b = 0$)** *for general ϕ, f, excluding the endpoint case $r = \tilde{r} = 2$ (so $p = \tilde{p} = \infty$), which is false in general — see the example of* [27]

• **mixed case ($a > 0$, $b \in \mathbb{R}$)** *for $\phi(\cdot)$ and $f(\cdot, t)$* radial, *but* including *the endpoint case $r = \tilde{r} = 2$, provided $m \geq 2$.*

• **pure Schrödinger case ($a = 0$, $b \neq 0$)** *holds for $\phi(\cdot)$ and $f(\cdot, t)$* radial, *but* not including *the endpoint case $r = \tilde{r} = 2$ (which is open), again provided $m \geq 2$.*

In addition, for all cases, *if $m \geq 2$ and ϕ and f are radial, we also have weighted L^2 estimates. Denoting $\|g\|_{L^{2,\pm 1}} := \||x|^{\pm 1} g\|_{L^2}$:*

$$\|\mathrm{e}^{-t(a+\mathrm{i}b)H}\varphi\|_{L_t^2 L_x^{2,-1}} \lesssim \|\varphi\|_{L^2}$$

$$\left\|\int_0^t \mathrm{e}^{-(t-s)(a+\mathrm{i}b)H} f(\cdot, s)\,\mathrm{d}s\right\|_{L_t^2 L_x^{2,-1} \cap L_t^r L_x^p} \lesssim \min(\|f\|_{L_t^{\tilde{r}'} L_x^{\tilde{p}'}}, \|f\|_{L_t^2 L_x^{2,1}})$$

(excluding $r = \tilde{r} = 2$ only if $a = 0$).

Remarks on the proofs of these estimates. The goal is to estimate $w(x, t)$, a solution of the linear inhomogeneous initial value problem

$$w_t + (a + \mathrm{i}b)Hu = f, \quad w(x, 0) = \varphi(x).$$

• **L^2 contraction for any $a \geq 0$.** We begin with this trivial observation, using $V \geq 0$. Multiply the equation for w by $\overline{w}$, take the real part, integrate in space and time, and use the Hölder inequality to produce the basic L^2 estimate:

$$\|w\|_{L_t^\infty L_x^2} \lesssim \|\varphi\|_{L^2} + \|f\|_{L_t^1 L_x^2}.$$

• **Estimates for $a > 0$, radial data, and $m \geq 2$.** These estimates are the easiest, coming as they do directly from "energy estimates." First, to compensate for the lack of Hardy inequality in dimension 2, change the function:

$$\widetilde{w}(x, t) := \mathrm{e}^{ii\theta} w(r, t)$$

so that

$$|w_r|^2 + |\frac{w}{r}|^2 = |\nabla \widetilde{w}|^2,$$

where now $\widetilde{w}(x, t)$ solves

$$\widetilde{w}_t + (a + \mathrm{i}b)(-\Delta + \widetilde{V})\widetilde{w}, \quad \widetilde{w}(x, 0) = \mathrm{e}^{\mathrm{i}\theta}\varphi(r),$$

and the key fact is that the "new" potential satisfies

$$r^2 \widetilde{V}(r) = m^2 - 2mh_3(r) \geq m^2 - 2m \geq 0 \quad \text{for } m \geq 2.$$

To get an L^∞_x estimate, we employ the simple embedding inequality

$$\|w\|_{L^\infty} \lesssim \|w_r\|_{L^2}^{1/2} \|\frac{w}{r}\|_{L^2}^{1/2}$$

for radial functions, established in [12]. So now, as above, multiply the equation by $\overline{\widetilde{w}}$, take the real part, integrate over space and time, and use $a > 0$ and $\widetilde{V} \geq 0$ to arrive at

$$\|w\|^2_{L^\infty_t L^2_x \cap L^2_t L^{2,-1}_x \cap L^\infty_t L^2_x} \leq \|\nabla \widetilde{w}\|^2_{L^2_t L^2_x} + \|\widetilde{w}\|^2_{L^\infty_t L^2_x} \lesssim \|\varphi\|^2_{L^2} + \|wf\|_{L^1_t L^1_x}$$

which, by Hölder and interpolation yields all the desired estimates in this case (radial data, $a > 0$, $m \geq 2$).

• **Estimates for $b = 0$.** Not surprisingly, we obtain the finest estimates in the "pure heat" case. The first step is to establish the time decay estimates

$$\|e^{-tH}\varphi\|_{L^p_x} \lesssim t^{-(1/a-1/p)/2}\|\varphi\|_{L^a}$$

for $1 \leq a \leq p \leq \infty$. This is done via the maximum principle, comparing with the the heat flow $e^{t\Delta}$ for which these estimates follow immediately from the explicit fundamental solution. A necessary preliminary step is to establish boundedness of solutions via an embedding inequality $\|w\|_{L^\infty} \lesssim \|Hw\|_{L^2} + \|w\|_{L^2}$. The second step is to follow [9] in applying the Marcinkiewicz interpolation theorem to the time decay estimates to generate the homogeneous space-time estimates

$$\|e^{-tH}\varphi\|_{L^r_t L^p} \leq C\|\varphi\|_{L^a}$$

for $1/r = 1/a - 1/p'$, $r \geq a > 1$. Finally, the corresponding nonhomogeneous estimates then follow from these, together with the the Hardy−Littlewood inequality.

• **Estimates for $a = 0$.** The "pure Schrödinger" case is the most delicate. Here we adopt the approach from [2, 3], working with radial functions to compensate for the lack of Hardy inequality in 2 dimensions. The idea is to first obtain the weighted L^2 estimates (in this context a "Kato smoothing" estimate) through a resolvent estimate which itself comes directly from estimates on the relevant equation, and by exploiting the positivity and repulsivity of the potential. The various other estimates can then be obtained by perturbative arguments from the reference operator $-\Delta + 1/r^2$, provided $m \geq 2$. We refer to [13] for details.

5. Conclusions and future directions

The map equations discussed in this article are of both physical and geometric interest, and yet it is only very recently that the global behavior of solutions is starting to be understood, and that only in very limited settings. Much work remains to be done. We single out a few pressing directions of inquiry:

• **Move beyond the equivariant setting.** To now, global results (e.g., blow-up or asymptotic behavior) for energy-critical map problems are confined to equivariant maps. It is important to remove this restrictive symmetry assumption, especially for stability and asymptotic questions. Among other things, this requires (1) understanding the energy landscape near the (now much larger) harmonic map family outside the equivariant class (note that some of this analysis described above is essentially one-dimensional in nature, and may not survive the removal of the symmetry assumption); (2) adapting the "Hasimoto transform" used above, which introduces "loss-of-regularity" problems.

• **Singularity formation in $m = 1$ Schrödinger maps.** This is the most obvious unanswered question. Blow-up arguments for the pure heat-flow with degree $m = 1$ rely heavily on the maximum principle, unavailable for the Landau – Lifshitz problem (with $b \neq 0$). On the other hand, the analysis described above breaks down (in several ways) for $m = 1$, as indeed it should if blow-up is possible.

• **The full physical model.** The Landau – Lifshitz equation described above is a "bare-bones" model, incorporating only the "exchange energy," and (for $a > 0$) dissipation. The question of how physically important effects such as anisotropy, and demagnetization (a nonlocal term) affect solutions has hardly been touched.

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