

Interior Regularity Criteria for Suitable Weak Solutions of the Navier-Stokes Equations

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Abstract: We present new interior regularity criteria for suitable weak solutions of the 3-D Navier-Stokes equations: a suitable weak solution is regular near an interior point z if either the scaled $L_{x,t}^{p,q}$ -norm of the velocity with $3/p + 2/q \leq 2$, $1 \leq q \leq \infty$, or the $L_{x,t}^{p,q}$ -norm of the vorticity with $3/p + 2/q \leq 3$, $1 \leq q < \infty$, or the $L_{x,t}^{p,q}$ -norm of the gradient of the vorticity with $3/p + 2/q \leq 4$, $1 \leq q, 1 \leq p$, is sufficiently small near z .

1. Introduction

We continue our study in [10] of the regularity problem for *suitable weak solutions* $(u, p) : \Omega \times I \rightarrow \mathbb{R}^3 \times \mathbb{R}$ of the three-dimensional incompressible Navier-Stokes equations (NS)

$$\begin{cases} u_t - \Delta u + (u \cdot \nabla)u + \nabla p = f & \text{in } \Omega \times I, \\ \operatorname{div} u = 0 \end{cases} \quad (1)$$

Here Ω is either a domain in \mathbb{R}^3 or the 3-dimensional torus \mathbb{T}^3 , I is a finite time interval, $u(x, t)$ is the velocity field and $p(x, t)$ is the pressure. We also denote the vorticity field by $w = \operatorname{curl} u$. By suitable weak solutions we mean functions which solve (1) in the sense of distributions and satisfy some integrability conditions and the local energy inequality (for details, see Definition 2.1 in Sect. 2). For a point $z = (x, t) \in \mathbb{R}^3 \times \mathbb{R}$ we denote

$$B_{x,r} = \{y \in \mathbb{R}^3 : |y - x| < r\}, \quad Q_{z,r} := B_{x,r} \times (t - r^2, t).$$

A solution u is said to be *regular at* $z \in \Omega \times I$ if $u \in L^\infty(Q_{z,r})$ for some $Q_{z,r} \subset \Omega \times I$, $r > 0$. Otherwise it is *singular at* z (see [2, p. 780]).

Although the existence of weak solutions was proved by Leray and Hopf [17, 11] in \mathbb{R}^3 and domains, it is not known whether the solution stays regular for all time even if all

the data are smooth. One type of condition ensuring regularity involves zero-dimensional integrals,

$$\|u\|_{L^{p,q}(\Omega \times I)} < \infty, \quad \frac{3}{p} + \frac{2}{q} = 1, \quad 3 \leq p \leq \infty, \tag{2}$$

where

$$\|u\|_{L^{p,q}(\Omega \times I)} = \|u\|_{L_t^q L_x^p(\Omega \times I)} = \left\| \|u(x, t)\|_{L_x^p(\Omega)} \right\|_{L_t^q(I)}. \tag{3}$$

These integrals have zero dimension if one assigns the dimensions 1, 2, and -1 to x , t and u . This is related to the scaling property of solutions of (NS): The map

$$\{u(x, t), p(x, t)\} \rightarrow \{\lambda u(\lambda x, \lambda^2 t), \lambda^2 p(\lambda x, \lambda^2 t)\} \quad (\lambda > 0), \tag{4}$$

sends a solution of (NS) to another solution, with a new force $\lambda^3 f(\lambda x, \lambda^2 t)$.

The first contributions in this direction, concerning uniqueness and regularity of weak solutions, were made by [20, 31, 32, 15] when $3/p + 2/q < 1$. The borderline cases $3/p + 2/q = 1$, $3 < p \leq \infty$, for different types of domains were later proved by [8, 33, 9, 37]. See [38, 34, 4] for results in the setting of Lorentz spaces. The endpoint case $(p, q) = (3, \infty)$ was recently resolved [7] (also see the references in [34, 7] for earlier results in subclasses). Similar regularity criteria have been established near the boundary [36, 12, 28].

In a series of papers [21]–[24], Scheffer began to study the partial regularity theory for (NS). His results were further generalized and strengthened in Caffarelli-Kohn-Nirenberg [2], which proved that the set \mathcal{S} of possible interior singular points of a suitable weak solution is of one-dimensional parabolic Hausdorff measure zero, i.e. $\mathcal{P}^1(\mathcal{S}) = 0$ (the estimate of the Hausdorff measure was improved by a logarithmic factor in [5]). The key to the analysis in [2] is the following regularity criterion: there is an absolute constant $\epsilon > 0$ such that, if u is a suitable weak solution of (NS) in $\Omega \times I$ and if for an interior point $z \in \Omega \times I$,

$$\limsup_{r \rightarrow 0_+} \frac{1}{r} \int_{Q_{z,r}} |\nabla u(y, s)|^2 dy ds \leq \epsilon, \tag{5}$$

then u is regular at z . See [18] for a simpler proof and [16] for more details. See [27, 29] for extensions when z lies on a flat or curved boundary.

The objective of this paper is to present new sufficient conditions for the regularity of suitable weak solutions to (NS) in the interior, in terms of the smallness of the scaled $L^{p,q}$ -norm of the velocity, vorticity or the gradient of the vorticity. We obtained such results in terms of the velocity either in the interior or on a flat boundary in [10]. We will assume that the force f belongs to a parabolic Morrey space $M_{2,\gamma}$, for some $\gamma > 0$, equipped with the norm

$$\|f\|_{M_{2,\gamma}(\Omega \times I)}^2 = \sup_{Q_{z,r} \subset \Omega \times I, r > 0} \frac{1}{r^{1+2\gamma}} \int_{Q_{z,r}} |f|^2 dz'. \tag{6}$$

(This space is trivial if $\gamma > 2$, and it contains $L^{\frac{5}{2-\gamma}}(\Omega \times I)$ for $\gamma \leq 2$.)

Suitable weak solutions will be defined in Definition 2.1 of Sect. 2.

Theorem 1.1 (Regularity Criteria). *Suppose the pair (u, p) is a suitable weak solution of (NS) in $\Omega \times I$ with force $f \in M_{2,\gamma}(\Omega \times I)$ for some $\gamma > 0$. Suppose $z = (x, t) \in \Omega \times I$ and $Q_{z,r} \subset \Omega \times I$. Then u is regular at z if one of the following conditions holds, for a small constant $\epsilon > 0$ depending only on p^* (or p, p^\sharp), q , and γ (but independent of $\|f\|_{M_{2,\gamma}}$).*

(i) **Velocity criteria.** $u \in L_{\text{loc}}^{p^*,q}$ near z and

$$\limsup_{r \rightarrow 0_+} r^{-\left(\frac{3}{p^*} + \frac{2}{q} - 1\right)} \|u - (u)_r\|_{L^{p^*,q}(Q_{z,r})} \leq \epsilon, \tag{7}$$

where $(u)_r(s) = \frac{1}{|B_r|} \int_{B_r} u(y, s) dy$, for some p^*, q satisfying

$$1 \leq 3/p^* + 2/q \leq 2, \quad 1 \leq p^*, q \leq \infty. \tag{8}$$

The same result holds if $u - (u)_r$ is replaced by u in (7).

(ii) **Velocity gradient criteria.** $\nabla u \in L_{\text{loc}}^{p,q}$ near z and

$$\limsup_{r \rightarrow 0_+} r^{-\left(\frac{3}{p} + \frac{2}{q} - 2\right)} \|\nabla u\|_{L^{p,q}(Q_{z,r})} \leq \epsilon, \tag{9}$$

for some p, q satisfying

$$2 \leq 3/p + 2/q \leq 3, \quad 1 \leq q \leq \infty. \tag{10}$$

(iii) **Vorticity criteria.** $w = \text{curl } u \in L_{\text{loc}}^{p,q}$ near z and

$$\limsup_{r \rightarrow 0_+} r^{-\left(\frac{3}{p} + \frac{2}{q} - 2\right)} \|w\|_{L^{p,q}(Q_{z,r})} \leq \epsilon, \tag{11}$$

for some p, q satisfying

$$2 \leq 3/p + 2/q \leq 3, \quad 1 \leq q \leq \infty, \quad (p, q) \neq (1, \infty). \tag{12}$$

(iv) **Vorticity gradient criteria.** $\nabla^2 u \in L_{\text{loc}}^{p^\sharp,q}$ near z and

$$\limsup_{r \rightarrow 0_+} r^{-\left(\frac{3}{p^\sharp} + \frac{2}{q} - 3\right)} \|\nabla w\|_{L^{p^\sharp,q}(Q_{z,r})} \leq \epsilon, \tag{13}$$

for some p^\sharp, q satisfying

$$3 \leq 3/p^\sharp + 2/q \leq 4, \quad 1 \leq q, \quad 1 \leq p^\sharp. \tag{14}$$

Furthermore, for $p^\sharp > 1$, ∇w can be replaced by $\text{curl } w$.

Comments for Theorem 1.1.

1. The region defined by (8) corresponds to the union of II and III in Fig. 1, including all borderlines. The region defined by (10) corresponds to IV, including all borderlines. The region defined by (12) also corresponds to IV, but without the corner point $(1/p, 1/q) = (1, 0)$. The region defined by (14) corresponds to V, including all borderlines.

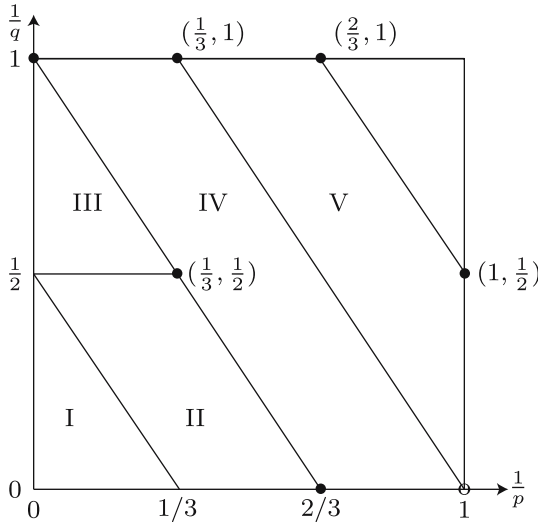


Fig. 1. Regularity Criteria

2. In (8), the lower bound $1 \leq 3/p^* + 2/q$ is only to ensure a non-positive exponent of r in (7). The true limit is the upper bound $3/p^* + 2/q \leq 2$. Similar comments apply to (10), (12) and (14).
3. The quantities in (7), (9), (11) and (13) are zero-dimensional, and are invariant under the scaling (4). Such quantities are useful in the regularity theory for (NS), see e.g. [2].
4. In [10], the authors obtained Theorem 1.1 (i) only for region II, without the borderline $q = 2$ (but the result is also valid on a flat boundary of Ω). Theorem 1.1 (i) extends it to region III, and in particular includes the point $(1/p, 1/q) = (1/3, 1/2)$. It does not further assume the smallness of the pressure, in contrast to, e.g., Theorem 2.2. Special cases $(1/p, 1/q) = (1/3, 1/3)$ and $(1/2, 0)$ were obtained in [39] and [30], respectively.
5. Theorem 1.1 (ii) contains the special case $(p, q) = (2, 2)$ of [2].
6. Theorem 1.1 (iii) contains the special case $(p, q) = (2, 2)$ of [39].

Theorem 1.1 implies many known regularity criteria. Some of them are summarized below. For simplicity we assume $f = 0$. The Lorentz space $L^{(p,\infty)}$ for $p < \infty$ is defined with the norm $\|v\|_{L^{(p,\infty)}} = \sup_{\sigma>0} \sigma \{|\{v| > \sigma\}|\}^{1/p}$.

Corollary 1.2. *Let u be a weak solution of (NS) in $\Omega \times I$ with $f = 0$ and $Q_{z_0,r_0} \subset \Omega \times I$ for some $r_0 > 0$. Then u is regular at z_0 if one of the following conditions holds.*

(i) *Zero-dimensional integrals of u [8, 33, 9, 37]. If*

$$u \in L^{p,q}(Q_{z_0,r_0}), \quad \frac{3}{p} + \frac{2}{q} = 1, \quad 3 < p \leq \infty, \tag{15}$$

or $u \in L^{3,\infty}(Q_{z_0,r_0})$ and $\|u\|_{L^{3,\infty}(Q_{z_0,r_0})}$ is sufficiently small.

(ii) *Lorentz spaces [38, 13, 34, 4]. If u is in $L^{(q,\infty)}((t_0 - r^2, t_0); L^{(p,\infty)}(B_{x_0,r}))$ with $3/p + 2/q = 1$, $3 < p < \infty$, and $\|u\|_{L_t^{(q,\infty)} L_x^{(p,\infty)}(Q_{z_0,r})}$ is sufficiently small.*

(iii) Zero-dimensional integrals of ∇u [1].

$$\nabla u \in L^{p,q}(Q_{z_0,r_0}), \quad \frac{3}{p} + \frac{2}{q} = 2, \quad \frac{3}{2} < p \leq \infty,$$

or $\nabla u \in L^{3/2,\infty}(Q_{z_0,r_0})$ and $\|\nabla u\|_{L^{3/2,\infty}(Q_{z_0,r_0})}$ is sufficiently small.

(iv) Zero-dimensional integrals of $w = \text{curl } u$ [3].

$$w \in L^{p,q}(Q_{z_0,r_0}), \quad \frac{3}{p} + \frac{2}{q} = 2, \quad \frac{3}{2} < p \leq \infty, \tag{16}$$

or $w \in L^{3/2,\infty}(Q_{z_0,r_0})$ and $\|w\|_{L^{3/2,\infty}(Q_{z_0,r_0})}$ is sufficiently small.

Comments for Corollary 1.2.

1. To prove Corollary 1.2 using Theorem 1.1, we need to show that u is suitable under the corresponding assumptions. It suffices to show that $|u|^2|\nabla u| \in L^1_{t,x}$, which justifies the integration by parts and thus one can prove the local energy inequality. In fact, it is enough to show $u \in L^4_{t,x}$ since $\iint |u|^2|\nabla u|dz \leq \|u\|_{L^4}^2 \|\nabla u\|_{L^2}$.

For (i), it follows from $\|u\|_{L^4}^2 \leq \|u\|_{L^{p,q}} \|u\|_{L^{2,\infty}}^{2/q} \|u\|_{L^{6,2}}^{3/p}$.

For (ii), since $3 < p < \infty$, one can choose p_1, q_1 so that

$$q_1 < q, \quad p_1 < p, \quad 1/p_1 + 1/q_1 \leq 1/2, \quad 3/p_1 + 1/q_1 \leq 1.$$

That is, $(1/p_1, 1/q_1)$ lies in region V of Fig. 2 of [10]. By the imbedding of $L^{(p,\infty)} \subset L^{p_1}$ and $L^{(q,\infty)} \subset L^{q_1}$, we have $u \in L^{p_1,q_1}$. Interpolating with $u \in L^{2,\infty} \cap L^{6,2}$, we get $u \in L^4_{t,x}$.

For (iii), we have $\iint |u|^2|\nabla u|dz \leq \|u\|_{L^{2,\infty}}^{2/q} \|u\|_{L^{6,2}}^{3/p} \|\nabla u\|_{L^{p,q}}$.

For (iv), since $\|\nabla u\|_{L^{p,q}(Q_r)} \leq C\|w\|_{L^{p,q}(Q_{2r})} + C\|u\|_{L^{p,q}(Q_{2r})}$ (see Remark 3.7), it follows from (iii).

2. Strictly speaking, one also needs to show that $p \in L^{3/2}$ so that (u, p) is suitable. But this has already been done [35, 18]. By [18, Lem. 3.4], one has $\nabla p \in L^{5/3}_x L^{15/14}_x(Q_r)$ for every weak solution in Q_r . Let $\tilde{p}(x, t) = p(x, t) - \int_{B_r} p(x, t) dx$. The new pair (u, \tilde{p}) is suitable since the local energy inequality (22) remains the same if one replaces p by \tilde{p} , and $\tilde{p} \in L^{5/3}_{t,x}(Q_r)$ by Poincaré inequality.
3. We now complete the proof of Corollary 1.2. For (ii), since $3 < p < \infty$, one can choose $q_2 < q$, $p_2 < p$, and $3/p_2 + 2/q_2 = 2$. Being small in $L^{(q,\infty)}_t L^{(p,\infty)}_x(Q_{z_0,r})$ implies smallness in the scaled norm $\frac{1}{r}L^{q_2}L^{p_2}(Q_r)$ by imbedding. Then one applies Theorem 1.1. For the rest, one imbeds $L^{p,q}$ to $L^{p_2,q}$ for some suitable $p_2 < p$.
4. Corollary 1.2 (i) is due to several authors, already quoted above. Theorem 1.1 does not imply the end point case $u \in L^{3,\infty}(Q_{z_0,r_0})$ without smallness assumption, for which see [7].
5. For Corollary 1.2 (ii), [38] proved regularity for small u in the classes $L^{(q,\infty)}_t L^p_x(Q_r)$ with $3 < p < \infty$, [13] in the class $L^\infty_t L^{(3,\infty)}_x$ (see [14] for improvement), [34] in the classes $L^{(q,\infty)}_t L^{(p,\infty)}_x(\Omega \times I)$ with $3 < p < \infty$, [4] in the classes $L^{(q,\infty)}_t L^{(p,\infty)}_x$ with $3 < p < \infty$ and the classes $L^{(p,\infty)}_x L^{(q,\infty)}_t$ with $3 \leq p < \infty$. It follows from

these results, in particular, that u is regular at z_0 if it satisfies, for $\theta \in [0, 1]$ and some $\epsilon = \epsilon(\theta) > 0$,

$$\lim_{r \rightarrow 0} \operatorname{ess\,sup}_{Q_{z_0,r}} |t - t_0|^{\theta/2} |x - x_0|^{1-\theta} |u(x, t)| \leq \epsilon. \tag{17}$$

Our Theorem 1.1 does not cover the endpoint cases $p = 3, \infty$, except the cases $\theta = 0, 1$ in (17) when suitability is assumed.

6. Corollary 1.2 (iii) was proved in [1] for the cases $3/2 < p < \infty$. The endpoints $p = 3/2$ and $p = \infty$ were not obtained in [1]. The $p = 3/2$ case without the smallness assumption follows from [7] and imbedding.
7. Corollary 1.2 (iv) was proved in [3, Prop. 2]. The main result in [3, Th. 1] shows regularity near z_0 assuming only two components of the vorticity belonging to $L^{p,q}_{x,t}$. Again, the $p = 3/2$ case without the smallness assumption follows from [7] and Remark 3.7.

A major motivation for the study of such regularity criteria is to improve the partial regularity result of [2]. For example, Constantin [6] proved, when $\Omega = \mathbb{T}^3$, the existence of suitable weak solutions satisfying

$$\nabla w \in L^{4/3-\epsilon}(\Omega \times I), \quad \forall 0 < \epsilon \ll 1. \tag{18}$$

Note that the integral $\iint |\nabla w|^{4/3-\epsilon} dz$ has dimension $1 + 3\epsilon$. Combining this estimate with Theorem 1.1 (iv), we find that the parabolic Hausdorff dimension of the singular set \mathcal{S} of u is at most one. This is slightly weaker than the CKN theorem that the one-dimensional parabolic Hausdorff measure of \mathcal{S} is actually zero. Note that Scheffer [25, 26] constructed examples satisfying the local energy inequality and their dimensions of singular sets are arbitrarily close to one. Thus the CKN result is optimal for functions satisfying only the local energy inequality. However, the proof of (18) uses the equation for the vorticity, which may not be satisfied by Scheffer’s examples. Therefore there might be hope to prove other a priori estimates for w and thus improve the partial regularity.

The rest of this paper is organized as follows. In Sect. 2 we introduce some scaling invariant functionals, recall the notion of suitable weak solutions and a regularity criterion involving the scaled norms of velocity and pressure. In Sect. 3 we establish some estimates regarding the velocity, pressure and vorticity, and prove Theorem 1.1.

We finally correct a typo in Lemma 15 of [10]: In the second line from the bottom of p. 615 and the third line of p. 616, the function $C(r)$ should be replaced by $\hat{C}(r) = r^{-2} \int_{Q_r} |u - \frac{1}{|Q_r|} \int_{Q_r} u|^3 dz$.

2. Preliminaries

In this section we introduce the notation, review suitable weak solutions, and recall a regularity criterion involving scaled norms.

We start with the notation. Let Ω be either an open domain in \mathbb{R}^3 or the 3-dimensional torus \mathbb{T}^3 , and I be a finite time interval. By $N = N(\alpha, \beta, \dots)$ we denote a constant depending on the prescribed quantities α, β, \dots , which may change from line to line. For $1 \leq q \leq \infty$, $W^{k,q}(\Omega)$ denote the usual Sobolev spaces, i.e. $W^{k,q}(\Omega) = \{f \in L^q(\Omega) : D^\alpha f \in L^q(\Omega), 0 \leq |\alpha| \leq k\}$. We denote by $\int_E f$ the average of f on E ; i.e., $\int_E f = \int_E f/|E|$. For a function $f(x, t)$, $E \subset \Omega$ and $J \subset I$, we denote $\|f\|_{L^p,q(E \times J)} = \|f\|_{L^q L^p(E \times J)} = \left\| \|f\|_{L^p(E)} \right\|_{L^q(J)}$.

Next, we define several scaling-invariant functionals similar to those in [2, 18, 16, 27]. For a suitable weak solution (u, p) and $z = (x, t) \in \Omega \times I$, let

$$\begin{aligned}
 A(r) &:= \sup_{t-r^2 \leq s < t} \frac{1}{r} \int_{B_{x,r}} |u(y, s)|^2 dy, & E(r) &:= \frac{1}{r} \int_{Q_{z,r}} |\nabla u(y, s)|^2 dy ds, \\
 C(r) &:= \frac{1}{r^2} \int_{Q_{z,r}} |u(y, s)|^3 dy ds, & \tilde{C}(r) &:= \frac{1}{r^2} \int_{Q_{z,r}} |u(y, s) - (u)_r(s)|^3 dy ds, \\
 D(r) &:= \frac{1}{r^2} \int_{Q_{z,r}} |p(y, s)|^{\frac{3}{2}} dy ds,
 \end{aligned}$$

where $(u)_r(s) = \frac{1}{|B_{x,r}|} \int_{B_{x,r}} u(\cdot, s) dy$. Let p, q and p^* be numbers satisfying

$$\frac{3}{p} + \frac{2}{q} = 3, \quad 1 \leq q \leq \infty, \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}. \tag{19}$$

Recall $w = \nabla \times u$ is the vorticity field of u . We define

$$\begin{aligned}
 \tilde{G}(r) &:= \frac{1}{r} \|u(y, s) - (u)_r(s)\|_{L^q_s L^{p^*}_y(Q_{z,r})}, & G_1(r) &:= \frac{1}{r} \|\nabla u(y, s)\|_{L^q_s L^p_y(Q_{z,r})}, \\
 W(r) &:= \frac{1}{r} \|w(y, s)\|_{L^q_s L^p_y(Q_{z,r})}.
 \end{aligned}$$

When $1 \leq q \leq 2$, we also define

$$W_1(r) := \frac{1}{r} \|\nabla w(y, s)\|_{L^q_s L^{p^\sharp}_y(Q_{z,r})}, \quad \tilde{W}_1(r) := \frac{1}{r} \|\text{curl } w(y, s)\|_{L^q_s L^{p^\sharp}_y(Q_{z,r})},$$

where p^\sharp is the number satisfying, for p, q as in (19),

$$\frac{3}{p^\sharp} + \frac{2}{q} = 4, \quad \frac{1}{p} = \frac{1}{p^\sharp} - \frac{1}{3}, \quad 1 \leq p^\sharp \leq \frac{3}{2}. \tag{20}$$

We now define suitable weak solutions for the (NS).

Definition 2.1. Suppose that f belongs to the parabolic Morrey space $M_{2,\gamma}(\Omega \times I)$ for some $\gamma \in (0, 2]$. A pair (u, p) is a **suitable weak solution** to the Navier-Stokes equations (1) in $\Omega \times I$ with force f if the following conditions are satisfied:

(a) The functions $u : \Omega \times I \rightarrow \mathbb{R}^3$ and $p : \Omega \times I \rightarrow \mathbb{R}$ satisfy

$$u \in L^\infty(I; L^2(\Omega)) \cap L^2(I; W^{1,2}(\Omega)), \quad p \in L^{\frac{3}{2}}(\Omega \times I). \tag{21}$$

(b) u and p solve (1) in $\Omega \times I$ in the sense of distributions.

(c) u and p satisfy the local energy inequality

$$\begin{aligned}
 & \int_\Omega |u(x, t)|^2 \phi(x, t) dx + 2 \int_{t_0}^t \int_\Omega |\nabla u(x, t')|^2 \phi(x, t') dx dt' \\
 & \leq \int_{t_0}^t \int_\Omega \left(|u|^2 (\partial_t \phi + \Delta \phi) + (|u|^2 + 2p) u \cdot \nabla \phi + 2f \cdot u \phi \right) dx dt' \tag{22}
 \end{aligned}$$

for all $t \in I = (t_0, t_1)$ and all nonnegative functions $\phi \in C_0^\infty(\Omega \times I)$. \square

In this definition we impose no initial or boundary condition for u .

The main difference between suitable weak solutions and *Leray-Hopf weak solutions* (see [2, p.779]) is the additional condition of the local energy inequality (22). The existence of suitable weak solutions is proved in [22, 2]. Definition 2.1 is the slightly modified version used in [18]. As remarked in [2, p. 823], it is an open question if all weak solutions are suitable.

Next we recall a local regularity criterion, which is a refined version of [2, Prop. 1], and is formulated in the present form with $f = 0$ in [19, 18], and proved with nonzero $f \in M_{2,\gamma}$ in [16, Prop. 2.8].

Theorem 2.2. *There exists $\epsilon > 0$ depending only on $\gamma > 0$ (and independent of $\|f\|_{M_{2,\gamma}}$), and $r_0 > 0$ depending on $\|f\|_{M_{2,\gamma}}$, such that if (u, p) is a suitable weak solution of (NS) with $f \in M_{2,\gamma}$, then u is regular at $z = (x, t) \in \Omega \times I$ if*

$$C(r) + D(r) < \epsilon \text{ for some } r \in (0, r_0). \tag{23}$$

An important feature of (23) is that it requires only one r , not infinitely many r . We will prove our regularity criteria based on this theorem. For our proof in the next section, in order to get (23), it suffices to assume $\gamma > -1$. The assumption $\gamma > 0$ is made in order to apply Theorem 2.2.

3. Local Interior Regularity

In this section, we present the proof of Theorem 1.1. Through the entire section, we assume (u, p) is a suitable weak solution in $\Omega \times I$. Without loss of generality, we assume $z = (0, 0)$ and $Q_r = Q_{(0,0),r} \subset \Omega \times I$. By Hölder inequality, it suffices to consider borderline exponents, i.e., those exponents p, p^*, p^\sharp and q satisfying (19) and (20). Denote $m_\gamma = \|f\|_{M_{2,\gamma}}$.

Lemma 3.1. *Suppose $Q_{2r} \subset \Omega \times I$ and $0 < r \leq m_\gamma^{-1/(1+\gamma)}$. Then*

$$A(r) + E(r) \leq N[1 + C(2r) + D(2r)].$$

Proof. By choosing suitably localized ϕ in the local energy inequality (22), we get

$$A(r) + E(r) \leq N \left(C^{\frac{2}{3}}(2r) + C(2r) + \frac{1}{r^2} \|u\|_{L^3(Q_{2r})} \|p\|_{L^{\frac{3}{2}}(Q_{2r})} + r \int_{Q_{2r}} |f|^2 dz' \right)$$

which is bounded by $N[1 + C(2r) + D(2r) + r^{2(\gamma+1)}m_\gamma^2]$. \square

Lemma 3.2. *Suppose $u \in L^{p^*,q}(Q_r)$ with $3/p^* + 2/q = 2, 1 \leq q \leq \infty$, then*

$$\tilde{C}(r) \leq NA^{\frac{1}{q}}(r)E^{1-\frac{1}{q}}(r)\tilde{G}(r).$$

Proof. Let $\alpha = (2p^* - 3)/3p^*$ and $\beta = 1/p^*$. Note $1/3 = \alpha/2 + \beta/6 + (1 - \alpha - \beta)/p^*$. Using the Hölder inequality and Sobolev imbedding, we obtain

$$\begin{aligned} \|u - (u)_r\|_{L^3(B_r)} &\leq N \|u\|_{L^2(B_r)}^\alpha \|u - (u)_r\|_{L^6(B_r)}^\beta \|u - (u)_r\|_{L^{p^*}(B_r)}^{1-\alpha-\beta} \\ &\leq N \|u\|_{L^2(B_r)}^\alpha \|\nabla u\|_{L^2(B_r)}^\beta \|u - (u)_r\|_{L^{p^*}(B_r)}^{\frac{1}{3}}, \end{aligned}$$

where we used $1 - \alpha - \beta = 1/3$. Raising to the third power, integrating in time and dividing both sides by r^2 , we get

$$\begin{aligned} \tilde{C}(r) &\leq \frac{N}{r^2} \int_{-r^2}^0 \|u\|_{L^2(B_r)}^{3\alpha} \|\nabla u\|_{L^2(B_r)}^{3\beta} \|u - (u)_r\|_{L^{p^*}(B_r)} dt \\ &\leq \frac{N}{r^2} r^{\frac{3}{2}\alpha} A^{\frac{3}{2}\alpha}(r) \left(\int_{-r^2}^0 \|\nabla u\|_{L^2(B_r)}^2 dt \right)^{\frac{3\beta}{2}} \left(\int_{-r^2}^0 \|u - (u)_r\|_{L^{p^*}(B_r)}^q dt \right)^{\frac{1}{q}}, \end{aligned}$$

which equals $NA^{\frac{1}{q}}(r)E^{1-\frac{1}{q}}(r)\tilde{G}(r)$. \square

Lemma 3.3. *Suppose $0 < 2r \leq \rho$ and $Q_\rho \subset \Omega \times I$. Then*

$$C(r) \leq N \left(\frac{r}{\rho}\right) C(\rho) + N \left(\frac{\rho}{r}\right)^2 \tilde{C}(\rho).$$

Proof. This follows from the Hölder inequality:

$$C(r) \leq \frac{N}{r^2} \int_{Q_r} (|(u)_\rho|^3 + |u - (u)_\rho|^3) dz' \leq N \left(\frac{r}{\rho}\right) C(\rho) + N \left(\frac{\rho}{r}\right)^2 \tilde{C}(\rho).$$

\square

Lemma 3.4. *Suppose $0 < 2r \leq \rho$ and $Q_\rho \subset \Omega \times I$. Then*

$$D(r) \leq N \left(\frac{\rho}{r}\right)^2 (\tilde{C}(\rho) + \rho^{\frac{3}{2}(\gamma+1)} m_\gamma^{\frac{3}{2}}) + N \left(\frac{r}{\rho}\right) D(\rho). \tag{24}$$

Proof. Let $\phi(x) \geq 0$ be supported in B_ρ with $\phi = 1$ in $B_{\rho/2}$. The divergence of (1) gives $-\Delta p = \partial_i \partial_j (u_i u_j) - \nabla \cdot f$ in the sense of distributions. Let

$$p_1(x, t) := \int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|} \{ \partial_i \partial_j [(u_i - (u_i)_\rho)(u_j - (u_j)_\rho)\phi] - \nabla \cdot (f\phi) \} (y, t) dy$$

and $p_2(x, t) := p(x, t) - p_1(x, t)$. Due to $\operatorname{div} u = 0$, $\Delta p_2 = 0$ in $B_{\rho/2}$. By the mean value property of harmonic functions,

$$\frac{1}{r^2} \int_{B_r} |p_2|^{\frac{3}{2}} dx \leq \frac{Nr}{\rho^3} \int_{B_{\rho/2}} |p_2|^{\frac{3}{2}} dx \leq \frac{Nr}{\rho^3} \int_{B_\rho} |p|^{\frac{3}{2}} dx + \frac{Nr}{\rho^3} \int_{B_\rho} |p_1|^{\frac{3}{2}} dx.$$

By Calderon-Zygmund and potential estimates,

$$\frac{r}{\rho^3} \int_{B_\rho} |p_1|^{\frac{3}{2}} dx \leq \frac{1}{r^2} \int_{B_\rho} |p_1|^{\frac{3}{2}} dx \leq \frac{N}{r^2} \int_{B_\rho} |u - (u)_\rho|^3 + \frac{N\rho^{9/4}}{r^2} \left(\int_{B_\rho} |f|^2 dx \right)^{\frac{3}{4}}.$$

Adding these estimates, integrating in time, and using $\int_{-r^2}^0 \frac{\rho^{9/4}}{r^2} \left(\int_{B_\rho} |f|^2 dx \right)^{\frac{3}{4}} dt \leq Nr^{-3/2} m_\gamma^{3/2} \rho^{3+3\gamma/2}$, we get

$$\frac{1}{r^2} \int_{Q_r} |p|^{\frac{3}{2}} dz' \leq \frac{1}{r^2} \int_{Q_r} |p_1|^{\frac{3}{2}} + |p_2|^{\frac{3}{2}} dz' \leq \text{RHS of (24)}.$$

\square

Now we are ready to prove Theorem 1.1 (i).

Proof of Theorem 1.1 (i). It suffices to prove the borderline cases $3/p^* + 2/q = 2$ and $1 \leq q \leq \infty$. The other cases follow by Hölder inequality. Suppose $0 < 4r \leq \rho$. By Lemmas 3.2 and 3.4, and by Lemma 3.3, we get

$$\begin{aligned} C(r) + D(r) &\leq N \left(\frac{r}{\rho}\right) \left(C\left(\frac{\rho}{2}\right) + D\left(\frac{\rho}{2}\right)\right) + N \left(\frac{\rho}{r}\right)^2 \left(\tilde{C}\left(\frac{\rho}{2}\right) + \rho^{\frac{3}{2}(\gamma+1)} m^{\frac{3}{\gamma}}\right) \\ &\leq N \left(\frac{r}{\rho}\right) \left(C\left(\frac{\rho}{2}\right) + D\left(\frac{\rho}{2}\right)\right) + N \left(\frac{\rho}{r}\right)^2 \\ &\quad \times \left(A^{\frac{1}{q}}\left(\frac{\rho}{2}\right) E^{1-\frac{1}{q}}\left(\frac{\rho}{2}\right) \tilde{G}\left(\frac{\rho}{2}\right) + \rho^{\frac{3}{2}(\gamma+1)} m^{\frac{3}{\gamma}}\right). \end{aligned}$$

Suppose $\rho \leq m_\gamma^{-1/(\gamma+1)}$. By Lemma 3.1,

$$N \left(\frac{\rho}{r}\right)^2 A^{\frac{1}{q}}\left(\frac{\rho}{2}\right) E^{1-\frac{1}{q}}\left(\frac{\rho}{2}\right) \tilde{G}\left(\frac{\rho}{2}\right) \leq N \left(\frac{\rho}{r}\right)^2 (1 + C(\rho) + D(\rho)) \tilde{G}(\rho).$$

Combining the above estimates, we obtain

$$\begin{aligned} C(r) + D(r) &\leq N_2 \left(\left(\frac{r}{\rho}\right) + \left(\frac{\rho}{r}\right)^2 \tilde{G}(\rho)\right) (C(\rho) + D(\rho)) \\ &\quad + N_2 \left(\frac{\rho}{r}\right)^2 \left(\tilde{G}(\rho) + \rho^{\frac{3}{2}(\gamma+1)} m^{\frac{3}{\gamma}}\right). \end{aligned}$$

Choose $\theta \in (0, 1/4)$ so that $N_2\theta < 1/4$. We fix $r_0 < \min\{1, \frac{1}{m_\gamma}, \frac{1}{m_\gamma} (\frac{\epsilon\theta^2}{8N_2})^{2/3}\}^{1/(\gamma+1)}$ such that $\tilde{G}(r) < \frac{\theta^2}{1+8N_2} \min\{1, \epsilon\}$ for all $r \leq r_0$, where ϵ is the constant in Theorem 2.2. Replacing r and ρ by θr and r , respectively, we get

$$C(\theta r) + D(\theta r) \leq \frac{1}{2} (C(r) + D(r)) + \frac{\epsilon}{4}, \quad \forall r < r_0.$$

By iteration,

$$C(\theta^k r) + D(\theta^k r) \leq \frac{1}{2^k} (C(r) + D(r)) + \frac{\epsilon}{2}, \quad \forall r < r_0.$$

Thus, for k sufficiently large, $C(\theta^k r) + D(\theta^k r) \leq \epsilon$, from which z is a regular point due to Theorem 2.2.

The last statement of Theorem 1.1 (i), that one can replace $u - (u)_r$ by u , is because $\|u - (u)_r\|_{L^{p^*,q}} \leq N \|u\|_{L^{p^*,q}}$. \square

The following modification of Lemma 3.2 is all that is needed to prove Theorem 1.1 (ii).

Lemma 3.5. *Suppose $0 < 2r \leq \rho$ and $Q_\rho \subset \Omega \times I$. Then*

$$\tilde{C}(r) \leq N A^{1/q}(r) E^{1-1/q}(r) G_1(r). \tag{25}$$

Proof. The proof is similar to that of Lemma 3.2. When $1 \leq p < 3$, using the same exponents $\alpha = 1 - 1/p$ and $\beta = 1/p - 1/3$, we have

$$\begin{aligned} \|u - (u)_r\|_{L^3(B_r)}^3 &\leq N \|u\|_{L^2(B_r)}^{3\alpha} \|u - (u)_r\|_{L^6(B_r)}^{3\beta} \|u - (u)_r\|_{L^{p^*(B_r)}}^{3(1-\alpha-\beta)} \\ &\leq N \|u\|_{L^2(B_r)}^{2/q} \|\nabla u\|_{L^2(B_r)}^{2-2/q} \|\nabla u\|_{L^p(B_r)}. \end{aligned}$$

If $p = 3$ (and $q = 1$), by Gagliardo-Nirenberg and Poincaré inequalities,

$$\begin{aligned} \|u - (u)_r\|_{L^3(B_r)}^3 &\leq N \|u - (u)_r\|_{L^2(B_r)}^2 \|\nabla u\|_{L^3(B_r)} + \frac{N}{r^{3/2}} \|u - (u)_r\|_{L^2(B_r)}^3 \\ &\leq N \|u\|_{L^2(B_r)}^2 \|\nabla u\|_{L^3(B_r)}. \end{aligned}$$

Integrating in time and applying the Hölder inequality, we get (25). \square

Proof of Theorem 1.1 (ii). The proof is the same as that for Theorem 1.1 (i): we only need to replace Lemma 3.2 by Lemma 3.5, and replace the quantity $\tilde{G}(r)$ by $G_1(r)$. \square

The next lemma shows that the gradient of the velocity can be controlled by the vorticity. This is the key to Theorem 1.1 (iii).

Lemma 3.6. *Suppose $0 < 2r \leq \rho$ and $Q_\rho \subset \Omega \times I$. Suppose $\nabla u \in L_{x,t}^{p,q}(Q_\rho)$ with $\frac{3}{p} + \frac{2}{q} = 3$ and $1 \leq q < \infty$. Then*

$$G_1(r) \leq N \left(\frac{\rho}{r}\right) W(\rho) + N \left(\frac{r}{\rho}\right)^{\frac{3}{p}-1} G_1(\rho). \tag{26}$$

Furthermore, if $p = 3$ (so $q = 1$), then

$$G_1(r) \leq N \left(\frac{\rho}{r}\right) W(\rho) + N \left(\frac{r}{\rho}\right) G_1(\rho) + g(u; r), \tag{27}$$

where $g(u; r) \rightarrow 0$ as $r \rightarrow 0$.

Proof. Choose a standard cut off function ϕ supported in B_ρ such that $\phi = 1$ in $B_{3\rho/4}$. Define

$$v(x, t) := \int_{\mathbb{R}^3} \nabla_x \frac{1}{4\pi|x-y|} \times w(y, t)\phi(y)dy, \quad h = u - v.$$

Note that $\Delta_x h(x, t) = 0$ in $B_{3\rho/4}$.

We give the proof of (26) first. By the mean value property of harmonic functions, for each fixed time t ,

$$\|\nabla h\|_{L^p(B_r)} \leq N \left(\frac{r}{\rho}\right)^{3/p} \|\nabla h\|_{L^p(B_{\rho/2})} \leq N \left(\frac{r}{\rho}\right)^{3/p} \left(\|\nabla u\|_{L^p(B_\rho)} + \|\nabla v\|_{L^p}\right).$$

On the other hand, due to Calderon-Zygmund estimates, for each fixed time,

$$\|\nabla v\|_{L^p} \leq N \|w\|_{L^p(B_\rho)}.$$

Combining these estimates, we obtain

$$\|\nabla u\|_{L^p(B_r)} \leq \|\nabla v\|_{L^p(B_r)} + \|\nabla h\|_{L^p(B_r)} \leq N \|w\|_{L^p(B_\rho)} + N \left(\frac{r}{\rho}\right)^{\frac{3}{p}} \|\nabla u\|_{L^p(B_\rho)}.$$

Taking L^q -norm in time and dividing both sides by r , we get (26).

To prove (27), set $p = 3$ (so $q = 1$), use the above estimate for ∇v , and modify the estimate for ∇h as follows:

$$\|\nabla h\|_{L^3(B_r)} \leq \|\nabla h - (\nabla h)_r\|_{L^3(B_r)} + \|(\nabla h)_r\|_{L^3(B_r)}. \tag{28}$$

The second term in (28) is just $Nr|(\nabla h)_r|$. For the first term in (28), use the Poincaré-Sobolev inequality, the mean-value property, and an interior estimate:

$$\begin{aligned} \|\nabla h - (\nabla h)_r\|_{L^3(B_r)} &\leq N \left\| \nabla^2 h \right\|_{L^{3/2}(B_r)} \leq N \left(\frac{r}{\rho}\right)^2 \left\| \nabla^2 h \right\|_{L^{3/2}(B_{\rho/2})} \\ &\leq N \left(\frac{r}{\rho}\right)^2 \|\nabla h\|_{L^3(B_\rho)} \leq N \left(\frac{r}{\rho}\right)^2 [\|\nabla u\|_{L^3(B_\rho)} + \|\nabla v\|_{L^3(B_\rho)}] \\ &\leq N \left(\frac{r}{\rho}\right)^2 \|\nabla u\|_{L^3(B_\rho)}. \end{aligned} \tag{29}$$

Combine this estimate with the above estimate for $\|\nabla v\|_{L^3}$, divide by r , and integrate in time to get

$$G_1(r) \leq N \frac{\rho}{r} W(\rho) + N \frac{r}{\rho} G_1(\rho) + N \int_{-r^2}^0 |(\nabla h)_r| dt. \tag{30}$$

Since h (and hence ∇h) is harmonic in $B_{3\rho/4}$, $(\nabla h)_r = (\nabla h)_{\rho/2}$, and so

$$|(\nabla h)_r| = |(\nabla h)_{\rho/2}| \leq \frac{N}{\rho} \|\nabla h\|_{L^3(B_{\rho/2})}.$$

Thus

$$g(u; r) := N \int_{-r^2}^0 |(\nabla h)_r| dt \leq \frac{N}{\rho} \int_{-r^2}^0 \|\nabla u\|_{L^3(B_\rho)} dt.$$

Since $\nabla u \in L^{3,1}(Q_\rho)$, we have $g(u, r) \rightarrow 0$ as $r \rightarrow 0$, and so (30) yields (27). \square

Remark 3.7. By similar argument, if $w = \text{curl } u \in L_{loc}^{p,q}$ near z , then so is ∇u , since $\|\nabla u\|_{L^{p,q}(Q_r)} \leq N \|w\|_{L^{p,q}(Q_\rho)} + N \|u\|_{L^{p,q}(Q_\rho)}$ if $0 < r < \rho \leq 2r$.

Proof of Theorem 1.1 (iii). It suffices to prove the borderline cases $3/p + 2/q = 3$ and $1 < p \leq 3$. The other cases follow by Hölder’s inequality. If $p < 3$, we use the estimate (26), and if $p = 3$, we use the refined estimate (27). Choose $\theta \in (0, 1/4)$ so that if $p < 3$, then $N\theta^{3/p-1} < 1/2$, where N is the constant in (26), and if $p = 3$, $N\theta < 1/2$, where N is the constant from (27). Replace r, ρ by θr and r , respectively. Note that $G_1(r)$ is finite by Remark 3.7. The estimate (26) ($p < 3$) or (27) ($p = 3$) then implies

$$G_1(\theta r) \leq \frac{N}{\theta} W(r) + \frac{1}{2} G_1(r) + \begin{cases} 0 & \text{if } p < 3 \\ Ng(u; \theta r) & \text{if } p = 3 \end{cases}.$$

Choose r_0 so that $\sup_{r < r_0} W(r) < \frac{\theta\epsilon}{8N}$, and (if $p = 3$) $g(u; r_0) < \frac{\epsilon}{8N}$, where ϵ is the constant in Theorem 1.1 (ii). Then for $r \leq r_0$, we have

$$G_1(\theta r) \leq \frac{1}{2}G_1(r) + \frac{\epsilon}{4}.$$

Iterating this estimate, we obtain, for all $r \leq r_0$,

$$G_1(\theta^k r) \leq \frac{1}{2^k}G_1(r) + \frac{\epsilon}{2}. \tag{31}$$

Choose an integer $k_0 \geq 3 + \sup_{\theta r_0 < r < r_0} \log_2 \frac{G_1(r)}{\epsilon}$. Then $G_1(r) \leq \epsilon$ for $r < \theta^{k_0} r_0$. The regularity of u at $z = (0, 0)$ now follows from Theorem 1.1 (ii). \square

In the next lemma, we show that vorticity and the gradient of velocity can be controlled by the gradient of vorticity in scaled norms, which is the key for Theorem 1.1 (iv).

Lemma 3.8. *Suppose $0 < 2r \leq \rho$ and $Q_\rho \subset \Omega \times I$. Suppose $1 \leq q \leq 2$, and p, p^\sharp satisfy (19) and (20). If $\nabla w \in L_{x,t}^{p^\sharp, q}(Q_\rho)$, then*

$$W(r) \leq N \left(\frac{\rho}{r}\right) W_1(\rho) + N \left(\frac{r}{\rho}\right)^{\frac{3}{p}-1} W(\rho). \tag{32}$$

Furthermore, if $1 \leq q < 2$, we have

$$G_1(r) \leq N \left(\frac{\rho}{r}\right) \tilde{W}_1(\rho) + N \left(\frac{r}{\rho}\right)^{3/p} G_1(\rho) + g(u; r) \tag{33}$$

with $g(u; r) \rightarrow 0$ as $r \rightarrow 0$.

Proof. Statement (32) follows from Sobolev imbedding:

$$\begin{aligned} \|w\|_{L^p(B_r)} &\leq \|(w)_\rho\|_{L^p(B_r)} + \|w - (w)_\rho\|_{L^p(B_\rho)} \\ &\leq N \left(\frac{r}{\rho}\right)^{\frac{3}{p}} \|w\|_{L^p(B_\rho)} + N \|\nabla w\|_{L^{p^\sharp}(B_\rho)}. \end{aligned}$$

Taking the L^q norm in time and dividing both sides by r , we get (32).

The proof of (33) is similar to that of the second part of Lemma 3.6. Choose a standard cut off function ϕ supported in B_ρ such that $\phi = 1$ in $B_{3\rho/4}$. Define 2-tensors

$$V(x, t) := \int_{\mathbb{R}^3} \nabla_x \frac{1}{4\pi|x-y|} (\text{curl } w(y, t)) \phi(y) dy, \quad H := \nabla u - V.$$

Note that since $\nabla \cdot u = 0$, $-\Delta u = \text{curl } w$, and so $\Delta_x H(x, t) = 0$ in $B_{3\rho/4}$. Potential estimates give, if $p^\sharp > 1$, (i.e. $q < 2$),

$$\|V\|_{L^p} \leq N \|\text{curl } w\|_{L^{p^\sharp}(B_\rho)}.$$

The same estimate as in (29) gives

$$\begin{aligned} \|H - (H)_r\|_{L^p(B_r)} &\leq N \left(\frac{r}{\rho}\right)^{\frac{3}{p^\sharp}} \|H\|_{L^p(B_\rho)} \\ &\leq N \left(\frac{r}{\rho}\right)^{\frac{3}{p^\sharp}} \left[\|\nabla u\|_{L^p(B_\rho)} + \|\operatorname{curl} w\|_{L^{p^\sharp}(B_\rho)} \right]. \end{aligned}$$

Using $\frac{1}{r} \|(H)_r\|_{L^p(B_r)} = Nr^{\frac{3}{p}-1} |(H)_r| = Nr^{\frac{3}{p}-1} |H|_{r=0}$ together with the last two estimates, we find (33) with

$$g(u; r) := Nr^{\frac{3}{p}-1} \left(\int_{-r^2}^0 (|H|_{r=0})^q dt \right)^{1/q}.$$

Now arguing as in the proof of Lemma 3.6,

$$g(u; r) = Nr^{\frac{3}{p}-1} \left(\int_{-r^2}^0 |(H)_{\rho/2}|^q dt \right)^{1/q} \leq \frac{Nr^{\frac{3}{p}-1}}{\rho^{3/p}} \left(\int_{-r^2}^0 (\|\nabla u\|_{L^p(B_\rho)})^q dt \right)^{1/q},$$

and since $\nabla u \in L^{p,q}(Q_\rho)$, we have $g(u, r) \rightarrow 0$ as $r \rightarrow 0$. \square

Proof of Theorem 1.1 (iv). It suffices to prove the borderline cases $3/p^\sharp + 2/q = 4$. The other cases follow by Hölder inequality. We also assume $z = (0, 0)$.

We first consider $1 < q \leq 2$. Let ϵ be the constant in Theorem 1.1 (iii) and we suppose $W_1(r) \leq \epsilon/4$ for any $r < r_0$. Our assertion follows a procedure similar to the proof in Theorem 1.1 (iii). Since $\frac{3}{p} - 1 > 0$ in (32), we replace r and ρ by θr and r , respectively, after choosing $\theta \in (0, 1/2)$ appropriately, and then iterate (32). This procedure leads to the conclusion that $W(\theta^k r) < \epsilon/2$ for $r < r_0$ and k sufficiently large, and so $W(r) < \epsilon$ for $r < \theta^k r_0$. Theorem 1.1 (iii) then implies the regularity.

For $1 \leq q < 2$, we can use (33) instead. Arguing just as in the second part of the proof of Theorem 1.1(iii), we conclude that $G_1(r)$ can be made small enough to apply Theorem 1.1(ii), provided $\tilde{W}_1(r) \leq W_1(r)$ can be made arbitrarily small. \square

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