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Asymptotic Stability, Concentration, and Oscillation in Harmonic Map Heat-Flow, Landau-Lifshitz, and Schrödinger Maps on $\mathbb{R}^2$

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Abstract: We consider the Landau-Lifshitz equations of ferromagnetism (including the harmonic map heat-flow and Schrödinger flow as special cases) for degree m equivariant maps from $\mathbb{R}^2$ to $\mathbb{S}^2$. If $m \geq 3$, we prove that near-minimal energy solutions converge to a harmonic map as $t \to \infty$ (*asymptotic stability*), extending previous work (Gustafson et al., Duke Math J 145(3), 537–583, 2008) down to degree $m = 3$. Due to slow spatial decay of the harmonic map components, a new approach is needed for $m = 3$, involving (among other tools) a "normal form" for the parameter dynamics, and the 2D radial double-endpoint Strichartz estimate for Schrödinger operators with sufficiently repulsive potentials (which may be of some independent interest). When $m = 2$ this asymptotic stability may fail: in the case of heat-flow with a further symmetry restriction, we show that more exotic asymptotics are possible, including infinite-time concentration (blow-up), and even "eternal oscillation".

Contents

1. Introduction and Results

The *Landau-Lifshitz* (sometimes *Landau-Lifshitz-Gilbert*) equation describing the dynamics of an 2D isotropic ferromagnet is (eg. [13])

$$\vec{u}_t = a_1(\Delta\vec{u} + |\nabla\vec{u}|^2\vec{u}) + a_2\vec{u} \times \Delta\vec{u}, \qquad a_1 \geq 0, \quad a_2 \in \mathbb{R}, \tag{1.1}$$

where the *magnetization vector* $\vec{u} = \vec{u}(t,x) = (u_1, u_2, u_2)$ is a 3-vector with normalized length, so can be considered a map into the 2-sphere $\mathbb{S}^2$:

$$\vec{u} : [0,T) \times \mathbb{R}^2 \to \mathbb{S}^2 := \{\vec{u} \in \mathbb{R}^3 \mid |\vec{u}| = 1\}. \tag{1.2}$$

The special case $a_2 = 0$ of (1.1) is the very well-studied *harmonic map heat-flow* into $\mathbb{S}^2$, while the special case $a_1 = 0$ is known as the *Schrödinger flow* (or *Schrödinger map*) equation, the geometric generalization of the linear Schrödinger equation for maps into the Kähler manifold $\mathbb{S}^2$.

In order to exhibit the simple geometry of (1.1) more clearly, we introduce, for $\vec{u} \in \mathbb{S}^2$, the tangent space

$$T_{\vec{u}}\mathbb{S}^2 := \vec{u}^\perp = \{\vec{\xi} \in \mathbb{R}^3 \mid \vec{u} \cdot \vec{\xi} = 0\} \tag{1.3}$$

to the sphere $\mathbb{S}^2$ at $\vec{u}$. For any vector $\vec{v} \in \mathbb{R}^3$, we define two operations on vectors:

$$J^{\vec{v}} := \vec{v}\times, \qquad P^{\vec{v}} := -J^{\vec{v}}J^{\vec{v}}. \tag{1.4}$$

For $\vec{u} \in \mathbb{S}^2$, $P^{\vec{u}}$ projects vectors orthogonally onto $T_{\vec{u}}\mathbb{S}^2$, while $J^{\vec{u}}$ is a $\pi/2$ rotation (complex structure) on $T_{\vec{u}}\mathbb{S}^2$. Denoting

$$a = a_1 + ia_2 \in \mathbb{C}, \tag{1.5}$$

the Landau-Lifshitz equation (1.1) may be written

$$\vec{u}_t = P_a^{\vec{u}}\Delta\vec{u}, \qquad P_a^{\vec{u}} := a_1P^{\vec{u}} + a_2J^{\vec{u}}. \tag{1.6}$$

The energy associated to (1.1) is simply the Dirichlet functional

$$\mathcal{E}(\vec{u}) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\vec{u}|^2 dx, \tag{1.7}$$

and (1.6) formally yields the energy identity

$$\mathcal{E}(\vec{u}(t)) + 2a_1\int_0^t\int_{\mathbb{R}^2} |P^{\vec{u}}\Delta\vec{u}(s,x)|^2 dx ds = \mathcal{E}(\vec{u}(0)) \tag{1.8}$$

implying, in particular, energy non-increase if $a_1 > 0$, and energy conservation if $a_1 = 0$ (Schrödinger map).

To a finite-energy map $\vec{u} : \mathbb{R}^2 \to \mathbb{S}^2$ is associated the *degree*

$$\deg(\vec{u}) := \frac{1}{4\pi} \int_{\mathbb{R}^2} \vec{u}_{x_1} \cdot J^{\vec{u}} \vec{u}_{x_2} dx. \tag{1.9}$$

If $\lim_{|x|\to\infty} \vec{u}(x)$ exists (which will be the case below), we may identify $\vec{u}$ with a map $\mathbb{S}^2 \to \mathbb{S}^2$, and if the map is smooth, $\deg(\vec{u})$ is the usual Brouwer degree (in particular, an integer). It follows immediately from expression (1.9) that the energy is bounded from below by the degree:

$$\mathcal{E}(\vec{u}) = \frac{1}{2} \int_{\mathbb{R}^2} |\vec{u}_{x_1} - J^{\vec{u}} \vec{u}_{x_2}|^2 + 4\pi \deg(\vec{u}) \geq 4\pi \deg(\vec{u}), \tag{1.10}$$

and equality here is achieved exactly at *harmonic maps* solving the first-order equations

$$\vec{u}_{x_1} = J^{\vec{u}} \vec{u}_{x_2} \tag{1.11}$$

which, in stereographic coordinates

$$\mathbb{S}^2 \ni \vec{u} \quad \longleftrightarrow \quad \frac{u_1 + i u_2}{1 - u_3} \in \mathbb{C} \cup \{\infty\} \tag{1.12}$$

are the Cauchy-Riemann equations, and the solutions are rational functions. These harmonic maps are critical points of the energy $\mathcal{E}$ and, in particular, static solutions of the Landau-Lifshitz equation (1.1).

In this paper we specialize to the class of *m-equivariant maps*, for some $m \in \mathbb{Z}^+$:

$$\vec{u}(t,x) = e^{m\theta R} \vec{v}(t,r), \qquad \vec{v} : [0,T) \times [0,\infty) \to \mathbb{S}^2 \tag{1.13}$$

with notations

$$R := J^{\vec{k}} = \vec{k} \times, \quad \vec{k} = (0,0,1), \tag{1.14}$$

and polar coordinates

$$x_1 + i x_2 = r e^{i\theta}. \tag{1.15}$$

In terms of the radial profile map $\vec{v} = (v_1, v_2, v_3)$, the energy is

$$\mathcal{E}(\vec{u}) = \pi \int_0^\infty \left(|\vec{v}_r|^2 + \frac{m^2}{r^2}(v_1^2 + v_2^2) \right) r dr. \tag{1.16}$$

Finite energy implies $\vec{v}$ is continuous in r and $\lim_{r\to 0} \vec{v} = \pm\vec{k}$, $\lim_{r\to\infty} \vec{v} = \pm\vec{k}$ (see [11] for details). We force non-trivial topology by working in the class of maps

$$\Sigma_m := \{\vec{u} = e^{m\theta R} \vec{v}(r) \mid \mathcal{E}(\vec{u}) < \infty,\ \vec{v}(0) = -\vec{k},\ \vec{v}(\infty) = \vec{k}\}. \tag{1.17}$$

It is easy to check that the degree of such maps is m:

$$\deg \restriction_{\Sigma_M} \equiv m. \tag{1.18}$$

The harmonic maps saturating inequality (1.10) which also lie in Σ_m are those corresponding to βz^m ($\beta \in \mathbb{C}^\times = \mathbb{C}\setminus\{0\}$) in stereographic coordinates (1.12). In the representation $\mathbb{S}^2 \subset \mathbb{R}^3$, the harmonic map corresponding to z^m is given by

$$e^{m\theta R}\vec{h}(r), \quad \vec{h} = (h_1, 0, h_3), \quad h_1 = \frac{2}{r^m + r^{-m}}, \quad h_3 = \frac{r^m - r^{-m}}{r^m + r^{-m}}. \tag{1.19}$$

The full two-dimensional family of m-equivariant harmonic maps in Σ_m is then generated by rotation and scaling, so for $s > 0$ and $\alpha \in \mathbb{R}$, we denote

$$\mu = m \log s + i\alpha, \quad \vec{h}[\mu] = e^{\alpha R}\vec{h}^s, \quad \vec{h}^s = \vec{h}(r/s). \tag{1.20}$$

The harmonic map $e^{m\theta R}h[\mu]$ corresponds under stereographic projection to $e^{-\overline{\mu}}z^m$.

We are concerned here with basic global properties of solutions of the Landau-Lifshitz equations (1.1), especially the possible formation of singularities, and the long-time asymptotics.

For finite-energy solutions of (1.1) in 2 space dimensions, finite-time singularity formation is only known to occur in the case of the 1-equivariant harmonic map heat-flow ($a_2 = 0$) – the first such result [5] was for the problem on a disk with Dirichlet boundary conditions (this was extended to Σ_1 on $\mathbb{R}^2$ in [10]). Examples of finite-time blow-up for different target manifolds (not the physical case $\mathbb{S}^2$) are also known (eg. [22]).

For the Schrödinger case ($a_1 = 0$), it is known that small-energy solutions remain regular (this was proved first in [6] for equivariant maps, and then in [2] without symmetry restriction). In the present setting, the energy is *not* small – indeed by (1.10) and (1.18),

$$\mathcal{E} \restriction_{\Sigma_m} \geq 4\pi m. \tag{1.21}$$

A self-similar blow-up solution, which however carries infinite energy, is constructed in [7].

In the recent works [9,10,12], it was shown that when $m \geq 4$, solutions of (1.1) in Σ_m with near minimal energy ($\mathcal{E}(\vec{u}) \approx 4\pi m$) are globally regular, and converge asymptotically to a member $e^{m\theta R}\vec{h}[\mu]$ of the harmonic map family. In particular, the harmonic maps are *asymptotically stable*. The analysis there fails to extend to $m \leq 3$, due to the slower spatial decay of $\frac{d}{d\mu}\vec{h}[\mu]$ (a point which we hope to clarify below). With a new approach, we can now handle the case $m = 3$ as well:

Theorem 1.1. *Let $m \geq 3$, $a = a_1 + ia_2 \in \mathbb{C}\setminus\{0\}$, and $a_1 \geq 0$. Then there exists $\delta > 0$ such that for any $\vec{u}(0, x) \in \Sigma_m$ with $\mathcal{E}(\vec{u}(0)) \leq 4m\pi + \delta^2$, we have a unique global solution $\vec{u} \in C([0,\infty); \Sigma_m)$ of (1.1), satisfying $\nabla\vec{u} \in L^2_{t,loc}([0,\infty); L^\infty_x)$. Moreover, for some $\mu \in \mathbb{C}$ we have*

$$\|\vec{u}(t) - e^{m\theta R}\vec{h}[\mu]\|_{L^\infty_x} + a_1\mathcal{E}(\vec{u}(t) - e^{m\theta R}\vec{h}[\mu]) \to 0 \quad as \ t \to \infty. \tag{1.22}$$

In short, every solution with energy close to the minimum converges to one of the harmonic maps uniformly in x as $t \to \infty$. Even for the higher degrees $m \geq 4$, this result is stronger than the previous ones [12,10,9], where the convergence was given only in time average.[1] Note that in the dissipative case ($a_1 > 0$), solutions converge to

[1] The statements in the previous papers do not follow directly from Theorem 1.1, but are implied by the proof in this paper.

a harmonic map also in the energy norm, while this is impossible for the conservative Schrödinger flow ($a_1 = 0$).

The analysis for the case $m = 2$ seems trickier still, and we have results only in the special case of the harmonic map heat-flow ($a_2 = 0$) with the further restriction that the image of the radial profile map $\vec{v}(r)$ remain on a great circle: $v_2 \equiv 0$ (though of course the map $\vec{u}(x)$ itself covers the full sphere m times) – this is a condition which is preserved by the evolution only for the heat-flow. These results show, in particular, that the strong asymptotic stability result of Theorem 1.1 for $m \geq 3$ is no longer valid; instead, more exotic asymptotics are possible, including infinite-time concentration (blow-up) and "eternal oscillation":

Theorem 1.2. *Let $m = 2$ and $a > 0$. Then there exists $\delta > 0$ such that for any $\vec{u}(0, x) = e^{2\theta R}\vec{v}(0, r) \in \Sigma_2$ with $\mathcal{E}(\vec{u}(0)) \leq 8\pi + \delta^2$, and $v_2(0, r) \equiv 0$, we have a unique global solution $\vec{u} \in C([0, \infty); \Sigma_2)$ satisfying $\nabla\vec{u} \in L^2_{t,loc}([0, \infty); L^\infty_x)$. Moreover, for some continuously differentiable $s : [0, \infty) \to (0, \infty)$ we have*

$$\|\vec{u}(t) - e^{m\theta R}\vec{h}(r/s(t))\|_{L^\infty_x} + \mathcal{E}(\vec{u}(t) - e^{m\theta R}\vec{h}(r/s(t))) \to 0 \quad as \quad t \to \infty. \tag{1.23}$$

In addition, we have the following asymptotic formula for $s(t)$:

$$(1 + o(1)) \log(s(t)) = \frac{2}{\pi} \int_1^{\sqrt{at}} \frac{v_1(0, r)}{r} dr + O_c(1), \tag{1.24}$$

where as $t \to \infty$, $o(1) \to 0$ and $O_c(1)$ converges to some finite value. In particular there are initial data yielding each of the following types of asymptotic behavior:

(1) $s(t) \to \exists\, s_\infty \in (0, \infty)$.
(2) $s(t) \to 0$.
(3) $s(t) \to \infty$.
(4) $0 = \liminf s(t) < \limsup s(t) < \infty$.
(5) $0 < \liminf s(t) < \limsup s(t) = \infty$.
(6) $0 = \liminf s(t) < \limsup s(t) = \infty$.

Estimate (1.23) shows that these solutions do converge asymptotically to the *family* of harmonic maps. However, the evolution along this family, described by the parameter $s(t)$, does not necessarily approach a particular map in Σ_2 (although it might – case (1)). The solution may in fact converge pointwise (but not uniformly) to a constant map $\pm\vec{k}$ (which has zero energy, zero degree, and lies outside Σ_2) as in (2)–(3) (this is infinite-time blow-up or concentration), or it may asymptotically "oscillate" along the harmonic map family, as in (4)–(6).

Note that the above classification (1)–(6) is stable against initial "local" perturbation. Namely, if two initial data $v^1(0)$ and $v^2(0)$ satisfy

$$\int_1^\infty \frac{|v^1_1(0, r) - v^2_1(0, r)|}{r} dr < \infty, \tag{1.25}$$

the corresponding solutions have the same asymptotic type among (1)–(6). More precisely, the difference of their scaling parameters converges in $(0, \infty)$. The point is that the energy just barely fails to control the above integral.

In particular, the oscillatory behavior in (4)–(6) is driven solely by the distribution around spatial infinity. In fact, if we replace the domain $\mathbb{R}^2$ by the disk $D = \{x \in \mathbb{R}^2 \mid |x| < 1\}$ with the same symmetry restriction with $m = 2$ and the same boundary conditions $v(t,0) = -\vec{k}$ and $v(t,1) = \vec{k}$, then it is known [1] (see also [8]) that all the solutions behave like (2), namely they concentrate at $x = 0$ as $t \to \infty$, provided that $v_3(0,r)$ has only one zero. The formula (1.24) suggests that we should always have (2) on D without the additional condition. Also, if we replace the domain $\mathbb{R}^2$ by $\mathbb{S}^2$, then we can rather easily show in the dissipative case $a_1 > 0$ that the solution converges to one harmonic map for all $m \in \mathbb{N}$, by the argument in this paper, or even those in the previous papers. We state the result on $\mathbb{S}^2$ in Appendix A with a sketch of the proof.

We should mention that existence of eternal oscillation of the same type was first shown in [18] for the semilinear heat equation of $u(t,x) : [0,\infty) \times \mathbb{R}^N \to \mathbb{R}$,

$$u_t - \Delta u = |u|^p u, \tag{1.26}$$

for very high dimensions and power[2] ($N \geq 11$ and $p > 4/(N - 4 - 2\sqrt{N-1})$), by using the comparison principle, but they did not obtain an asymptotic formula valid for all solutions, nor the asymptotic stability of the family of stationary solutions in a solution class containing the eternal oscillations.

There is another example in [21, Sect. 5] with less similarity to ours, but for the harmonic map heat flow, which shows existence of "eternal winding" around a compact 1-parameter family of harmonic maps from $\mathbb{S}^2$ to $\mathbb{S}^2 \times \mathbb{R}^2$ with some warped metric, where the analysis is reduced to an ODE on the target by the special choice of initial data. In this case, the weird behavior of the solutions is entirely due to the artificial choice of the metric on the target.

Compared with those results, we have the following advantages:

(1) The setting is very simple and physically natural.
(2) The asymptotic formula is explicit in terms of the initial data, and valid for all general solutions under the symmetry condition.

We want to emphasize also that our analysis works in the same way in the dissipative ($a_1 > 0$) and the dispersive ($a_1 = 0$) cases. We need $a_2 = 0$ in Theorem 1.2 only because the angular parameter $\alpha(t)$ gets beyond our control (hence we remove it by the constraint), but the rest of our arguments could work in the general case.[3]

1.1. The main difficulty and the main idea. The standard approach for asymptotic stability is to decompose the solution into a leading part with finite dimensional parameters varying in time, and the rest decaying in time either by dissipation or by dispersion. In our context, we want to decompose the solution in the form

$$\vec{v}(t) = \vec{h}[\mu(t)] + \check{v}(t) \tag{1.27}$$

such that the remainder $\check{v}(t)$ decays, and the parameter $\mu(t) \in \mathbb{C}$ converges as $t \to \infty$ (at least for Theorem 1.1). In favorable cases (the higher m, in our context), we can choose $\mu(t)$ such that all secular modes for $\check{v}(t)$ are absorbed into the time evolution of the main part $\vec{h}[\mu(t)]$. This means that the kernel of the linearized operator for $\check{v}(t)$ is spanned by the parameter derivatives of $\vec{h}[\mu]$, and hence we can put that component of $\partial_t \vec{v}(t)$ into $\partial_t \vec{h}[\mu(t)]$. This is good both for $\check{v}(t)$ and $\mu(t)$, because

[2] The power is bigger at least than the H^5 scaling critical exponent.

[3] We will use the parameter convergence in the proof of Theorem 1.1 in the dispersive case $a_1 = 0$ to fix our linearized operator. However it is possible to treat the linearized operator even with non-convergent parameter and $a_1 = 0$, if we assume one more regularity on the initial data. We do not pursue it here since the wild behavior of $\alpha(t)$ prevents us from using it.

(1) $\check{v}(t)$ will be free from secular modes, and so we can expect it to decay by dissipation or dispersion, at least at the linearized level.
(2) The decomposition is preserved by the linearized equation. Hence $\dot{\mu}(t)$ is affected by $\check{v}$ only superlinearly, i.e. at most in quadratic terms.

In particular, if we can get L^2 decay of $\check{v}$ in time, then $\dot{\mu}(t)$ becomes integrable in time, and so converges as $t \to \infty$. This is indeed the case for $m > 3$.

However, the above naive argument does not take into account the space-time behavior of each component. The problem comes from the fact that the decomposition and the decay estimate must be implemented in different function spaces, and they may be incompatible if the eigenfunctions decay too slowly at the spatial infinity.

In fact, the parameter derivative of $\vec{h}[\mu]$ is given by

$$d\vec{h}[\mu] = h_1^s e^{\alpha R}[(h_3^s, 0, -h_1^s)d\mu_1 + (0, 1, 0)d\mu_2], \tag{1.28}$$

and hence the eigenfunctions are $O(r^{-m})$ for $r \to \infty$, i.e. slower for lower m. On the other hand, the spatial decay property in the function space for the time decay estimate is essentially determined by the invariance of our problem under the scaling

$$\vec{v}(t, x) \mapsto \vec{v}(\lambda^2 t, \lambda x), \tag{1.29}$$

which maps solutions into solutions, preserving the energy. If we want L^2 decay in time (so that we can integrate quadratic terms in $\dot{\mu}$), then a function space with the right scaling is given by

$$\check{v}/r \in L_t^2 L_x^\infty. \tag{1.30}$$

To preserve such norms in x under the orthogonal projection, the eigenfunction must be in the dual space, for which $m > 3$ is necessary. Indeed, this is the essential reason for the restriction $m \geq 4$ in the previous works [10–12]. We emphasize that the above difficulty is common for the dissipative and dispersive cases, since they share the same scaling property. That is, the dissipation does not help with this issue, even though it gives us more flexibility in the form of decay estimates.

The main novelty of the present approach is the non-orthogonal decomposition

$$L_x^2 = (h_1^s) \oplus (\varphi^s)^\perp, \tag{1.31}$$

where $\varphi^s(r)$ is smooth and supported away from $r = 0$ and from $r = \infty$, so that the (non-orthogonal) projection may preserve the decay estimates. This is good for the remainder $\check{v}$, but not for the parameter μ —the decomposition is no longer preserved by the linearized evolution, since they have no particular relation. This implies that we get a new error term in $\dot{\mu}(t)$ which is linear in $\check{v}(t)$ (see Sect. 6). This contribution is handled by including it in a sort of "normal form" for the dynamics of the parameters $\mu(t)$, explained in Sect. 7. In particular, it is this new term which drives the non-trivial dynamics for the $m = 2$ heat-flow given in Theorem 1.2.

For the purely dispersive (Schrödinger map) case, one tool we use should be of some independent interest: the 2D radial "double-endpoint Strichartz estimate" for Schrödinger operators with sufficiently "repulsive" potentials (in the absence of a potential, the estimate is false). The proof is given in Sect. 10.2.

1.2. Organization of the paper. In Sect. 2, we use the "generalized Hasimoto transform" to derive the main equation used to obtain time-decay estimates of the remainder term. Section 3 gives the details of the solution decomposition described above, and addresses the inversion of the Hasimoto transform. The estimates for going back and forth between the different coordinate systems (the "Hasimoto" one of Sect. 2 and the decomposition of Sect. 3) are given in Sect. 4. Section 5 is devoted to establishing the time-decay (dispersive if $a_1 = 0$, diffusive if $a_1 > 0$) of the remainder term, using energy-, Strichartz-, and scattering-type estimates. The dynamics of the parameters $\mu(t)$ are derived and estimated in Sect. 6. The leading term in the equation for $\dot{\mu}$ is not integrable in time, and so Sect. 7 gives an integration by parts in time to identify (and estimate) a kind of "normal form" correction to $\mu(t)$, whose time derivative is integrable. At this stage, the proof of Theorem 1.1 for $m > 3$ is complete. A more subtle estimate of an error term for $m = 3$ is done in Sect. 8, completing the proof in that case. Finally, in Sect. 9, the normal form correction is analyzed in the case $m = 2$, $a_2 = 0$, $v_2 = 0$, in order to prove Theorem 1.2. Proofs of certain linear estimates (including the double-endpoint Strichartz) are relegated to Sect. 10. Appendix A states the analogous theorems for domain $\mathbb{S}^2$ and sketches the proofs.

At the end of each of the main sections, we will put a proposition summarizing the main contents of that section.

1.3. Some further notation. We distinguish inner products in $\mathbb{R}^3$ and $\mathbb{C}$ by

$$\vec{a} \cdot \vec{b} = \sum_{k=1}^{3} a_k b_k, \quad a \circ b = \operatorname{Re} a \operatorname{Re} b + \operatorname{Im} a \operatorname{Im} b. \tag{1.32}$$

Both will be used for $\mathbb{C}^3$ vectors too. The L^2_x inner-product is denoted

$$(f \mid g) = \int_{\mathbb{R}^2} f(x)\overline{g(x)}dx, \tag{1.33}$$

while (f, g) just denotes a pair of functions. For any radial function $f(r)$ and any parameter $s > 0$, we denote rescaled functions by

$$f^s(r) := f(r/s), \quad f^{s\!\!/}(r) := f(r/s)s^{-2}. \tag{1.34}$$

We denote the Fourier transform on $\mathbb{R}^2$ by $\mathcal{F}$, and, for radial functions, the Fourier-Bessel transform of order m by $\mathcal{F}_m$:

$$(\mathcal{F}f)(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^2} f(x)e^{-ix\cdot\xi}dx, \quad (\mathcal{F}_m)f(\rho) = \int_0^\infty J_m(r\rho)f(r)rdr, \tag{1.35}$$

where J_m is the Bessel function of order m. For $m \in \mathbb{Z}$ we have

$$J_m(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{im\theta - ir\sin\theta}d\theta, \quad \mathcal{F}[f(r)e^{im\theta}] = i^m(\mathcal{F}_m f)e^{im\theta}. \tag{1.36}$$

We denote the Laplacian Δ_x on the subspace spanned by d-dimensional spherical harmonics of order m by

$$\Delta_d^{(m)} := \partial_r^2 + (d-1)r^{-1}\partial_r - m(m+d-2)r^{-2}. \tag{1.37}$$

Finally, the space L^p_q is the dyadic version of $L^p(rdr)$ defined by the norm

$$\|f\|_{L^p_q} = \Big\| \|f(r)\{2^j < r < 2^{j+1}\}\|_{L^p(rdr)} \Big\|_{\ell^q_j(\mathbb{Z})}. \tag{1.38}$$

2. Generalized Hasimoto Transform

In this section, we recall from the previous papers [9–11] the equation for the remainder part, which is written in terms of a derivative vanishing exactly on the harmonic maps, and so independent of the decomposition. The equation was originally derived in [6] in the case of small energy solutions (hence with no harmonic map component), and called there the *generalized Hasimoto transform*.

Under the m-equivariance assumption (1.13), the Landau-Lifshitz equation (1.6) is equivalent to the following reduced equation for $\vec{v}(r, t)$:

$$\vec{v}_t = P_a^{\vec{v}} \left[\partial_r^2 + \frac{\partial_r}{r} + \frac{m^2}{r^2} R^2 \right] \vec{v}. \tag{2.1}$$

Define the operator $\partial_{\vec{v}}$ on vector-valued functions by

$$\partial_{\vec{v}} := \partial_r - \frac{m}{r} J^{\vec{v}} R. \tag{2.2}$$

Since for any vector $\vec{b}$, $J^{\vec{v}} R \vec{b} = \vec{k}(\vec{v} \cdot \vec{b}) - (\vec{k} \cdot \vec{v})\vec{b}$, we have $J^{\vec{v}} R = -v_3$ on the tangent space $T_{\vec{v}}\mathbb{S}^2 = \vec{v}^{\perp}$. For future use, we denote the corresponding operator on scalar functions by

$$L_{\vec{v}} := \partial_r + \frac{m v_3}{r}. \tag{2.3}$$

Then Eq. (2.1) can be factored as

$$\vec{v}_t = -P_a^{\vec{v}} D_{\vec{v}}^* \partial_{\vec{v}} \vec{v}, \tag{2.4}$$

where

$$D := P^{\vec{v}} \partial P^{\vec{v}} \tag{2.5}$$

will always denote a covariant derivative (which acts on $T_{\vec{v}}\mathbb{S}^2$-valued functions), and $*$ denotes the adjoint in $L^2(\mathbb{R}^2)$. Denote the right-most factor in (2.4) by

$$\vec{w} := \partial_{\vec{v}} \vec{v} = \vec{v}_r - \frac{m}{r} P^{\vec{v}} \vec{k}. \tag{2.6}$$

Then (2.4) becomes $\vec{v}_t = -P_a^{\vec{v}} D_{\vec{v}}^* \vec{w}$, and applying $D_{\vec{v}}$ to both sides yields

$$D_t \vec{w} = -P_a^{\vec{v}} D_{\vec{v}} D_{\vec{v}}^* \vec{w}. \tag{2.7}$$

Now we rewrite the equation for $\vec{w}$ by choosing an appropriate orthonormal frame field on $T_{\vec{v}}\mathbb{S}^2$, realized in $\mathbb{C}^3$. Let $\mathbf{e} = \mathbf{e}(t, r)$ satisfy

$$\operatorname{Re} \mathbf{e} \in \vec{v}^{\perp}, \quad |\operatorname{Re} \mathbf{e}| = 1, \quad \operatorname{Im} \mathbf{e} = J^{\vec{v}} \operatorname{Re} \mathbf{e}. \tag{2.8}$$

Let S, T be real scalar, and let q, ν be complex scalar, defined by

$$\vec{w} = q \circ \mathbf{e}, \quad P^{\vec{v}} \vec{k} = \nu \circ \mathbf{e}, \quad D_t \mathbf{e} = -iS\mathbf{e}, \quad D_r \mathbf{e} = -iT\mathbf{e}. \tag{2.9}$$

Then we have the general curvature relation

$$[D_r, D_t]\mathbf{e} = i(T_t - S_r)\mathbf{e} = i \det \left(\vec{v} \ \vec{v}_r \ \vec{v}_t \right) \mathbf{e}. \tag{2.10}$$

Using Eq. (2.4) for $\vec{v}$, we get

$$T_t - S_r = (\vec{w} + \frac{m}{r} P^{\vec{v}} \vec{k}) \cdot P^{\vec{v}}_{ia} D^*_{\vec{v}} \vec{w}. \tag{2.11}$$

Now we fix $\mathbf{e}$ by imposing

$$D_r \mathbf{e} = 0, \quad \mathbf{e}(r = \infty) = (1, i, 0). \tag{2.12}$$

(The unique existence of such $\mathbf{e}$ will be guaranteed by Lemma 4.1.) Then (2.11) yields

$$- S_r = (q + \frac{m}{r} \nu) \circ (ia L^*_{\vec{v}} q). \tag{2.13}$$

A key observation is that in the Schrödinger (non-dissipative) case $a = i$, we can pull out the derivative on q: $S_r = (\partial_r + \frac{2}{r})(\frac{1}{2}|q|^2 + \frac{m}{r}\nu \circ q)$, and so

$$S = -Q + \int_r^\infty 2Q \frac{dr}{r}, \qquad Q := \frac{1}{2}|q|^2 + \frac{m}{r}\nu \circ q = \frac{1}{2}|\vec{w}|^2 + \frac{m w_3}{r}. \tag{2.14}$$

The evolution equation (2.7) for w yields our equation for q:

$$(\partial_t + iS) q = -a L_{\vec{v}} L^*_{\vec{v}} q, \qquad S = \int_r^\infty (q + \frac{m}{r}\nu) \circ (ia L^*_{\vec{v}} q) dr. \tag{2.15}$$

This is the basic equation used to establish diffusive ($a_1 > 0$) or dispersive ($a_1 = 0$) decay estimates. The operator acting on q can be expanded as

$$L_{\vec{v}} L^*_{\vec{v}} = \partial_r^* \partial_r + \frac{(m-1)^2}{r^2} + \frac{2m(1 - v_3)}{r^2} + \frac{m}{r} w_3. \tag{2.16}$$

Following is a summary of this section:

Proposition 2.1. *Let $m \in \mathbb{N}$ and $\vec{u}(t, x) = e^{m\theta R}\vec{v}(t, r)$ be a (local) solution of the Landau-Lifshitz equation (1.1), and let $\mathbf{e}(t, r)$ be a complex orthonormal frame field on $T_{\vec{v}}\mathbb{S}^2$ satisfying*

$$D_r \mathbf{e} = 0, \quad \mathbf{e}(r = \infty) = (1, i, 0), \tag{2.17}$$

where D denotes the covariant derivative (2.5). Define $\vec{w}$, q and ν by

$$\vec{w} = \vec{v}_r - \frac{m}{r} P^{\vec{v}} \vec{k}, \quad q = \vec{w} \cdot \mathbf{e}, \quad \nu = P^{\vec{v}} \vec{k} \cdot \mathbf{e}. \tag{2.18}$$

Then they solve equations

$$(\partial_t + iS) q = -a L_v L^*_v q, \quad S = \int_r^\infty (q + \frac{m}{r}\nu) \circ (ia L^*_v q) dr, \tag{2.19}$$

where $L_v = \partial_r + m v_3/r$ and L^*_v is its adjoint. If $a = i$, the equation of S can be rewritten as

$$S = -Q + \int_r^\infty 2Q \frac{dr}{r}, \quad Q = \frac{1}{2}|q|^2 + \frac{m}{r}\nu \circ q. \tag{2.20}$$

We will use the above equations to derive decay estimates on the remainder $\vec{v} - \vec{h}[\mu]$ via q. The following two sections are devoted to the correspondence between q and the remainder (including the existence of $\mathbf{e}$), and then in Sect. 5 we derive the decay estimates.

3. Decomposition and Orthogonality

In this section, we investigate the interplay between the decay estimates and the orthogonality condition for the decomposition into the harmonic map and the remainder, illuminating the difference between the higher and the lower degrees.

We introduce coordinates for the decomposition of the original map

$$\vec{v} = \vec{h}[\mu] + \check{v}, \tag{3.1}$$

or more precisely for the remainder $\check{v}$, and a localized orthogonality condition which determines the decomposition. The choice of coordinates is the same as in the previous works [9,10,12], while the decomposition itself is different.

For each harmonic map profile $\vec{h}[\mu]$, $\mu = m \log s + i\alpha$, we introduce an orthonormal frame field

$$\mathbf{f} = \mathbf{f}[\mu] := e^{\alpha R}(-\vec{h}^s \times \vec{j} + i\vec{j}) \tag{3.2}$$

on the tangent space $T_{\vec{h}[\mu]}\mathbb{S}^2$, such that the parameter derivative of $\vec{h}[\mu]$ is given by

$$d\vec{h}[\mu] = h_1^s d\mu \circ \mathbf{f}. \tag{3.3}$$

We express the difference from the harmonic map in this frame by

$$z := \check{v} \cdot \mathbf{f}. \tag{3.4}$$

In other words $P^{\vec{h}[\mu]}\vec{v} = z \circ \mathbf{f}$, or $\check{v} = z \circ \mathbf{f} + \gamma \vec{h}[\mu]$, where we denote

$$\gamma := \sqrt{1 - |z|^2} - 1 = -O(|z|^2). \tag{3.5}$$

As explained in the Introduction, the orthogonality condition in the previous works

$$(z \mid h_1^s) = 0 \tag{3.6}$$

would not work for $m \le 3$ due to the slow decay of h_1^s for $r \to \infty$. Hence instead we determine the parameter μ by imposing *localized orthogonality*

$$(z \mid \varphi^s) = 0, \qquad \varphi^s = \varphi(r/s), \tag{3.7}$$

with some smooth localized function $\varphi(r) \in C_0^\infty((0,\infty);\mathbb{R})$, satisfying $(h_1 \mid \varphi) = 1$. The fact that $e^{m\theta R}\vec{h}[\mu]$ solves (1.11) means that

$$\partial_{\vec{h}}\vec{h} = 0, \tag{3.8}$$

and so we have

$$\vec{w} = \partial_{\vec{v}}\vec{v} = \check{v}_r + \frac{m}{r}(h_3^s \check{v} + \check{v}_3 \vec{v}) = L^s \check{v} + \frac{m}{r}\check{v}_3 \vec{v}. \tag{3.9}$$

Hence

$$L^s z = L^s \check{v} \cdot \mathbf{f} + \check{v} \cdot \mathbf{f}_r = \vec{w} \cdot \mathbf{f} - \frac{m}{r}\check{v}_3 z + \frac{m}{r} h_1^s \gamma. \tag{3.10}$$

In order to estimate z by $\vec{w}$ (or equivalently q), we introduce a right inverse of the operator $L^s = \partial_r + \frac{m}{r}h_3^s$, defined by

$$R_\varphi^s g := 2\pi h_1^s(r) \int_0^\infty \int_{r'}^r h_1^s(r'')^{-1} g(r'')dr'' \overline{\varphi^s}(r')h_1^s(r')r'dr'. \tag{3.11}$$

Then we have

$$L^s R_\varphi^s g = g, \qquad R_\varphi^s L^s g = g - h_1^s(g \mid \varphi^s), \tag{3.12}$$

hence $R_\varphi^s = (L^s)^{-1}$ on $(\varphi^s)^\perp$. Moreover we have the following uniform bounds

Lemma 3.1. *For all $p \in [1, \infty]$ and $|\theta| < m$, we have*

$$\|R_\varphi^s g\|_{r^\theta L_p^\infty} \lesssim \|\varphi\|_{r^{-\theta} L_{p'}^1} \|g\|_{r^{\theta+1} L_p^1}, \tag{3.13}$$

where the L_q^p norm is defined in (1.38). Moreover, the condition on φ is optimal in the following sense: if $\varphi \geq 0$, then $\varphi \in r^{-1}L_{p'}^1$ is necessary for R_φ^s to be bounded $r^{\theta-1}L_p^\infty \to \mathcal{D}'(0, \infty)$.

We give a proof in Sect. 10. Note that the above bounds are scaling invariant: denoting $D_s f := f(r/s)$, we have

$$R_\varphi^s = sD_s R_\varphi^1 D_s^{-1}, \quad L^s = s^{-1}D_s L^1 D_s^{-1}. \tag{3.14}$$

We can combine the estimates of the lemma with the embedding

$$r^{\theta_1} L_{q_1}^{p_1} \subset r^{\theta_2} L_{q_2}^{p_2} \iff \frac{2}{p_1} - \theta_1 = \frac{2}{p_2} - \theta_2, \; p_1 \geq p_2, \; q_1 \leq q_2. \tag{3.15}$$

The above lemma is used as follows. First note that the orthogonality $(\varphi^s \mid z) = 0$ implies that $z = R_\varphi^s L^s z$ because of (3.12). For the energy norm, we choose $\theta = 0$ and $p = 2$ in Lemma 3.1. Then

$$\|z/r\|_{L_x^2} \lesssim \|z\|_{L_2^\infty} \lesssim \|\varphi\|_{L_2^1} \|L^s z\|_{rL_2^1} \lesssim \|L^s z\|_{L_x^2}. \tag{3.16}$$

Since $|L^s - \partial_r| \lesssim 1/r$, we further obtain

$$R_\varphi^s : r^\theta L^2 \to r^\theta X, \quad (|\theta| < m), \tag{3.17}$$

where the space X is defined by the norm

$$\|z\|_X := \|z/r\|_{L_x^2} + \|z_r\|_{L_x^2}. \tag{3.18}$$

The Sobolev embedding $X \subset L^\infty$ is trivial by Schwarz:

$$\|z\|_{L_x^\infty}^2 \leq \|z/r\|_{L_x^2} \|z_r\|_{L_x^2}. \tag{3.19}$$

Hence we get by using (3.10),

$$\|z\|_X \lesssim \|L^s z\|_{L^2} \lesssim \|q\|_{L^2} + \|z\|_{L^\infty} \|z\|_X. \tag{3.20}$$

For L_t^2 estimates of z, we use Lemma 3.1 with $\theta = 1$ and $p = \infty$. Then we have

$$\|z/r\|_{L_p^\infty} \lesssim \|\varphi\|_{r^{-1}L_1^1} \|L^s z\|_{r^2 L_p^1} \lesssim \|L^s z\|_{L_p^\infty}, \tag{3.21}$$

for any $p \in [1, \infty]$, and so by using (3.10),

$$\|z/r\|_{L_t^2 L_p^\infty} \lesssim \|q\|_{L_t^2 L_p^\infty} + \|z\|_{L_{t,x}^\infty} \|z\|_{L_t^2 L_p^\infty}. \tag{3.22}$$

If we were to use h_1 instead of φ, then we would need $m > 3$ for the Strichartz-type bound (3.22), and $m > 2$ for the energy bound (3.16), by the last statement of the lemma.

As a summary of this section, we have

Proposition 3.2. *Let $m \geq 2$, $\vec{v}(r) \in \Sigma_m$ and, for some $\mu = m \log s + i\alpha \in \mathbb{C}$,*

$$\vec{v} = \vec{h}[\mu] + \check{v}, \quad z = \check{v} \circ \mathbf{f}, \tag{3.23}$$

where $\mathbf{f}$ is the orthonormal frame on $T_{\vec{h}[\mu]}\mathbb{S}^2$ defined in (3.2). Suppose that

$$(z|\varphi^s) = 0, \quad \varphi^s := \varphi(r/s) \tag{3.24}$$

for a fixed $\varphi \in C_0(0, \infty)$ satisfying $(h_1|\varphi) = 1$. Then we have the estimates

$$\begin{aligned} &\|z\|_X := \|z/r\|_{L_x^2} + \|z_r\|_{L_x^2} \lesssim \|q\|_{L_x^2} + \|z\|_{L_x^\infty} \|z\|_X, \\ &\|z/r\|_{L_p^\infty} \lesssim \|q\|_{L_x^\infty} + \|z\|_{L_x^\infty} \|z\|_{L_p^\infty} \quad (1 \leq p \leq \infty), \\ &\|z\|_{L_x^\infty} \lesssim \|z\|_X, \end{aligned} \tag{3.25}$$

where $q = \vec{w} \cdot \mathbf{e}$ is the same as in Proposition 2.1.

In the next section, we see that such an orthogonal decomposition uniquely exists for $\vec{v} \in \Sigma_m$ with energy close to the ground one, with small norms for q and z, so that we can dispose of the quadratic terms in the above estimates.

4. Coordinate Change

Before beginning the estimates for the evolution, we establish in this section the bi-Lipschitz correspondence between the different coordinate systems: $\vec{v}$ and (μ, q), including unique existence of the decomposition. It is valid for any map in our class Σ_m with energy close to the ground states.

For that purpose, we need to translate between the different frames $\mathbf{e}$ and $\mathbf{f}$. At each point (t, r), we define $\mathcal{M} = \mathbf{f} \otimes \mathbf{e} \in GL_{\mathbb{R}}(\mathbb{C})$, a real-linear map $\mathbb{C} \to \mathbb{C}$, by

$$\mathcal{M}z := \mathbf{f} \cdot (\mathbf{e} \circ z). \tag{4.1}$$

Its transpose ${}^t\mathcal{M} = \mathbf{e} \otimes \mathbf{f}$, defined by ${}^t\mathcal{M}z = \mathbf{e} \cdot (\mathbf{f} \circ z)$, is the adjoint in the sense that $(\mathcal{M}z) \circ w = z \circ ({}^t\mathcal{M}w)$. For any $\mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{C}^3$ we have

$$\mathbf{b} \cdot (\mathbf{c} \circ \mathbf{d}) = (\operatorname{Re} \mathbf{b} \cdot \mathbf{c}) \circ \mathbf{d} + (\operatorname{Im} \mathbf{b} \cdot \mathbf{c}) \circ \mathbf{d}. \tag{4.2}$$

Since $\mathbf{f}(\infty) = e^{-i\alpha}\mathbf{e}(\infty)$, and $\mathbf{f} \perp \vec{h}[\mu]$, we have

$$\mathcal{M}(\infty) = e^{-i\alpha}, \quad \mathcal{M}_r = \mathbf{f} \otimes \mathbf{e}_r + \mathbf{f}_r \otimes \mathbf{e} = -\mathbf{f} \otimes \check{v}(\mathbf{e} \cdot \vec{v}_r) - \frac{m}{r} h_1^s \check{v} \otimes \mathbf{e}. \tag{4.3}$$

Then $\mathbf{e}$ can be recovered from $\mathcal{M}$ by

$$\mathbf{e} = P^{\vec{h}[\mu]}\mathbf{e} + (\vec{h}[\mu]\cdot\mathbf{e})\vec{h}[\mu] = {}^t\mathcal{M}\mathbf{f} - (1+\gamma)^{-1}({}^t\mathcal{M}z)\vec{h}[\mu], \tag{4.4}$$

provided that $|\gamma| < 1$. We further introduce some spaces with (pseudo-)norms:

$$\begin{aligned} &\mathcal{E}_m(\vec{v}) = \mathcal{E}(e^{m\theta R}\vec{v}(r)), \quad |\mu|_C = \min(|\operatorname{Re}\mu|, 1) + dist(\operatorname{Im}\mu, 2\pi\mathbb{Z}),\\ &\|z\|_X = \|z/r\|_{L^2_x} + \|z_r\|_{L^2_x}, \quad \|\mathcal{M}\|_Y = \|\mathcal{M}_r\|_{L^1(dr)} + \|\mathcal{M}\|_{L^\infty_r},\\ &\Sigma_m(\delta) = \{\vec{v}(r) : [0,\infty] \to \mathbb{S}^2 \mid \vec{v}(0) = -\vec{k},\ \vec{v}(\infty) = \vec{k},\ \mathcal{E}_m(\vec{v}) \le 4m\pi + \delta^2\},\\ &L^2(\delta) = \{q(r) : [0,\infty) \to \mathbb{C} \mid \|q\|_{L^2_x} \le \sqrt{2}\delta\}, \quad C = \mathbb{C}/2\pi i\mathbb{Z}. \end{aligned} \tag{4.5}$$

The metric on C is defined such that

$$\|\vec{h}[\mu^1] - \vec{h}[\mu^2]\|_X \sim \|\vec{h}[\mu^1] - \vec{h}[\mu^2]\|_{L^\infty} \sim |\mu^1 - \mu^2|_C. \tag{4.6}$$

The following lemma is the goal of this section.

Lemma 4.1. *Let $m \in \mathbb{N}$ and $\varphi \in C_0^1(0,\infty)$ satisfy $(\varphi \mid h_1) = 1$. Then there exists $\delta > 0$ such that the system of equations*

$$\begin{aligned} &\vec{v} = \vec{h}[\mu] + \check{v}, \quad z = \vec{v}\cdot\mathbf{f}[\mu], \quad (z \mid \varphi^s) = 0, \quad \gamma = \sqrt{1-|z|^2} - 1,\\ &q = \mathbf{e}\circ(\vec{v}_r - \frac{m}{r}P^{\vec{v}}\vec{k}), \quad D_r\mathbf{e} = 0, \quad \mathbf{e}(\infty) = (1, i, 0), \end{aligned} \tag{4.7}$$

defines a bijection from $\vec{v} \in \Sigma_m(\delta)$ to $(\mu, q) \in \mathbb{C} \times L^2(\delta)$, which is unique under the condition $\|z\|_{L^\infty_x} \lesssim \delta$. $\check{v}$, z and $\mathbf{e}$ are also uniquely determined. Moreover, if $(\vec{v}^j, \dots, \mathbf{e}^j)$ with $j = 1, 2$ are such tuples given in this way, then we have

$$\begin{aligned} &\|\check{v}^1 - \check{v}^2\|_X + \|z^1 - z^2\|_X + \|\mathbf{e}^1 - \mathbf{e}^2\|_{L^\infty} + \|\mathcal{M}^1 - \mathcal{M}^2\|_Y\\ &\quad\lesssim \|\vec{v}^1 - \vec{v}^2\|_X \sim |\mu^1 - \mu^2|_C + \|q^1 - q^2\|_{L^2}, \end{aligned} \tag{4.8}$$

where $\mathcal{M}^j := \mathbf{f}^j \otimes \mathbf{e}^j$.

In particular, we have pointwise smallness,

$$\|\check{v}\|_{L^\infty_x} \sim \|z\|_{L^\infty_x} \lesssim \delta \ll 1, \tag{4.9}$$

so that we can neglect higher order terms in z or $\check{v}$.

Proof. We always assume (3.9), (3.4) and (3.5), which define the maps

$$\vec{v} \mapsto \vec{w} = q\circ\mathbf{e}, \quad (\vec{v}, \mu) \mapsto \check{v} \leftrightarrow z \mapsto \gamma, \quad (\check{v}, \mu) \mapsto \vec{v}, \tag{4.10}$$

with the Lipschitz continuity

$$\begin{aligned} &\|\vec{w}^1 - \vec{w}^2\|_{L^2} \lesssim \|\vec{v}^1 - \vec{v}^2\|_X, \quad \|\gamma^1 - \gamma^2\|_X \lesssim \|z^1 - z^2\|_X \sim \|\check{v}^1 - \check{v}^2\|_X,\\ &\Big|\|\vec{v}^1 - \vec{v}^2\|_X - \|\check{v}^1 - \check{v}^2\|_X\Big| \lesssim |\mu^1 - \mu^2|_C. \end{aligned} \tag{4.11}$$

The energy can be written as

$$2\mathcal{E}_m(\vec{v}) = \|\vec{v}_r\|_{L^2_x}^2 + \left\|\frac{m}{r}R\vec{v}\right\|_{L^2_x}^2 = \|\vec{v}_r\|_{L^2_x}^2 + \left\|\frac{m}{r}P^{\vec{v}}\vec{k}\right\|_{L^2_x}^2$$
$$= \|\vec{w}\|_{L^2_x}^2 + 2m(\vec{v}_r \mid P^{\vec{v}}\vec{k}/r) = \|\vec{w}\|_{L^2_x}^2 + 4\pi[v_3(\infty) - v_3(0)]. \quad (4.12)$$

Since $\|P^{\vec{k}}\vec{v}\|_X \lesssim \mathcal{E}_m(\vec{v})^{1/2}$, $X \subset L^\infty_x$ and $|\vec{v}| = 1$, the boundary conditions $v_3(0) = -1$ and $v_3(\infty) = 1$ make sense in the energy norm.

Next we consider a point orthogonality. Let $\vec{v} \in \Sigma_m(\delta)$. Since $v_3(0) < 0 < v_3(\infty)$ and $v_3(r)$ is continuous, we have $\vec{v}(s_0) = e^{i\alpha_0 R}\vec{h}(1)$ for some $\mu_0 = m\log s_0 + i\alpha_0$, so that $\vec{v} = \vec{h}[\mu_0] + \check{v}$ is a decomposition satisfying $(z \mid \varphi^{s_0}) = 0$ if $\varphi(r) = \delta(r-1)$. In this case $\vec{v}$ is recovered from $(\vec{w}, \mu_0)$ by solving the ODE:

$$L^s z = \vec{w}\cdot\mathbf{f}[\mu_0] - \frac{m}{r}\check{v}_3 z + \frac{m}{r}h_1^{s_0}\gamma, \quad z(s_0) = 0, \quad (4.13)$$

or the equivalent integral equation

$$z = R^{s_0}_{\delta(r-1)}\left[\vec{w}\cdot\mathbf{f}[\mu_0] - \frac{m}{r}\check{v}_3 z + \frac{m}{r}h_1^{s_0}\gamma\right]. \quad (4.14)$$

The uniform bound on $R^s_{\delta(r-1)}$ can be localized onto any interval $I \ni s_0$, because z is the solution of the above initial value problem. Hence we get, in the same way as in (3.16),

$$\|z\|_{rL^2_x\cap L^\infty_x(I)} \lesssim \|q\|_{L^2_x(I)} + \|z\|_{L^\infty_x(I)}\|z\|_{rL^2_x(I)}. \quad (4.15)$$

Since $z(s_0) = 0$ and $\|q\|_{L^2_x} \le \delta \ll 1$, we get by continuity in r for $I \to (0,\infty)$,

$$\|z\|_{X\cap L^\infty_x} \lesssim \|q\|_{L^2_x} \lesssim \delta. \quad (4.16)$$

Thus every $\vec{v} \in \Sigma_m(\delta)$ is close at least to some $\vec{h}[\mu_0]$, and we have $\vec{v}^1 - \vec{v}^2 \in X$ by (4.11). $\Sigma_m(\delta)$ is a complete metric space with this distance.

Now we take any $\varphi \in C^1_0(0,\infty)$ satisfying $(\varphi \mid h_1) = 1$, and look for μ around μ_0 solving the orthogonality

$$F(\mu) := (\vec{v}\cdot\mathbf{f}[\mu] \mid \varphi^{s}) = (\check{v}\cdot\mathbf{f}[\mu] \mid \varphi^{s}) = (z \mid \varphi^{s}) = 0. \quad (4.17)$$

Its derivative in μ is given by

$$dF = -(\vec{v}\cdot h_1^s\vec{h}[\mu] \mid \varphi^{s})d\mu - i(\vec{v}\cdot h_3^s\mathbf{f}[\mu] \mid \varphi^{s})d\alpha - (\vec{v}\cdot\mathbf{f}[\mu] \mid (r\partial_r + 2)\varphi^{s})\frac{ds}{s}$$
$$= -d\mu - (\check{v}\cdot\vec{h}[\mu] \mid h_1^s\varphi^{s})d\mu - (\check{v}\cdot\mathbf{f}[\mu] \mid (r\partial_r + 2)\varphi^{s}ds/s + ih_3^s\varphi^{s}d\alpha)$$
$$= -d\mu + O(\delta|d\mu|). \quad (4.18)$$

In particular we have

$$|F(\mu_0)| \lesssim \delta, \quad \frac{\partial F}{\partial\mu}(\mu_0) = -I + O(\delta). \quad (4.19)$$

In addition, both $F(\mu)$ and $\partial_\mu F$ are Lipschitz in $\vec{v}$. Therefore by the implicit mapping theorem, if $\delta > 0$ is small enough, there exists a unique $\mu \in \mathbb{C}$ for each v such that $F(\mu) = 0$ and $|\mu - \mu_0| \lesssim \delta$, and $\vec{v} \mapsto \mu$ is Lipschitz. Then

$$\|z\|_{L^\infty_x} \lesssim \|\vec{v} - \vec{h}[\mu_0]\|_{L^\infty_x} + |\mu_0 - \mu| \lesssim \delta \ll 1, \quad (4.20)$$

and so by the same argument as for (4.16), we get $\|z\|_X \lesssim \delta$, and in addition,

$$\|z^1 - z^2\|_X \lesssim |\mu^1 - \mu^2|_C + \|\vec{w}^1 - \vec{w}^2\|_{L^2}. \tag{4.21}$$

If we have two such $\mu = \mu_1, \mu_2$ with $\|z^j\|_{L^\infty} \lesssim \delta$, then

$$|\mu_1 - \mu_2|_C \sim \|\vec{h}[\mu_1] - \vec{h}[\mu_2]\|_{L^\infty_x} \lesssim \|\vec{v} - \vec{h}[\mu_1]\|_{L^\infty_x} + \|\vec{v} - \vec{h}[\mu_2]\|_{L^\infty_x} \lesssim \delta, \tag{4.22}$$

and so the implicit mapping theorem implies that $\mu_1 = \mu_2$. Thus we get a bijection $\vec{v} \mapsto (\mu, \vec{w})$ with the Lipschitz continuity

$$\|\vec{v}^1 - \vec{v}^2\|_X \sim |\mu^1 - \mu^2|_C + \|\vec{w}^1 - \vec{w}^2\|_{L^2_x}. \tag{4.23}$$

For the frame field $\mathbf{e}$, we consider the matrix $\mathcal{M} = \mathbf{f} \otimes \mathbf{e}$, together with the equivalent set of Eqs. (4.3) and (4.4). Integrating (4.3) from $r = \infty$, we get

$$\begin{aligned} &\|\mathcal{M} - e^{i\alpha}\|_Y \lesssim \|\check{v}/r\|_{L^2_x} \|\vec{v}_r\|_{L^2_x} + \|\check{v}/r\|_{L^2_x} \|h_1^s/r\|_{L^2_x} \lesssim \delta, \\ &\|\mathcal{M}^1 - \mathcal{M}^2\|_Y \lesssim |\mu^1 - \mu^2|_C + \delta \|\vec{v}^1 - \vec{v}^2\|_X + \delta \|\mathbf{e}^1 - \mathbf{e}^2\|_{L^\infty} + \|\check{v}^1 - \check{v}^2\|_X, \end{aligned} \tag{4.24}$$

while (4.4) provides

$$\|\mathbf{e}^1 - \mathbf{e}^2\|_{L^\infty_x} \lesssim \|\mathcal{M}^1 - \mathcal{M}^2\|_{L^\infty_x} + |\mu^1 - \mu^2|_C + \|z^1 - z^2\|_{L^\infty_x}. \tag{4.25}$$

Hence for fixed $\vec{v} \in \Sigma_m(\delta)$ (and μ), we can get $(\mathcal{M}, \mathbf{e}) \in Y \times L^\infty$ by the contraction mapping principle for the system of (4.3) and (4.4). Moreover we get

$$\|\mathcal{M}^1 - \mathcal{M}^2\|_Y + \|\mathbf{e}^1 - \mathbf{e}^2\|_{L^\infty_x} \lesssim \|\vec{v}^1 - \vec{v}^2\|_X. \tag{4.26}$$

If $(\mu, q) \in C \times L^2(\delta)$ is given, we consider the system of Eqs. (4.3), (4.4) and

$$z = R_\varphi^s \left[\mathcal{M} q - \frac{m}{r} \check{v}_3 z + \frac{m}{r} h_1^s \gamma \right], \tag{4.27}$$

which is equivalent to the q equation in (4.7) under the orthogonality $(z \mid \varphi^{s'}) = 0$. The last equation provides, through the uniform bound on R_φ^s,

$$\|z^1 - z^2\|_X \lesssim |\mu^1 - \mu^2|_C + \|q^1 - q^2\|_{L^2_x} + \delta \|\mathcal{M}^1 - \mathcal{M}^2\|_Y + \delta \|z^1 - z^2\|_X. \tag{4.28}$$

Combining this with (4.24) and (4.25), we get $(z, \mathcal{M}, \mathbf{e})$ for any fixed (μ, q) by the contraction mapping, and moreover they satisfy

$$\|z^1 - z^2\|_X + \|\mathcal{M}^1 - \mathcal{M}^2\|_Y + \|\mathbf{e}^1 - \mathbf{e}^2\|_{L^\infty} \lesssim |\mu^1 - \mu^2|_C + \|q^1 - q^2\|_{L^2_x}. \tag{4.29}$$

□

So far we have derived estimates at each fixed t, for the energy norms in the above lemma, and for the dispersive norms in Proposition 3.2. Now we turn to the main part of this paper, the analysis of the global dynamics.

5. Decay Estimates for the Remainder

In this section, we derive dissipative or dispersive space-time estimates of the remainder $\check{v}$ in terms of z, from Eq. (2.15) for q. First by the smallness of z, we obtain from (3.20) and (3.22),

$$\|z(t)\|_X \lesssim \|q(t)\|_{L^2} \lesssim \delta, \quad \|z/r\|_{L^\infty_p} \lesssim \|q\|_{L^\infty_p}, \tag{5.1}$$

for all $p \in [1, \infty]$. Next we estimate the factor S, by using

$$\|r\int_r^\infty fgdr\|_{L^\infty_1} \lesssim \sum_{j\in\mathbb{Z}}\sum_{k\geq j} 2^{j-k}\|fg\|_{L^1(r\sim 2^k)} \sim \|fg\|_{L^1} \leq \|f\|_{L^2}\|g\|_{L^2}. \tag{5.2}$$

Then from the expression in (2.15) for S, we have

$$\|S(t)\|_{L^2_1} \lesssim \|S(t)\|_{r^{-1}L^\infty_1} \lesssim (\|q\|_{L^2_x} + \|z\|_{rL^2_x} + 1)\|L^*_{\vec{v}}q\|_{L^2_x} \lesssim \|L^*_{\vec{v}}q\|_{L^2_x}. \tag{5.3}$$

In the dispersive case $a_1 = 0$, we avoid the derivative by using expression (2.14)

$$\|S(t)\|_{L^2_1} \lesssim \|S(t)\|_{r^{-1}L^\infty_1} \lesssim (\|q\|_{L^2_x} + \|z\|_{rL^2_x} + 1)\|q\|_{L^\infty_2} \lesssim \|q\|_{L^\infty_2} \quad (a_1 = 0). \tag{5.4}$$

For the time decay estimates, we treat the dissipative and the dispersive cases separately.

5.1. Dissipative L^2_t estimate. Here we assume $a_1 > 0$. By Eq. (2.15) of q, we have

$$\partial_t\|q\|^2_{L^2} = -2a_1\|L^*_{\vec{v}}q\|^2_{L^2}, \tag{5.5}$$

hence

$$\|q\|_{L^\infty_t L^2_x} + \|L^*_{\vec{v}}q\|_{L^2_t L^2_x} \lesssim \|q(0)\|_{L^2_x} \sim \delta. \tag{5.6}$$

Since $R^{s*}_\varphi L^{s*} = I$ and $R^{s*}_\varphi : L^2 \to rL^2$ by Lemma 3.1 and duality, we have

$$\|q\|_{rL^2_x} \lesssim \|L^{s*}q\|_{L^2_x} \lesssim \|L^*_{\vec{v}}q\|_{L^2_x} + \|\check{v}\|_{L^\infty}\|q\|_{rL^2_x}. \tag{5.7}$$

Since the last term can be absorbed by (4.9) smallness of $\check{v}$, we get

$$\|q\|_X \lesssim \|q/r\|_{L^2_x} + \|L^{s*}q\|_{L^2_x} \lesssim \|L^*_{\vec{v}}q\|_{L^2_x}. \tag{5.8}$$

So by using the bound (3.17) on R^s_φ, we obtain

$$\|z\|_{rL^2_t X} \lesssim \|q\|_{L^2_t X} \lesssim \|L^*_{\vec{v}}q\|_{L^2_{t,x}} \lesssim \|q(0)\|_{L^2_x} \sim \delta, \tag{5.9}$$

and also from (5.3),

$$\|S\|_{L^2_t L^2_x} \lesssim \delta. \tag{5.10}$$

5.2. Dissipative decay. Next we show the convergence $q \to 0$ as $t \to \infty$, by comparing it with the free evolution. For $T > 0$, let

$$q^T := q - e^{(t-T)a\Delta_2^{(m-1)}} q(T). \tag{5.11}$$

Then we have

$$q_t^T - a\Delta_2^{(m-1)} q^T = (iS - aV)q, \quad q^T(T) = 0, \tag{5.12}$$

where the potential $V(t, x)$ is given by

$$V = \frac{2m(1 - v_3)}{r^2} + \frac{m}{r} w_3. \tag{5.13}$$

Multiplying the equation with q^T, we get the energy identity

$$\begin{aligned} &\frac{1}{2}\|q^T\|_{L_x^2}^2 + \int_T^t a_1(\|q_r^T\|_{L_x^2}^2 + \|\frac{m-1}{r} q^T\|_{L_x^2}^2) dt \\ &= \operatorname{Re} \int_T^t (-aVq + iSq \mid q^T) dt, \end{aligned} \tag{5.14}$$

and hence by Schwarz, and using estimate (5.3) to put $S \in L_t^2 L_x^2$,

$$\begin{aligned} \|q^T\|_{L_{t>T}^\infty L_x^2 \cap L_{t>T}^2 X} &\lesssim \|q/r\|_{L_{t>T}^2 L_x^2} + \|Sq\|_{L_{t>T}^2 L_x^1} + \|q\|_{L_{t>T}^4 L_x^4}^2 \\ &\lesssim \|q/r\|_{L_{t>T}^2 L_x^2} + \|q\|_{L_t^\infty L_x^2} \|q\|_{L_{t>T}^2 X} \to 0 \quad (T \to \infty). \end{aligned} \tag{5.15}$$

Hence $\|q(t)\|_{L_x^2}$ can not converge to a positive number, since $e^{(t-T)a\Delta_2^{(m-1)}} q(T) \to 0$ as $t \to \infty$ for all $T > 0$. Thus we obtain

$$\|z(t)\|_X \lesssim \|q(t)\|_{L_x^2} \to 0 \quad (t \to \infty). \tag{5.16}$$

5.3. Dispersive L_t^2 estimate. Next we consider the case $a_1 = 0$ (and $a_2 \neq 0$). We set (with no loss of generality) $a = i$. Since the energy identity provides only L_x^2 bound on q, we have to work with the Strichartz estimate in a perturbative way. Denoting $H^s := L^s L^{s*}$, the equation of q is given by

$$q_t + iH^{s(0)} q = N_1 + N_2, \tag{5.17}$$

where

$$N_1 := -2am \frac{h_3^{s(0)} - h_3^{s(t)}}{r^2} q, \quad N_2 := iSq - 2am \frac{\check{v}_3}{r^2} q - am \frac{w_3}{r} q, \tag{5.18}$$

and S is given by (2.14). We have

$$|N_1| \lesssim |h_3(s(t)/s(0))||q|/r^2, \tag{5.19}$$

and so

$$\|N_1\|_{L_t^2 L_2^1} \lesssim \|h_3(s(t)/s(0))\|_{L_t^\infty} \|q\|_{L_t^2 L_2^\infty}. \tag{5.20}$$

Using (5.4), we have

$$\|Sq\|_{L^1_t L^2_x} \le \|S\|_{L^2_t L^2_x} \|q\|_{L^2_t L^\infty_x} \lesssim \|q\|^2_{L^2_t L^\infty_2}. \tag{5.21}$$

The other terms in N_2 are bounded in $L^1_t L^2_x$ by

$$\|q\|^2_{L^2_t L^\infty_2} + \|z/r\|_{L^2_t L^\infty_2} \|q\|_{L^2_t L^\infty_2} \lesssim \|q\|^2_{L^2_t L^\infty_2}. \tag{5.22}$$

Now we need the endpoint Strichartz estimate for H^s with fixed scaling s:

Lemma 5.1. *Let $H^s = L^s L^{s*} = -\Delta_2^{(m-1)} + 2mr^{-2}(1-h_3^s)$ and $m > 1$. Then we have*

$$\begin{gathered}\|e^{-iH^s t}\varphi\|_{L^\infty_t L^2_x \cap L^2_t L^\infty_2} \lesssim \|\varphi\|_{L^2_x} \\ \left\|\int_{-\infty}^t e^{-iH^s(t-t')} f(t')dt'\right\|_{L^\infty_t L^2_x \cap L^2_t L^\infty_2} \lesssim \|f\|_{L^1_t L^2_x + L^2_t L^1_2},\end{gathered} \tag{5.23}$$

uniformly for any fixed $s > 0$.

This lemma will be proved in Sect. 10.2. Hence if $|\log(s(t)/s(0))| \ll 1$ for all t, then we have

$$\|q\|_{L^\infty_t L^2_x \cap L^2_t L^\infty_2} \lesssim \|q(0)\|_{L^2} \sim \delta, \tag{5.24}$$

and also from (5.4)

$$\|S\|_{L^2_t L^2_x} \lesssim \delta. \tag{5.25}$$

5.4. Dispersive decay. Next we prove the following asymptotics of scattering type for q and z:

$$e^{-it\Delta_2^{(m-1)}} q(t) \to \exists q_+ \text{ in } L^2_x, \quad z \to 0 \text{ in } L^\infty_x \quad (t \to \infty). \tag{5.26}$$

For the scattering of q, we further expand the equation

$$q_t - i\Delta_2^{(m-1)} q = N_0 + N_2, \tag{5.27}$$

where N_2 is as in (5.18), and

$$N_0 := -2am\frac{1-h_3^s}{r^2} q. \tag{5.28}$$

Then the global Strichartz bound implies that

$$\|N_0\|_{L^2_t L^1_2(T,\infty)} \to 0, \quad \|N_2\|_{L^1_t L^2_x(T,\infty)} \to 0 \tag{5.29}$$

as $T \to \infty$. By Strichartz (for $\Delta_2^{(m-1)}$) once again, we get the scattering of q.

For the vanishing of z, we use the inversion formula

$$z = R^s_\varphi g, \quad g = \mathcal{M}q + r^{-1}m(h_1^s \gamma - \check{v}_3 z). \tag{5.30}$$

Since R^s_φ is bounded $L^2_x \to L^\infty$, the latter two terms contribute at most with $\|z\|_{L^\infty_x}\|z/r\|_{L^2_x} \ll \|z\|_{L^\infty_x}$, hence we may drop them. Also we may replace q by its asymptotic free solution $q^\infty := e^{it\Delta_2^{(m-1)}}q_+$. Moreover we may approximate q_+ by nicer functions. Hence we assume that $\widehat{q} := \mathcal{F}_{m-1}q_+ \in C_0^\infty(0,\infty)$. Then we may further replace the free solution with the stationary phase part:

$$\begin{aligned} q^\infty(t,r) &= C_m t^{-1} e^{ir^2/(4t)} \int_0^\infty J_{m-1}(r\rho/(2t)) e^{i\rho^2/(4t)} q_+(\rho)\rho d\rho \\ &= C_m t^{-1} e^{ir^2/(4t)} \widehat{q}(r/(2t)) + \mathcal{R}, \end{aligned} \tag{5.31}$$

where the error is bounded by Plancherel

$$\|\mathcal{R}\|_{L^2_x} \sim \|(1-e^{ir^2/(4t)})q_+\|_{L^2_x} \lesssim t^{-1}\|r^2 q_+\|_{L^2_x} \to 0. \tag{5.32}$$

Now that spatially local vanishing is clear (eg. it follows from $\|R^s_\varphi Mq(t)\|_{rL^\infty} \lesssim \|q^\infty(t)\|_{L^\infty} \to 0$), we may extract the leading term of R^s_φ for large x. We assume that $s(t) \in L^\infty_t$ and $\operatorname{supp}\varphi^s \subset (0,b)$ for a fixed $b \in (0,\infty)$. Then for $r > b$ we have

$$\begin{aligned} (R^s_\varphi g)(r) &= o(1) + \int_b^r \frac{h^s_1(r)}{h^s_1(r'')} g(r'')dr'' \\ &= o(1) + \int_b^r (r'/r)^m g(r')dr' \quad \text{as } r \to \infty. \end{aligned} \tag{5.33}$$

Thus we are reduced to showing that

$$G\chi := \int_b^r (\rho/r)^m \mathcal{M}(t,\rho) t^{-1} e^{i\rho^2/(4t)} \chi(\rho/t)d\rho \to 0 \text{ in } L^\infty_r \tag{5.34}$$

for any $\chi \in C_0^\infty(0,\infty)$. By partial integration on $(\rho/t)e^{i\rho^2/(4t)}$, we have

$$\begin{aligned} r^m G\chi &= (i/2)[\rho^{m-1}\mathcal{M}(\rho)e^{i\rho^2/(4t)}\chi(\rho/t)]_b^r \\ &\quad - \int_b^r \big[(m-1)\mathcal{M}(\rho)\chi(\rho/t)/\rho + \mathcal{M}(\rho)\chi'(\rho/t)/t \\ &\quad + \mathcal{M}_r(\rho)\chi(\rho/t)\big]\rho^{m-1}e^{i\rho^2/(4t)}d\rho. \end{aligned} \tag{5.35}$$

The right-hand side is bounded by r^m/t, using $|\chi(\rho/t)| \lesssim \rho/t$ for the first, second and fourth terms, $|\chi'(\rho/t)| \lesssim 1$ for the third, and $\mathcal{M}_r \in L^\infty_t L^1(dr)$ for the fourth term. Thus we obtain $\|z(t)\|_{L^\infty_x} \to 0$.

Thus we have obtained the following a priori estimates in this section

Proposition 5.2. *Let $m \ge 2$ and $\vec{u}(t,x) = e^{m\theta R}v(t,r)$ be a solution of (1.1) on $0 < t < T$ with $\vec{u}(0) \in \Sigma_m$ and $\mathcal{E}(\vec{u}(0)) \le 4m\pi + \delta^2$ for some small $\delta > 0$. Let q, z, S be as in Proposition 2.1, and let $\mu(t)$ be given by Lemma 4.1.*

(I) *If $a_1 > 0$, then we have*

$$\|z\|_{L^\infty_t(0,T;X)\cap L^2_t(0,T;rX)} \lesssim \|q\|_{L^\infty_t(0,T;L^2_x)\cap L^2_t(0,T;X)} \lesssim \|q(0)\|_{L^2_x} \sim \delta. \tag{5.36}$$

Moreover, if $T = \infty$ then

$$\|z(t)\|_X \lesssim \|q(t)\|_{L^2_x} \to 0 \quad (t \to \infty). \tag{5.37}$$

(II) *If* $a_1 = 0$ *and* $s(t) = s(0) + O(\delta)$, *then we have*

$$\|z\|_{L_t^\infty(0,T;X)\cap L_t^2(0,T;rL_2^\infty)} \lesssim \|q\|_{L_t^\infty(0,T;L_x^2)\cap L_t^2(0,T;L_2^\infty)} \lesssim \|q(0)\|_{L_x^2} \sim \delta. \tag{5.38}$$

Moreover, if $T = \infty$ *and* $s(t)$ *converges as* $t \to \infty$, *then*

$$\|z(t)\|_{L_x^\infty} \to 0, \quad \|q(t) - e^{-it\Delta_2^{(m-1)}} q_+\|_{L_x^2} \to 0, \quad (t \to \infty) \tag{5.39}$$

for some radial $q_+ \in L_x^2$. $\Delta_2^{(m-1)}$ *is the* $(m-1)$*-equivariant Laplacian, see* (1.37).

Note that the decay of z is transferred to the remainder $\check{v}$ by Lemma 4.1. By using the above arguments and Lemma 4.1, it is easy to see that the solution is global unless $s(t) \to 0$ in finite time (for a detailed proof, see [11, Sect. 3]). The remaining sections are therefore devoted to the analysis of the parameter dynamics, which is the most novel part of this paper.

6. Parameter Evolution

It remains to control the asymptotic behavior of the parameter $\mu(t)$ of the harmonic map part of the solution. Its evolution is determined by differentiating the localized orthogonality condition

$$0 = \partial_t(z \mid \varphi^s) = (\check{v}_t \cdot \mathbf{f} \mid \varphi^s) + (\check{v} \cdot \mathbf{f}_t \mid \varphi^s) + (z \mid \partial_t \varphi^s), \tag{6.1}$$

and each term on the right is expanded by using

$$\begin{aligned} \check{v}_t &= \vec{v}_t - \vec{h}[\mu]_t = -(aL_{\vec{v}}^* q) \circ \mathbf{e} - h_1^s \dot{\mu} \circ \mathbf{f}, \\ \mathbf{f}_t &= -ih_3^s \dot{\alpha} \mathbf{f} - h_1^s \dot{\mu} h[\mu], \quad \partial_t \varphi^s = -\frac{\dot{s}}{s} r \partial_r \varphi^s. \end{aligned} \tag{6.2}$$

Plugging this into the above and then dividing it by s^2, we get

$$\dot{\mu} = -(\mathcal{M}aL_{\vec{v}}^* q \mid \varphi^{\not s}) - (h_1^s \dot{\mu} \gamma \mid \varphi^{\not s}) - (z \mid (\frac{\dot{\mu}_1}{m} r\partial_r - i\dot{\mu}_2 h_3^s)\varphi^{\not s}). \tag{6.3}$$

The last two terms are bounded by

$$|\dot{\mu}| \|z\|_{L^\infty} (\|\varphi\|_{L^1} + \|r\partial_r \varphi\|_{L^1}), \tag{6.4}$$

and so absorbed by the left-hand side since $\|z\|_{L^\infty} \lesssim \delta \ll 1$.

Since $|\nu| = |P^v \vec{k}| \lesssim h_1^s + |\check{v}|$ and hence

$$|\vec{v}_r| \lesssim |q| + |z|/r + h_1^s/r, \tag{6.5}$$

we get from (4.3),

$$|\mathcal{M}_r| \lesssim |qz| + |z|^2/r + |zh_1^s|/r. \tag{6.6}$$

The leading (first in the r.h.s) term in (6.3) can be estimated, using $[\mathcal{M}a, L_{\vec{v}}^*] = \mathcal{M}_r a$, as follows:

$$\begin{aligned} |(\mathcal{M}aL_{\vec{v}}^* q \mid \varphi^{\not s})| &\lesssim s^{-1}(\|\mathcal{M}_r\|_{L_x^2} + \|\mathcal{M}\|_{L_x^\infty})\|q\|_{rL_x^2} \\ &\lesssim s^{-1}(\|q\|_{L_x^2} + \|z/r\|_{L_x^2} + 1)\|q\|_{rL_x^2}. \end{aligned} \tag{6.7}$$

Hence using that $\|z\|_{L^\infty_x} \lesssim \|z\|_X \lesssim \|q\|_{L^2} \lesssim \delta \ll 1$, we get

$$\|s\dot{\mu}\|_{L^2_t} \lesssim \|q/r\|_{L^2_{t,x}} \lesssim \|q\|_{L^2_t L^\infty_2}. \tag{6.8}$$

Then the last two terms of (6.3) are bounded in L^1_t by

$$\|s\dot{\mu}\|_{L^2_t} \|z/r\|_{L^2_t L^\infty_x} \|(r|\varphi| + r^2|\varphi_r|)^{\not{s}}\|_{L^1_x} \lesssim \|q\|^2_{L^2_t L^\infty_2}, \tag{6.9}$$

where we used (5.1). Thus we have obtained

Proposition 6.1. *Let $\vec{v}$, q, μ, φ and $\mathcal{M}$ as in Proposition 5.2 and Lemma 4.1. Then $\mu(t)$ satisfies*

$$\dot{\mu} = -(\mathcal{M}aL^*_{\vec{v}}q \mid \varphi^{\not{s}}) + error, \tag{6.10}$$

where $L^*_{\vec{v}} = -\partial_r - 1/r + mv_3/r$, and

$$\|s\dot{\mu}\|_{L^2_t(0,T)} \lesssim \|q\|_{L^2_t(0,T;L^\infty_2)}, \quad \|error\|_{L^1_t(0,T)} \lesssim \|q\|^2_{L^2_t(0,T;L^\infty_2)}. \tag{6.11}$$

Thus our problem is reduced to the global behavior of the above term on the right, which is linear in q.

7. Partial Integration for the Parameter Dynamics

Now we want to integrate in t the right-hand side of (6.3), which is not bounded in L^1_t. The key idea is to employ the q Eq. (2.15), by identifying a factor of $L_{\vec{v}}L^*_{\vec{v}}q$, through a partial integration in space.

For the spatial integration, we first freeze the phase factor $\mathcal{M}$. Since $\vec{h}[\mu] = \vec{v} = -\vec{k}$ at $r = 0$, we have $\mathcal{M}(t,0) = e^{i\widetilde{\alpha}}$, i.e. $\mathbf{f}(t,0) = e^{i\widetilde{\alpha}}\mathbf{e}(t,0)$ for some real $\widetilde{\alpha}(t)$. Then $D_t\mathbf{f}(t,0) = i\widetilde{\alpha}'(t)\mathbf{f}(t,0) - iS(t,0)\mathbf{f}(t,0)$, and so

$$\widetilde{\alpha}'(t) = S(t,0) + \alpha'(t). \tag{7.1}$$

We decompose

$$\mathcal{M} = e^{i\widetilde{\alpha}} + \check{\mathcal{M}}, \tag{7.2}$$

and rewrite the leading term of (6.3) as follows. Let $c = \|h_1\|^{-2}_{L^2}$. Since $L_{\vec{v}} = L^s + m\check{v}_3/r$ and $L^s h^s_1 = 0$, we have

$$\begin{aligned}(\mathcal{M}aL^*_{\vec{v}}q \mid \varphi^{\not{s}}) &= ae^{i\widetilde{\alpha}}(L^*_{\vec{v}}q \mid \varphi^{\not{s}}) + (\check{\mathcal{M}}aL^*_{\vec{v}}q \mid \varphi^{\not{s}}) \\ &= ae^{i\widetilde{\alpha}}\left[(L^*_{\vec{v}}q \mid (\varphi - ch_1)^{\not{s}}) + (mq\check{v}_3/r \mid c{h_1}^{\not{s}})\right] \\ &\quad + (\check{\mathcal{M}}aq \mid L_{\vec{v}}\varphi^{\not{s}}) + (\check{\mathcal{M}}_r aq \mid \varphi^{\not{s}}).\end{aligned} \tag{7.3}$$

The second term is bounded by $\|q\check{v}r^{-3}\|_{L^1_x} \lesssim \|q/r\|_{L^2_x}\|z/r^2\|_{L^2_x}$, and the last two terms are bounded by

$$\|q/r\|_{L^2_x}(\|\mathcal{M}_r/r\|_{L^2_x} + \|\check{\mathcal{M}}/r\|_{L^\infty_x}), \tag{7.4}$$

where the last factor is further bounded by using that $\check{\mathcal{M}} = 0$ at $r = 0$,

$$\|\check{\mathcal{M}}/r\|_{L^\infty_x} \lesssim \|\mathcal{M}_r/r\|_{L^2_x} \lesssim \|q/r\|_{L^2_x} + \|z/r^2\|_{L^2_x} \lesssim \|q\|_{L^\infty_2}. \tag{7.5}$$

We further rewrite the remaining (main) term. By the definition of c, we have

$$(\varphi - ch_1 \mid h_1) = 1 - c\|h_1\|^2_{L^2} = 0, \tag{7.6}$$

and so we have

$$\varphi^s - ch_1^s = L^{s*}R^{s*}_\varphi(\varphi^s - ch_1^s), \tag{7.7}$$

where the operator R^s_φ was defined in (3.11). Let

$$\psi := R^*_\varphi(\varphi - ch_1) = -\frac{c}{m-1}r^{1-m} + O(r^{1-3m}) \quad (r \to \infty), \tag{7.8}$$

where the asymptotic form easily follows from the fact that

$$\psi(r) = -c(h_1(r)r)^{-1}\int_r^\infty h_1(r')^2 r' dr' \quad (r \gg 1). \tag{7.9}$$

Then we have, by using Eq. (2.15) for q,

$$\begin{aligned}(-aL^*_{\vec{v}}q|(\varphi - ch_1)^{s\!\!/}) &= (-aL^s L^*_{\vec{v}}q \mid \psi^s/s) \\ &= (q_t - iSq \mid \psi^s/s) + (amq \mid L_{\vec{v}}\check{v}_3 r^{-1}\psi^s/s),\end{aligned} \tag{7.10}$$

and, using (3.9), the last term is bounded by

$$\|q(|q| + |\check{v}/r|)r^{-2}\|_{L^1_x} \lesssim (\|q/r\|_{L^2_x} + \|z/r^2\|_{L^2_x})^2 \lesssim \|q\|^2_{L^\infty_2}. \tag{7.11}$$

For $m \geq 2$, $\psi \in L^2_\infty$, and so

$$\|(Sq \mid \psi^s/s)\|_{L^1_t} \lesssim \|S\|_{L^2_t L^2_x}\|q\|_{L^2_t L^\infty_2} \lesssim \delta\|q\|_{L^2_t L^\infty_2}, \tag{7.12}$$

either by (5.10) or (5.25). Thus we have obtained

$$\|\dot{\mu} - e^{i\widetilde{\alpha}}(q_t \mid \psi^s/s)\|_{L^1_t} \lesssim \delta\|q\|_{L^2_t L^\infty_2}. \tag{7.13}$$

Integrating by parts in t, the leading term is rewritten as

$$\begin{aligned}e^{i\widetilde{\alpha}}(q_t \mid \psi^s/s) = \partial_t(e^{i\widetilde{\alpha}}q \mid \psi^s/s) - i(s\dot{\alpha} + sS(t,0))(e^{i\widetilde{\alpha}}q \mid \psi^{s\!\!/}) \\ +\dot{s}(e^{i\widetilde{\alpha}}q \mid (r\partial_r + 1)\psi^{s\!\!/}).\end{aligned} \tag{7.14}$$

The last term can be bounded in L^1_t by using (6.8),

$$\|s\dot{\mu}\|_{L^2_t}\|(q \mid (r\partial_r + 1)\psi^{s\!\!/})\|_{L^2_t} \lesssim \|q\|_{L^2_t L^\infty_2}\|(q \mid (r\partial_r + 1)\psi^{s\!\!/})\|_{L^2_t}. \tag{7.15}$$

If $m = 2$ or $m > 3$, then $(r\partial_r + 1)\psi \in L^1$, and so the above is further bounded by $\|q\|^2_{L^2_t L^\infty_2}$. When $m = 3$, we need some extra effort to bound the last factor in L^2_t – this is done in the next section.

If $m > 2$, we have for the leading term

$$|(e^{i\widetilde{\alpha}}q \mid \psi^s/s)| \lesssim \|q\|_{L^2_x}\|\psi\|_{L^2_x} \lesssim \|q(0)\|_{L^2_x}, \tag{7.16}$$

while for $m = 2$ this term can be infinite from the beginning. We will show in Sect. 9 that the time difference $[(e^{i\widetilde{\alpha}}q \mid \psi^s/s)]_0^t$ can be controlled for finite t, but still may become unbounded as $t \to \infty$ for some initial data.

This also means that the second to last term of (7.14) is beyond our control when $m = 2$, and so in this case we force it to vanish by making the assumptions $a_2 = 0$ and $v_2 = 0$. For the other cases ($m > 2$), we should estimate $S(t, 0)$, for which we use in the Schrödinger case ($a = i$) that

$$S = -Q + \int_r^\infty \frac{2Q}{r}dr, \quad Q = \frac{1}{2}|q|^2 + \frac{m}{r}w_3 = O(|q|^2 + |z/r|^2 + qh_1^s/r), \tag{7.17}$$

since $|w_3| \lesssim |q||\nu|$ and $|\nu| \lesssim h_1^s + |z|$. Thus we get at each t, using (5.1),

$$\|S\|_{L^\infty_x} \lesssim \|q\|^2_{L^\infty_2} + s^{-1}\|q\|_{L^\infty}. \tag{7.18}$$

Then the second term in (7.14) is bounded in L^1_t,

$$\begin{aligned} &\|q\|^2_{L^2_t L^\infty_2}\|(q \mid \psi^s/s)\|_{L^\infty_t} + (\|s\dot{\alpha}\|_{L^2_t} + \|q\|_{L^2_t L^\infty_x})\|(q \mid \psi^{s\!\!/})\|_{L^2_t} \\ &\lesssim \|q\|_{L^2_t L^\infty_2}(\delta + \|(q \mid \psi^{s\!\!/})\|_{L^2_t}), \end{aligned} \tag{7.19}$$

where we used (6.8). If $m > 3$, then $\psi \in L^1$ and hence the last factor $\|(q \mid \psi^{s\!\!/})\|_{L^2_t}$ is bounded by $\|q\|_{L^2_t L^\infty_2} \lesssim \delta$. Its estimate for $m = 3$ is deferred to the next section.

In the dissipative case $a_1 > 0$, we estimate simply by (2.15) at each t,

$$\|S\|_{L^\infty_x} \lesssim (\|q/r\|_{L^2_x} + \|z/r^2\|_{L^2_x} + \|h_1^s/r^2\|_{L^2_x})\|L^*_v q\|_{L^2_x}, \tag{7.20}$$

and hence the second term in L^1_t is bounded by

$$\begin{aligned} &\|q\|_{L^2_t L^\infty_2}\|q\|_{L^2_t X} + (\|s\dot{\mu}\|_{L^2_t} + \|q\|_{L^2_t X})\|(q \mid \psi^{s\!\!/})\|_{L^2_t} \\ &\lesssim \|q\|_{L^2_t L^\infty_2}(\delta + \|(q \mid \psi^{s\!\!/})\|_{L^2_t}), \end{aligned} \tag{7.21}$$

where we used (6.8) and (5.9).

Thus we have obtained all the necessary estimates to prove Theorem 1.1 when $m > 3$. In summary, we have

Proposition 7.1. *Under the same assumptions as for Proposition 6.1, we have*

$$\dot{\mu} = \partial_t(e^{i\widetilde{\alpha}}q|\psi^s/s) - is\dot{\alpha}(e^{i\widetilde{\alpha}}q|\psi^{s\!\!/}) + \dot{s}(e^{i\widetilde{\alpha}}q|(r\partial_r + 1)\psi^{s\!\!/}) + error, \tag{7.22}$$

where $\|error\|_{L^1_t(0,T)} \lesssim \delta\|q\|_{L^2_t(0,T;L^\infty_2)}$. *Moreover, if $m = 2$ or $m > 3$, then the second to last term can be included in the error.*

The proof of Theorem 1.1 for $m = 3$ will be complete once we show

$$\|(q \mid \psi^{s\!\!/})\|_{L^2_t} + \|(q \mid (r\partial_r + 1)\psi^{s\!\!/})\|_{L^2_t} \lesssim \delta, \tag{7.23}$$

which will be done in Sect. 8. This estimate together with the above proposition implies the convergence of $\mu(t) = \mu(0) + O(\delta)$, closing all the estimates and the assumptions in the previous sections. For Theorem 1.2, it remains to derive the asymptotic formula (1.24) from the leading term $(q \mid \psi^s/s)$, and to show that all of the asymptotic behavior (1)–(6) can be realized by the choice of the initial data $\vec{u}(0, x)$ – this is done in Sect. 9.

8. Special Estimates for $m = 3$

In this section we finish the proof of Theorem 1.1 by showing (7.23). It suffices to estimate the leading term for $r \to \infty$:

$$\|(r^{-2}\chi(r) \mid q)\|_{L_t^2} \lesssim \delta, \tag{8.1}$$

with $\chi \in C^\infty$ satisfying $\chi(r) = 0$ for $r < 1$ and $\chi(r) = 1$ for $r > 2$, since the rest decays at slowest $O(r^{-8}) \in L_x^1$, for which we can simply use $q \in L_t^2 L_x^\infty$. Once the above is proved, we can conclude that

$$\|\mu\|_{L_t^\infty} \lesssim |\mu(0)| + \|q\|_{L_t^\infty L_x^2 \cap L_t^2 L_2^\infty}. \tag{8.2}$$

The boundedness of μ and the scattering of q imply that the "normal form" correction $(e^{i\tilde{\alpha}} q \mid \psi^s/s)$ converges to zero, and so $\mu(t)$ is convergent as $t \to \infty$.

To estimate (8.1), we use perturbation from the free evolution $e^{at\Delta_2^{(2)}}$:

$$\dot{q} - a\Delta_2^{(2)} q = N_0 + N_2, \tag{8.3}$$

where N_0 and N_2 are as in (5.28) and (5.18), satisfying

$$N_0 \in r^{-2}\langle r/s\rangle^{-4} L_t^2 L_x^\infty, \quad N_2 \in L_t^1 L_x^2. \tag{8.4}$$

For the contribution of N_2 as well as the initial data, we use the following estimate.

Lemma 8.1. *For any $l > 0$, any $a \in \mathbb{C}^\times$ with $\operatorname{Re} a \geq 0$, and any functions $g(r)$, $f(r)$, and $F(t, r)$, we have*

$$\begin{aligned} \|(g \mid e^{at\Delta_2^{(l)}} f)\|_{L_t^2(0,\infty)} &\lesssim \|r^2 g\|_{L_x^\infty} \|f\|_{L_x^2}, \\ \|(g \mid \int_{-\infty}^t e^{a(t-s)\Delta_2^{(l)}} F(s)ds)\|_{L_t^2(\mathbb{R})} &\lesssim \|r^2 g\|_{L_x^\infty} \|F\|_{L_t^1 L_x^2}. \end{aligned} \tag{8.5}$$

Proof. We start with the estimate for the free part. Let $\widehat{g} = \mathcal{F}_l g$ and $\widehat{f} = \mathcal{F}_l f$. The above L_t^2 norm equals by Plancherel in space,

$$\|(\widehat{g} \mid e^{-atr^2} \widehat{f})\|_{L_t^2(0,\infty)} \sim \|\int_0^\infty e^{-at\sigma} G(\sigma) d\sigma\|_{L_t^2(0,\infty)}, \tag{8.6}$$

where we put

$$G(\sigma) := \widehat{g}(\sigma^2)\overline{\widehat{f}(\sigma^2)}. \tag{8.7}$$

If $a_1 > 0$, then (8.6) is bounded by Minkowski,

$$\begin{aligned} &\leq \|\int_0^\infty e^{-a_1 t\sigma} |G(\sigma)| d\sigma\|_{L_t^2(0,\infty)} \leq \int_0^\infty e^{-a_1\sigma} \|t^{-1} G(\sigma/t)\|_{L_t^2(0,\infty)} d\sigma \\ &\leq \|G\|_{L_\sigma^2(0,\infty)} \int_0^\infty \sigma^{-1/2} e^{-a_1\sigma} d\sigma \lesssim \|G\|_{L_\sigma^2(0,\infty)}. \end{aligned} \tag{8.8}$$

If $a_1 = 0$, then $a_2 \neq 0$ and (8.6) is bounded by Plancherel in t,

$$\leq \|\int_0^\infty e^{-ia_2 t\sigma} G(\sigma) ds\|_{L_t^2(\mathbb{R})} \sim \|G\|_{L_\sigma^2(0,\infty)}. \tag{8.9}$$

Thus in both cases we obtain

$$\|(g \mid e^{at\Delta_2^{(l)}} f)\|_{L_t^2(0,\infty)} \lesssim \|G(\sigma)\|_{L_\sigma^2(\mathbb{R})} \lesssim \|\widehat{g}\|_{L^\infty}\|\widehat{f}\|_{L^2} \sim \|\widehat{g}\|_{L^\infty}\|f\|_{L_x^2}. \tag{8.10}$$

Then the first desired estimate follows from

$$|\widehat{g}(\rho)| \le \int_0^\infty |J_l(r\rho)||g(r)|r dr \le \|r^2 g\|_{L_x^\infty} \int_0^\infty |J_l(r)|\frac{dr}{r} \sim \|r^2 g\|_{L_x^\infty}, \tag{8.11}$$

since $|J_l(r)| \lesssim \min(r^l, r^{-1/2})$ for $r > 0$.

By duality, the estimate on the Duhamel term is equivalent to

$$\|\int_0^\infty \lambda(s+t)e^{\overline{a}s\Delta_2^{(l)}} g(x)ds\|_{L_t^\infty L_x^2} \lesssim \|r^2 g\|_{L_x^\infty}\|\lambda\|_{L_t^2}, \tag{8.12}$$

which is equivalent to

$$\|\int_0^\infty \lambda(t)e^{\overline{a}t\Delta_2^{(l)}} g(x)dt\|_{L_x^2} \lesssim \|r^2 g\|_{L_x^\infty}\|\lambda\|_{L_t^2}, \tag{8.13}$$

which is dual to the first estimate. □

For the potential part N_0, we transfer the equation to $\mathbb{R}^6$ by $u = r^{-2}q$ and consider

$$u_t - a\Delta_6^{(0)} u = r^{-2}N_0. \tag{8.14}$$

Then thanks to the decay of the potential, we have

$$r^{-2}N_0 \in L_t^2 L_x^{10/7}(\mathbb{R}^6) \subset L_t^2 \dot{H}_{3/2}^{-1/5}(\mathbb{R}^6), \tag{8.15}$$

as long as $s(t)$ is away from 0 and ∞. Then by the endpoint Strichartz or the energy estimate on $\mathbb{R}^6$, the corresponding Duhamel term is bounded in $L_t^2 \dot{H}_3^{-1/5}(\mathbb{R}^6)$, and since $|\nabla_x r^{-4}\chi(r)| \lesssim r^{-5}$, we have $r^{-4}\chi \in \dot{H}_{5/4}^1(\mathbb{R}^6) \subset \dot{H}_{3/2}^{1/5}(\mathbb{R}^6)$. Thus to summarize, we have

$$\|(r^{-2}\chi \mid q)\|_{L_t^2} \lesssim \|q(0)\|_{L_x^2} + \|q\|_{L_t^2 L_{2,x}^\infty}. \tag{8.16}$$

Completion of the proof of Theorem 1.1. Let initial data $\vec{u}(0)$ be specified as in Theorem 1.1. The existence of a unique local-in-time solution $\vec{u}(t)$ in the given spaces can be deduced by working in the (μ, q) variables (using the bijection of Lemma 4.1) and using estimates similar to those of Sects. 5 and 6. The details are carried out in the Schrödinger case $(a = i)$ in [12], and carry over to the general case in a straightforward way (in fact, there are well-established methods for energy-space local existence in the dissipative case, starting with the pioneering work [19] on the heat-flow). It follows from this local theory that the solution continues as long as $\mu(t)$ is bounded and q is bounded in $L_t^\infty L_x^2 \cap L_t^2 L_2^\infty$.

For $m > 3$, the estimates of the previous four sections give the boundedness of q and μ which ensure the solution is global, as well as the convergence of $\mu(t)$. The convergence to a harmonic map then follows from the estimates of Sect. 5. □

9. Special Estimates for $m = 2$, $a > 0$, $v_2 = 0$

Let $m = 2$ and (with no further loss of generality) $a = 1$. By the bijective correspondence $\vec{v} \leftrightarrow (\mu, q)$, it is clear that $v_2 = 0$ is equivalent to $\mu, q \in \mathbb{R}$. It remains to control the leading term for the parameter dynamics

$$(q \mid \psi^s/s). \tag{9.1}$$

In particular, we will show that this can diverge to $\pm\infty$, or oscillate between them for certain initial data.

First by the asymptotics for $r \to \infty$, we have $\psi + cr_{1<}^{-1} \in L_x^2$, where we denote

$$r_{a<}^{-1} = \begin{cases} r^{-1} & (r > a) \\ 0 & (r \le a) \end{cases} \quad r_{<b}^{-1} = \begin{cases} r^{-1} & (r < b) \\ 0 & (r \ge b) \end{cases} \quad r_{a<b}^{-1} = \begin{cases} r^{-1} & (a < r < b) \\ 0 & (\text{otherwise}) \end{cases}. \tag{9.2}$$

Hence we may replace $s^{-1}\psi^s$ by $-cr_{s<}^{-1}$ modulo $o(1)L_t^\infty$.

Next we want to replace q by the free solution $q^0 := e^{t\Delta_2^{(1)}}q(0)$. For that we use the following pointwise estimate to bound $q - q_0$:

Lemma 9.1. *Let* $\operatorname{Re} a > 0$ *and* $l \ge 1$. *Then for any function* $g(r)$ *satisfying* $|g(r)| \le \langle r\rangle^{-1}$, *we have*

$$|e^{at\Delta_2^{(l)}} g(r)| \lesssim \min(1, 1/r, r/t). \tag{9.3}$$

Proof. Let $a_1 = \operatorname{Re} a$. By using the explicit kernel we have

$$e^{2at\Delta_2^{(l)}} g(r) = \frac{C}{t}\int_0^\infty \int_{-\pi}^{\pi} e^{-a\frac{r^2-2rq\cos\theta+q^2}{t}+il\theta} g(q) q\, d\theta\, dq, \tag{9.4}$$

and the integral in θ can be rewritten by partial integration on $e^{i\theta}$ as

$$\int_{-\pi}^{\pi} \frac{rq}{ilt} \sin\theta e^{-a\frac{r^2-2rq\cos\theta+q^2}{t}+il\theta} g(q) q\, d\theta\, dq. \tag{9.5}$$

The double integral for $|q - r| > r/2$ is bounded by using the second form by

$$\int_0^\infty \frac{rq}{t^2} e^{-a_1\frac{r^2+q^2}{4t}} \min(q, 1)\, dq \lesssim re^{-a_1\frac{r^2}{4t}} \min(t^{-1/2}, t^{-1}) \lesssim \min(1, r/t, 1/r), \tag{9.6}$$

and that for $|q - r| < r/2$ is bounded by using the first form by

$$\begin{aligned} t^{-1}\int_0^\infty \int_{-\pi}^{\pi} e^{-a_1\frac{(r-q)^2}{t}} e^{-a_1\frac{r^2\theta^2}{8t}}\, d\theta \min(r, 1)\, dq &\lesssim t^{-1}t^{1/2}(t/r^2)^{1/2} \min(r, 1) \\ &\lesssim \min(1, 1/r), \end{aligned} \tag{9.7}$$

and by the second form by $\lesssim r^2/t \times 1/r = r/t$. $\square$

The nonlinear part of q contributes as

$$(r_{s<}^{-1} \mid q - q^0) = -\int_0^t (e^{\Delta_2^{(m-1)}(t-t')} r_{s(t)<}^{-1} \mid V(t')q(t'))dt', \tag{9.8}$$

where the potential term is given by

$$V = \frac{2m(1-v_3)}{r^2} + \frac{m}{r}w_3 = \frac{2m(1-h_3^s)}{r^2} + O(\check{v}_3/r^2) + O(q/r). \tag{9.9}$$

The contribution from the last two parts is estimated with the r^{-1} bound from the above lemma, thus bounded by

$$(\|\check{v}_3/r^2\|_{L^2_{t,x}} + \|q/r\|_{L^2_{t,x}})\|q/r\|_{L^2_{t,x}} \lesssim \|q(0)\|_{L^2_x}^2. \tag{9.10}$$

We need to be more careful to estimate the other term $q(1-h_3^s)/r^2$. First by Schwarz and the pointwise estimate, we have

$$\begin{aligned} &\left|\int_0^t (e^{a\Delta_2^{(m-1)}(t-t')} r_{s(t)<}^{-1} \mid r^{-2} q(1-h_3^s))dt'\right|^2 \\ &\leq \|q/r\|_{L^2_{t,x}}^2 \int_0^t \int_0^\infty \min(s(t)^{-1}, r^{-1}, r/(t-t'))^2 \langle r/s(t')\rangle^{-4m} \frac{dr}{r} dt', \end{aligned} \tag{9.11}$$

where we also used that $|1 - h_3(r)| \lesssim \langle r\rangle^{-2m}$. It suffices to bound the last double integral. Let $\tau = t - t'$. For $0 < \tau < s(t)^2$, the r integral is bounded by

$$\int_0^{\tau/s(t)} \frac{r^2}{\tau^2}\frac{dr}{r} + \int_{\tau/s(t)}^{s(t)} \frac{1}{s(t)^2}\frac{dr}{r} + \int_{s(t)}^\infty \frac{1}{r^2}\frac{dr}{r} \lesssim s(t)^{-2}(1 + \log(s(t)^2/\tau)), \tag{9.12}$$

hence its τ integral is bounded by

$$\int_0^{s(t)^2} s(t)^{-2}(1 + \log(s(t)^2/\tau))d\tau = 1 + \int_0^1 |\log\theta| d\theta < \infty. \tag{9.13}$$

For $s(t)^2 < \tau$, the r integral is bounded by

$$\int_0^{s(t')} \frac{r^2}{\tau^2}\frac{dr}{r} + \int_{s(t')}^\infty \frac{r^2 s(t')^{4m}}{\tau^2 r^{4m}}\frac{dr}{r} \lesssim \frac{s(t')^2}{\tau^2}, \tag{9.14}$$

and its τ integral is bounded by the square of

$$\begin{aligned} \|s(t')/\tau\|_{L^2_\tau(s(t)^2,t)} &\lesssim \|s(t)/\tau\|_{L^2_\tau(s(t)^2,t)} + \|(s(t) - s(t-\tau))/\tau\|_{L^2_\tau(s(t)^2,t)} \\ &\lesssim 1 + \int_0^1 \|\dot{s}(t-\theta\tau)\|_{L^2_\tau(s(t)^2,t)} d\theta \\ &\lesssim 1 + \int_0^1 \|\dot{s}\|_{L^2} \theta^{-1/2} d\theta \lesssim 1. \end{aligned} \tag{9.15}$$

Thus we obtain

$$|(r_{s(t)<}^{-1} \mid q - q^0)| \lesssim \|q/r\|_{L^2_{t,x}} (\|\check{v}_3/r^2\|_{L^2_{t,x}} + \|\dot{s}\|_{L^2_t} + 1) \lesssim \|q(0)\|_{L^2}, \tag{9.16}$$

namely we may replace q by the free solution q^0 in the leading asymptotic term.

Furthermore, we can freeze the scaling parameter because

$$|(r^{-1}_{s(t)<} - r^{-1}_{s(0)<} \mid q^0)| \lesssim \|r^{-1}_{s(t)<} - r^{-1}_{s(0)<}\|_{L^2}\|q^0(t)\|_{L^2} \lesssim |[\log s]_0^t|^{1/2} o(1) \lesssim o(1)(|[\log s]_0^t| + 1). \tag{9.17}$$

Thus we obtain

$$(1 + o(1))[2 \log s]_0^t = -c[(r^{-1}_{s(0)<} \mid q^0)]_0^t + O(1), \quad (t \to \infty), \tag{9.18}$$

where $O(1)$ is convergent.

The leading term is further rewritten in the Fourier space by using that

$$\mathcal{F}_1[r^{-1}_{s(0)<}](\rho) = \rho^{-1} \int_{s(0)\rho}^{\infty} J_1(r)dr = \rho^{-1} J_0(s(0)\rho) = \rho^{-1}_{<1/s(0)} + s(0)R(s(0)\rho), \quad \exists R \in L^2_x. \tag{9.19}$$

Let $\widehat{q_0} := \mathcal{F}_1 q(0)$. By Plancherel we have $\|q(0)\|_{L^2_x} = \|\widehat{q_0}\|_{L^2_x}$ and

$$\begin{aligned} [-(r^{-1}_{s(0)<} \mid q^0)]_0^t &= ((1 - e^{-tr^2})r^{-1}_{<1/s(0)} \mid \widehat{q_0}) + O(1) \\ &= (r^{-1}_{1/\sqrt{t}<1/s(0)} \mid \widehat{q_0}) + O(1) \\ &= (\mathcal{F}_1 r^{-1}_{1/\sqrt{t}<1/s(0)} \mid q(0)) + O(1) \\ &= 2\pi \int_{s(0)}^{\sqrt{t}} q(0, r)dr + O(1). \end{aligned} \tag{9.20}$$

Thus we obtain (using that $c = \|h_1\|^{-2}_{L^2_x} = \pi^{-2}$),

$$(1 + o(1))[\log s]_0^t = \frac{1}{\pi} \int_{s(0)}^{\sqrt{t}} q(0, r)dr + O(1), \tag{9.21}$$

and the error term $O(1)$ converges to a finite value as $t \to \infty$.

Completion of the proof of Theorem 1.2. As in the proof of Theorem 1.1, we now have all the estimates to conclude the solution is global (in particular, $\mu(t)$ remains finite by the above formula and estimates), and the convergence to the harmonic map family follows from the estimates of Sect. 5. It remains to consider the asymptotics of $s(t)$.

Since $q(0) \in L^2_x$ does not require $\int_1^\infty |q(0, r)|dr < \infty$, it is easy to make up $q(0) \in L^2$, for any given $s(0) \in (0, \infty)$, such that the first term on the right of (9.21) attains arbitrarily given lim sup $\geq$ lim inf $\in [-\infty, \infty]$ as $t \to \infty$. In particular, all of the asymptotic behaviors (1)-(6) in Theorem 1.2 can be realized by appropriate choices of $(q(0), s(0))$, for which Lemma 4.1 ensures existence of corresponding initial data $\vec{u}(0) \in \Sigma_2$.

Using that $v_2 = 0$, we can further rewrite the leading term in terms of $\vec{v}$. Since $\mathbf{e} = (v_3, i, -v_1)$, we have

$$q = \vec{w} \cdot \mathbf{e} = v_1 v_{3r} - v_3 v_{1r} + \frac{2v_1}{r} = -\beta_r + \frac{2v_1}{r}, \tag{9.22}$$

where β is defined by $\vec{v} = (\cos\beta, 0, \sin\beta)$. Hence we have

$$(1 + o(1))[\log s]_0^t = \frac{2}{\pi} \int_{s(0)}^{\sqrt{t}} \frac{v_1(0, r)}{r} dr + O(1), \tag{9.23}$$

where $O(1)$ converges as $t \to \infty$. □

10. Proofs of the Key Linear Estimates

10.1. Uniform bound on the right inverse R_φ.

Proof of Lemma 3.1. Let $s = 1$ and omit it. It suffices to prove

$$\begin{aligned} \|R_\varphi g\|_{r^\theta L^\infty} &\lesssim \|\varphi\|_{r^{-\theta} L^1} \|g\|_{r^{\theta+1} L^1_\infty}, \\ \|R^*_\varphi f\|_{r^{-\theta-1} L^\infty} &\lesssim \|\varphi\|_{r^{-\theta} L^1_\infty} \|f\|_{r^{-\theta} L^1_\infty}. \end{aligned} \tag{10.1}$$

From this we get by duality,

$$\begin{aligned} \|R_\varphi g\|_{r^\theta L^\infty_1} &\lesssim \|\varphi\|_{r^{-\theta} L^1_\infty} \|g\|_{r^{\theta+1} L^1}, \\ \|R^*_\varphi f\|_{r^{-\theta-1} L^\infty_1} &\lesssim \|\varphi\|_{r^{-\theta} L^1} \|f\|_{r^{-\theta} L^1}, \end{aligned} \tag{10.2}$$

and the bilinear complex interpolation covers the intermediate cases.

It remains to prove (10.1). We rewrite the kernel of R_φ,

$$R_\varphi g = \iint \frac{h_1(r)}{h_1(r'')} \chi(r, r', r'') \overline{\varphi}(r') h_1(r') r' g(r'') dr'' dr', \tag{10.3}$$

where $\chi(r)$ is defined by

$$\chi(r, r', r'') = \begin{cases} 1 & (r' < r'' < r), \\ -1 & (r < r'' < r'), \\ 0 & (\text{otherwise}). \end{cases} \tag{10.4}$$

We decompose the double integral dyadically such that $r \sim 2^j$, $r'' \sim 2^k$ and $r' \sim 2^l$, and let

$$\begin{aligned} &A_j = 2^{-\theta j} \|R_\varphi g\|_{L^\infty(r\sim 2^j)}, \quad B_j = 2^{(\theta+1)j} \|R^*_\varphi f\|_{L^\infty(r\sim 2^j)}, \\ &\varphi_l = 2^{\theta l} \|\varphi\|_{L^1(r\sim 2^l)}, \quad g_k = 2^{(-\theta-1)k} \|g\|_{L^1(r\sim 2^k)}, \quad f_k = 2^{\theta k} \|f\|_{L^1(r\sim 2^k)}. \end{aligned} \tag{10.5}$$

For R_φ, we have

$$A_j \lesssim \sum_{\substack{j-1\le k\le l+1 \\ l-1\le k\le j+1}} 2^{-m|j|-\theta j+m|k|+\theta k-m|l|-\theta l} \varphi_l g_k. \tag{10.6}$$

The sums over k are bounded for $j-1 \le k \le l+1$ and for $l-1 \le k \le j+1$ respectively by

$$2^{-m|j|-\theta j-m|l|-\theta l} \varphi_l \sup_k g_k \times \begin{cases} \max(2^{(-m+\theta)j}, 1) \max(2^{(m+\theta)l}, 1), \\ \max(2^{(-m+\theta)l}, 1) \max(2^{(m+\theta)j}, 1), \end{cases} \tag{10.7}$$

and since the exponential factors are bounded, after summation over l we get

$$\|R_\varphi g\|_{r^\theta L^\infty} \lesssim \sum_l \sup_k \varphi_l g_k, \tag{10.8}$$

as desired. For R_φ^* in (10.1), we have

$$B_j \lesssim \sum_{\substack{k-1\le j\le l+1\\ l-1\le j\le k+1}} 2^{m|j|+\theta j-m|k|-\theta k-m|l|-\theta l}\varphi_l f_k. \tag{10.9}$$

Then the sums over k and l are bounded in both cases by

$$2^{m|j|+\theta j}\sup_{k,l}\varphi_l f_k \min(2^{mj-\theta j},1)\min(2^{-mj-\theta j},1), \tag{10.10}$$

and hence $\|R_\varphi^* f\|_{r^{-\theta-1}L^\infty} \lesssim \sup_{l,k}\varphi_l g_k$, as desired.

Next we show the optimality. Let $b\in\mathbb{Z}$, and choose any g which is piecewise constant on each dyadic interval $(2^j, 2^{j+1})$, $\operatorname{supp} g\subset[2^b,\infty)$, and $g\ge 0$. Then for $0<r\le 2^b$ we have

$$\begin{aligned} R_\varphi g(r) &= h(r)\int_{2^b}^\infty\int_{2^b}^a h(s)^{-1}\varphi(a)h(a)ag(s)dsda\\ &\gtrsim h(r)\sum_{j\ge b}\sum_{k=b}^{j-1}2^{m|k|+\theta k}g_k 2^{-m|j|-\theta j}\varphi_j \gtrsim h(r)\sum_{j\ge b} g_{j-1}\varphi_j, \end{aligned} \tag{10.11}$$

where we denote $g_k = \|g\|_{r^{\theta-1}L^\infty(r\sim 2^k)}$ and $\varphi_j = \|\varphi\|_{r^{-\theta}L^1(r\sim 2^j)}$. Choosing a test function $\psi\in C_0^\infty(0,\infty)$ satisfying $\psi\ge 0$, $\operatorname{supp}\psi\subset(0,2^b)$ and $(h\mid\psi)=1$, we see that $\varphi_j\in\ell_j^{p'}(j>b)$ is necessary since we can choose arbitrary non-negative $g_k\in\ell_k^p(k>b)$. Similarly by choosing $\operatorname{supp} g\subset(0,2^b]$ and $\operatorname{supp}\psi\subset(2^b,\infty)$, we see that $\varphi_j\in\ell_j^{p'}(j<b)$ is also necessary. □

10.2. Double endpoint Strichartz estimate. Lemma 5.1 holds for more general radial potentials. We call

$$\left\| r^{-1}\int_0^t e^{i(t-s)H}f(s)ds\right\|_{L^2_{t,x}} \lesssim \|rf\|_{L^2_{t,x}} \tag{10.12}$$

the Kato estimate for the operator H, and

$$\|u\|_{L_t^2(L_2^\infty)} \lesssim \|u(0)\|_{L_x^2} + \|iu_t + Hu\|_{L_t^2(L_2^1)} \tag{10.13}$$

the double endpoint Strichartz estimate for H. Lemma 5.1 is a consequence of the following.

Theorem 10.1. *For any $m>0$, the double endpoint Strichartz (10.13) holds for radially symmetric $u(t,x)=u(t,|x|)$ and $H=\Delta_2^{(m)}=\partial_r^2+r^{-1}\partial_r-m^2r^{-2}$.*

Lemma 10.2. *Suppose H_0 and $H = H_0 + V$ are both self-adjoint on $L^2(\mathbb{R}^2)$ and $|x|^2 V(x) \in L^\infty(\mathbb{R}^2)$. Assume that the Kato estimate (10.12) holds for H, and that the double endpoint Strichartz estimate (10.13) holds for H_0. Then we have the double endpoint Strichartz also for H. The same is true when we restrict all functions to radially symmetric ones, if V is also symmetric.*

Corollary 10.3. *Let $V = V(|x|) \in C^1(\mathbb{R}^2 \setminus \{0\})$ be a radially-symmetric function with $|x|^2 V \in L^\infty(\mathbb{R}^2)$, and suppose $H = -\Delta + V$ is self-adjoint on $L^2(\mathbb{R}^2)$. Let $f(t,x) = f(t,|x|)$ be radial. Then*

(1) *the Kato estimate (10.12) holds for H if and only if the double-endpoint estimate (10.13) holds for H,*
(2) *both estimates hold provided*

$$\inf_{r>0} r^2 V(r) > 0, \quad \inf_{r>0} -r^2 (rV(r))_r > 0. \tag{10.14}$$

Since our linearized operator H^s satisfies (10.14), the above implies Lemma 5.1.

Proof of Corollary 10.3. The first statement follows directly from Theorem 10.1 and Lemma 10.2. For the second statement: the methods of [4], adapted to the 2-dimensional radial setting (detailed in [12]), imply that conditions (10.14) yield the resolvent estimate (10.15), hence the Kato estimate, and the double endpoint estimate. □

Remark 1. While the double-endpoint estimate (10.13) always implies the Kato estimate (10.12), the reverse implication does not hold in general. For example, consider $H = -\Delta$ acting on $2D$ functions with zero angular average. The Kato estimate in this case can be verified, for example, by using the methods of [4] to establish the resolvent estimate

$$\sup_{\lambda \neq 0} \|(H - \lambda)^{-1} \varphi\|_{L^{2,-1}_x} \lesssim \|\varphi\|_{L^{2,1}_x}, \tag{10.15}$$

from which the Kato estimate follows by Plancherel in t (see [12] for details). On the other hand, if the double-endpoint estimate were to hold for zero-angular-average functions, so would the endpoint homogeneous estimate. Since the latter is known to hold for radial functions (see Tao [20]), it would therefore hold for all $2D$ functions, which is false (see Montgomery-Smith [16], also see [20]). Alternatively, a constructive counterexample is given by placing delta functions of the same mass but opposite sign at $(1, 0)$ and $(0, 1)$ in the plane.

Proof of Theorem 10.1. Following [15], we use the identity

$$\iint_{s<t} F(s,t)\,ds\,dt = C \int_0^\infty \frac{dr}{r} \int_{\mathbb{R}} \frac{da}{r} \int_{a-3r}^{a-r} ds \int_{a+r}^{a+3r} dt\, F(s,t),$$

for the decomposition, where $C > 0$ is some explicit positive constant. Define the bilinear operators I_j for $j \in \mathbb{Z}$ by

$$I_j(f,g) := \int_{2^j}^{2^{j+1}} \frac{dr}{r} \int_{\mathbb{R}} \frac{da}{r} \int_{a-3r}^{a-r} ds \int_{a+r}^{a+3r} dt \big\langle e^{i(t-s)\Delta_2^{(m)}} f(s) \big| g(t) \big\rangle_x,$$

where f and g are radial (i.e. $f(s) = f(|x|, s)$, etc.).

The desired estimate follows from

$$\sum_{j\in\mathbb{Z}} |I_j(f,g)| \lesssim \|f\|_{L_t^2 L_x^1} \|g\|_{L_t^2 L_x^1}.$$

Using the $1/t$ decay for $\|e^{it\Delta_2^{(m)}}\|_{L^1\to L^\infty}$, we can easily bound the supremum of the summand. To get summability, we need decay both faster and slower than $1/t$. In fact we have, for $\varphi = \varphi(|x|)$ radial,

$$\|e^{it\Delta_2^{(m)}}\varphi\|_{L_x^{\infty,\mu}} \lesssim |t|^{-1+\mu} \|\varphi\|_{L_x^{1,-\mu}}, \quad -m \le \mu \le 1/2. \tag{10.16}$$

This follows easily from the explicit fundamental solution

$$(e^{it\Delta_2^{(m)}}\varphi)(r) = \frac{c_m}{t}\int_0^\infty e^{i(r^2+\rho^2)/4t} J_m\left(\frac{r\rho}{2t}\right)\phi(\rho)\rho d\rho$$

(c_m a constant) in terms of the Bessel function J_m of the first kind, for which

$$\sup_{s>0} s^\mu J_m(s) < \infty, \quad -m \le \mu \le 1/2.$$

Next, when the decay is slower than $1/t$, namely if we choose $\mu > 0$ in (10.16), then we get a non-endpoint Strichartz estimate using the Hardy-Littlewood-Sobolev inequality in time:

$$\|\int_{\mathbb{R}} e^{-is\Delta_2^{(m)}} f(s)ds\|_{L_x^2} \lesssim \|f\|_{L_t^{p'} L_x^{1,-\alpha}}, \tag{10.17}$$

for $0 < \alpha \le 1/2$ and $1/p = 1/2 - \alpha/2$.

Now the rest of the proof follows along the lines of Keel-Tao [14]. We will prove that

$$|I_j(f,g)| \lesssim 2^{j(\alpha+\beta)/2} \|f\|_{L_t^2 L_x^{1,-\alpha}} \|g\|_{L_t^2 L_x^{1,-\beta}}, \tag{10.18}$$

for

$$-m \le \alpha = \beta < 0 \tag{10.19}$$

and for

$$0 < \alpha, \quad \beta \le 1/2. \tag{10.20}$$

For the first exponents (10.19), we use the decay estimate (10.16) and the $L^{\infty,\alpha} - L^{1,-\alpha}$ duality at each (s,t). Then we get

$$\begin{aligned}
|I_j(f,g)| &\lesssim \int_{2^j}^{2^{j+1}} \frac{dr}{r} \int_{\mathbb{R}} \frac{da}{r} \int_{a-3r}^{a-r} ds \int_{a+r}^{a+3r} dt \frac{\|f(s)\|_{L_x^{1,-\alpha}} \|g(t)\|_{L_x^{1,-\alpha}}}{|t-s|^{1-\alpha}} \\
&\lesssim \int_{2^j}^{2^{j+1}} \frac{dr}{r} \int_{\mathbb{R}} \frac{da}{r} 2^{j\alpha} \|f\|_{L_t^2(a-3r,a-r;L_x^{1,-\alpha})} \|g\|_{L_t^2(a+r,a+3r;L_x^{1,-\alpha})} \\
&\lesssim \int_{2^j}^{2^{j+1}} \frac{dr}{r} 2^{j\alpha} \frac{1}{r} \|f\|_{L_{a,t}^2(-3r<t-a<-r;L_x^{1,-\alpha})} \|g\|_{L_{a,t}^2(r<t-a<3r;L_x^{1,-\alpha})} \\
&\lesssim 2^{j\alpha} \|f\|_{L_t^2 L_x^{1,-\alpha}} \|g\|_{L_t^2 L_x^{1,-\alpha}},
\end{aligned}$$

where we used Hölder for s, t, a.

For the second exponents (10.20), we use the non-endpoint Strichartz (10.17) for both integrals in s and t, after applying the Schwartz inequality in x. Then we get

$$
\begin{aligned}
|I_j(f,g)| &\lesssim \int_{2^j}^{2^{j+1}} \frac{dr}{r} \int_{\mathbb{R}} \frac{da}{r} \|f\|_{L_t^{p'}(a-3r,a-r;L_x^{1,-\beta})} \|g\|_{L_t^{q'}(a+r,a+3r;L_x^{1,-\alpha})} \\
&\lesssim \int_{2^j}^{2^{j+1}} \frac{dr}{r} \int_{\mathbb{R}} \frac{da}{r} 2^{j(\alpha+\beta)/2} \|f\|_{L_t^2(a-3r,a-r;L_x^{1,-\beta})} \|g\|_{L_t^2(a+r,a+3r;L_x^{1,-\alpha})},
\end{aligned}
$$

and the rest is the same as above, where $1/p' = 1/2 + \alpha/2$ and $1/q' = 1/2 + \beta/2$. Thus we get (10.18) both for (10.19) and (10.20). By bilinear complex interpolation (cf. [3]), we can extend the region (α, β) to the convex hull:

$$
\alpha > \frac{m}{m+\frac{1}{2}}\left(\beta - \frac{1}{2}\right), \quad \beta > \frac{m}{m+\frac{1}{2}}\left(\alpha - \frac{1}{2}\right), \quad \alpha, \beta < \frac{1}{2}. \tag{10.21}
$$

The only property we need is that this set includes a neighborhood of $(0,0)$, where we are looking for the summability.

Now we use bilinear interpolation (see [3, Exercise 3.13.5(b)] and [17])

$$
\begin{aligned}
&T : X_i \times X_j \to Y_{i+j} \quad (i, j, i+j \in \{0,1\}) \\
&\quad \implies T : X_{\theta_1,r_1} \times X_{\theta_2,r_2} \to Y_{\theta_1+\theta_2,r_0} \quad 1/r_0 = 1/r_1 + 1/r_2,
\end{aligned}
$$

where $X_{\theta,r} := (X_0, X_1)_{\theta,r}$ denotes the real interpolation space.

The above bound (10.18) can be written as

$$
\|I(f,g)\|_{\ell_\infty^{-(\alpha+\beta)/2}} \lesssim \|f\|_{L_t^2 L_x^{1,-\alpha}} \|g\|_{L_t^2 L_x^{1,-\beta}},
$$

where ℓ_p^α denotes the weighted space over $\mathbb{Z}$:

$$
\|a\|_{\ell_p^\alpha} := \|2^{j\alpha} a_j\|_{\ell_j^p(\mathbb{Z})}.
$$

Hence the bilinear interpolation implies that

$$
\|I(f,g)\|_{\ell_1^{-(\alpha+\beta)/2}} \lesssim \|f\|_{L_t^2 L_2^{1,-\alpha}} \|g\|_{L_t^2 L_2^{1,-\beta}}, \tag{10.22}
$$

for all (α, β) in (10.21), where

$$
\|\varphi\|_{L_q^{p,s}}^q := \sum_{k\in\mathbb{Z}} \|2^{ks}\varphi\|_{L^p(|x|\sim 2^k)}^q,
$$

and we used the interpolation property of weighted spaces (cf. [3]):

$$
\begin{aligned}
(\ell_\infty^\alpha, \ell_\infty^\beta)_{\theta,q} &= \ell_q^{(1-\theta)\alpha+\theta\beta}, \quad \alpha \neq \beta, \\
(L^{p,\alpha}, L^{p,\beta})_{\theta,q} &= L_q^{p,(1-\theta)\alpha+\theta\beta}, \quad \alpha \neq \beta.
\end{aligned}
$$

By choosing $\alpha = \beta = 0$ in (10.22), we get the desired result. □

Proof of Lemma 10.2. By time translation, we can replace the interval of integration in (10.12) and (10.13) by $(-\infty, t)$. Then by taking the dual, we can also replace it by (t, ∞). Adding those two, we can replace it by $\mathbb{R}$. Then the standard TT^* argument implies that

$$\|e^{iHt}\varphi\|_{L_t^2(L^{2,-1})} \lesssim \|\varphi\|_{L^2}, \quad \|e^{iH_0t}\varphi\|_{L_t^2(L_2^\infty)} \lesssim \|\varphi\|_{L^2}.$$

Now let

$$u = \int_{-\infty}^t e^{i(t-s)H} f(s)ds.$$

Then the Duhamel formula for the equation

$$iu_t + H_0 u = f - Vu$$

implies that

$$u = \int_{-\infty}^t e^{i(t-s)H_0}(f - Vu)(s)ds.$$

Applying (10.12) for H and (10.13) for H_0, and using $L^{2,1} \subset L_2^1$, we get

$$\begin{aligned} \|u\|_{L_t^2(L_2^\infty)} &\lesssim \|f - Vu\|_{L_t^2(L_2^1)} \\ &\lesssim \|f\|_{L_t^2(L_2^1)} + \|r^2 V\|_{L_x^\infty} \|u\|_{L_t^2(L^{2,-1})} \lesssim \|f\|_{L_t^2(L^{2,1})}. \end{aligned} \tag{10.23}$$

Then by duality we also get

$$\|u\|_{L_t^2(L^{2,-1})} \lesssim \|f\|_{L_t^2(L_2^1)}.$$

Feeding this back into (10.23), we get

$$\|u\|_{L_t^2(L_2^\infty)} \lesssim \|f\|_{L_t^2(L_2^1)}.$$

The estimate on $e^{iHt}\varphi$ is simpler, or can be derived from the above by the TT^* argument. □

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Appendix A. Landau-Lifshitz Maps from $\mathbb{S}^2$

The same stability problem on $\mathbb{S}^2$, instead of $\mathbb{R}^2$, is much easier in the dissipative case, because the eigenfunctions get additional decay from the curved metric on $\mathbb{S}^2$. Indeed we have convergence for all $m \geq 1$:

Theorem A.1. *Let $m \geq 2$, $a \in \mathbb{C}$ and* $\operatorname{Re} a > 0$*. Then there exists $\delta > 0$ such that for any $\vec{u}(0,x) \in \Sigma_m$ with $\mathcal{E}(\vec{u}(0)) \leq 4m\pi + \delta^2$, we have a unique global solution $\vec{u} \in C([0,\infty);\Sigma_m)$ satisfying $\nabla\vec{u} \in L^2_{t,loc}([0,\infty);L^\infty_x)$. Moreover, for some $\mu_\infty \in \mathbb{C}$ we have*

$$\|\vec{u}(t) - e^{m\theta R}\vec{h}[\mu_\infty]\|_{L^\infty_x} + \mathcal{E}(\vec{u}(t) - e^{m\theta R}\vec{h}[\mu_\infty]) \to 0 \quad (t \to \infty). \tag{A.1}$$

Our proof does not give a uniform bound on δ if $m = 1$, but we have

Theorem A.2. *Let $m = 1$, $a \in \mathbb{C}$,* $\operatorname{Re} a \geq 0$ *and $\mu_0 \in \mathbb{C}$. Then there exists $\delta > 0$ such that for any $\vec{u}(0,x) \in \Sigma_1$ with $\mathcal{E}(\vec{u}(0) - \vec{h}[\mu_0]) \leq \delta^2$, we have a unique global solution $\vec{u} \in C([0,\infty);\Sigma_1)$ satisfying $\nabla\vec{u} \in L^2_{t,loc}([0,\infty);L^\infty_x)$. Moreover, for some $\mu_\infty \in \mathbb{C}$ we have*

$$\|\vec{u}(t) - e^{m\theta R}\vec{h}[\mu_\infty]\|_{L^\infty_x} + \mathcal{E}(\vec{u}(t) - e^{m\theta R}\vec{h}[\mu_\infty]) \to 0 \quad (t \to \infty). \tag{A.2}$$

The proof is essentially a small subset of that in the $\mathbb{R}^2$ case, so we just indicate necessary modifications.

Outline of Proof. By the stereographic projection, we can translate the problem to $\mathbb{R}^2$ with the metric $g(x)dx^2$, where $g(x) = (1 + r^2/4)^{-2}$. The harmonic maps are the same, while the evolution equation is changed to

$$q_t = iSq - aL_{\vec{v}}g^{-1}L^*_{\vec{v}}q, \quad S = \int_\infty^r g^{-1}(q + \frac{m}{r}\nu) \circ iaL^*_{\vec{v}}q dr. \tag{A.3}$$

In this setting we can use the "standard" orthogonality to decide μ:

$$0 = (z \mid gh_1^s), \tag{A.4}$$

since $gh_1 \in \langle r\rangle^{-1}L^1$. The energy identity

$$\partial_t \|q\|^2_{L^2_x} = -2a_1(g^{-1}L^*_{\vec{v}}q \mid L^*_{\vec{v}}q) \tag{A.5}$$

implies the a priori bound on q:

$$\|q\|_{L^\infty_t L^2_x} + \|g^{-1/2}L^*_{\vec{v}}q\|_{L^2_t L^2_x} \lesssim \|q(0)\|_{L^2_x} \sim \delta. \tag{A.6}$$

Since $g^{-1/2} \gtrsim r$, we get (by using $q = R^{s*}_\varphi L^{s*}q$ as on $\mathbb{R}^2$),

$$\|q\|_{L^2_x} \lesssim \|rL^*_{\vec{v}}q\|_{L^2_x} \lesssim \|g^{-1/2}L^*_{\vec{v}}q\|_{L^2_x} \in L^2_t. \tag{A.7}$$

Then by the orthogonality (A.4) we have $z = R^s_{\phi_s}L^s z$ with

$$\phi_s := s^2 g(rs)h_1/(gh_1^s \mid h_1^s), \tag{A.8}$$

and so

$$\|z/r\|_{L^2_x} \lesssim \|L^s z\|_{L^2_x}\|\phi_s\|_{L^1_2}. \tag{A.9}$$

Since

$$\int_0^\infty g(rs)\min(r,1/r)^{-m} r dr \sim \begin{cases} \min(1,s^{-2}) & (m > 2) \\ \min(s^{-1},s^{-2}) & (m = 1) \end{cases}, \tag{A.10}$$

we have

$$\|\phi_s\|_{L^1_2} \sim \begin{cases} 1 & (m \geq 2) \\ \max(s^{-1}, 1) & (m = 1) \end{cases}. \tag{A.11}$$

Anyway, if δ is small enough (depending on s), we get by the same argument as on $\mathbb{R}^2$,

$$\|z\|_X \lesssim \|q\|_{L^2_x} \in L^2_t \cap L^\infty_t. \tag{A.12}$$

Differentiating the orthogonality, we get

$$\begin{aligned} \dot\mu(h_1^s \mid gh_1^s) &= -(\mathcal{M}aL^*_{\check{v}}q \mid h_1^s) - (gz \mid (\frac{\dot\mu_1}{m} r\partial_r + i\dot\mu_2 h_3^s)h_1^s) \\ &= -((\mathcal{M}_r + \frac{m\check{v}_3}{r}\mathcal{M})aq \mid h_1^s) \\ &\quad -(gz \mid \{\frac{\dot\mu_1}{m}(r\partial_r + m) + i\dot\mu_2(h_3^s - 1)\}h_1^s), \end{aligned} \tag{A.13}$$

where on the second equality we used that $L^s h_1^s = 0$ and $(gz \mid h_1^s) = 0$. Using that

$$|(r\partial_r + m)h_1| + |(h_3 - 1)h_1| \lesssim \min(r^{m-1}, r^{-3m-1}) \lesssim \langle r \rangle^{-4}, \tag{A.14}$$

we can bound the last term in (A.13) by

$$|\dot\mu| \|z\|_{L^\infty_x} \min(s^2, 1), \tag{A.15}$$

which is much smaller than the term on the left. The second to last term in (A.13) is bounded at each t by

$$\|q\|^2_{L^2_x} \|h_1^s\|_{L^\infty_x} \lesssim \|q\|^2_{L^2_x}. \tag{A.16}$$

If $m > 1$, we can improve this for $s < 1$ as follows. By the same argument as on $\mathbb{R}^2$, we have

$$\|q\|_{L^\infty_{2,x}} \lesssim \|L^*_{\check{v}}q\|_{L^2_x} \lesssim \|g^{-1/2}L^*_{\check{v}}q\|_{L^2_x}, \tag{A.17}$$

where we need $m > 1$ for the boundedness of $R^s_{\phi_s} : r^2 L^1_2 \to rL^\infty_2$ and $R^{s*}_\varphi : rL^1_2 \to L^\infty_2$. Then we can replace the above estimate in the region $r < 1$ by

$$\|q\|_{L^\infty_2 \cap L^2_x} \|h_1^s\|_{L^1_\infty + L^\infty_x} \lesssim \|q\|^2_{L^2_x} \min(s^2, 1). \tag{A.18}$$

Thus we obtain

$$\|\dot\mu\|_{L^1_t} \lesssim \begin{cases} \delta^2 & (m \geq 2) \\ C(s)\delta^2 & (m = 1) \end{cases}, \tag{A.19}$$

and hence if $\delta > 0$ is small enough, we get the desired convergence as on $\mathbb{R}^2$. □

Remark 2. It is a natural question whether one can prove a weaker asymptotic stability as in Theorem A.2 also on $\mathbb{R}^2$. It is impossible in the energy space, at least in the heat flow case ($a > 0$), because of the presence of blow-up solutions arbitrarily close to the ground state, together with the scaling invariance of the energy space. It is however quite likely that the stability holds for sufficiently localized initial perturbation. This requires weighted estimates on the linearized evolution, which will be pursued in a forthcoming paper.

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