STABILITY IN H^1 OF THE SUM OF K SOLITARY WAVES FOR SOME NONLINEAR SCHRÖDINGER EQUATIONS

YVAN MARTEL, FRANK MERLE, and TAI-PENG TSAI

Abstract

In this article we consider nonlinear Schrödinger (NLS) equations in $\mathbb{R}^d$ for $d = 1, 2$, and 3. We consider nonlinearities satisfying a flatness condition at zero and such that solitary waves are stable. Let $R_k(t, x)$ be K solitary wave solutions of the equation with different speeds $v_1, v_2, \dots, v_K$. Provided that the relative speeds of the solitary waves $v_k - v_{k-1}$ are large enough and that no interaction of two solitary waves takes place for positive time, we prove that the sum of the $R_k(t)$ is stable for $t \geqslant 0$ in some suitable sense in H^1. To prove this result, we use an energy method and a new monotonicity property on quantities related to momentum for solutions of the nonlinear Schrödinger equation. This property is similar to the L^2 monotonicity property that has been proved by Martel and Merle for the generalized Korteweg–de Vries (gKdV) equations (see [12, Lem. 16, proof of Prop. 6]) and that was used to prove the stability of the sum of K solitons of the gKdV equations by the authors of the present article (see [15, Th. 1(i)]).

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1. Introduction

We first consider nonlinear Schrödinger (NLS) equations in $\mathbb{R}$ of the form

$$\begin{cases} \mathrm{i}\, \partial_t u = -\partial_x^2 u - f(|u|^2)u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $u_0 \in H^1(\mathbb{R})$ and $f : \mathbb{R} \to \mathbb{R}$ is a function of class C^1 such that $f(0) = 0$, satisfying assumption (A1) in Theorem 1. We denote

$$F(s) = \int_0^s f(s')\, ds'.$$

The NLS equation set in $\mathbb{R}^2$ or $\mathbb{R}^3$ is also considered later in the introduction.

Recall that Ginibre and Velo [7] proved that equation (1.1) is locally well posed in $H^1(\mathbb{R})$; for any $u_0 \in H^1$, there exist $T > 0$ and a unique maximal solution $u \in C([0,T), H^1)$ of (1.1) on $[0,T)$. Moreover, either $T = +\infty$ or $T < +\infty$, and then $\lim_{t\to T} \|\partial_x u(t)\|_{L^2} = +\infty$. Finally, H^1-solutions of (1.1) satisfy the following three conservation laws; for all $t \in [0,T)$,

- L^2-norm:

$$\int |u(t)|^2 = \int |u_0|^2, \tag{1.2}$$

- Energy:

$$E\big(u(t)\big) = \int |\partial_x u(t)|^2 - \int F\big(|u(t)|^2\big) = E(u_0), \tag{1.3}$$

- Momentum:

$$\operatorname{Im} \int \partial_x u(t)\, \overline{u}(t) = \operatorname{Im} \int \partial_x u_0\, \overline{u}_0. \tag{1.4}$$

It is also well known that equation (1.1) admits the following symmetries.

- Space-time translation invariance: If $u(t,x)$ satisfies (1.1), then for any $t_0, x_0 \in \mathbb{R}$, $w(t,x) = u(t - t_0, x - x_0)$ also satisfies (1.1).
- Phase invariance: If $u(t,x)$ satisfies (1.1), then for any $\gamma_0 \in \mathbb{R}$, $w(t,x) = u(t,x)e^{\mathrm{i}\gamma_0}$ also satisfies (1.1).
- Galilean invariance: If $u(t,x)$ satisfies (1.1), then for any $v_0 \in \mathbb{R}$,

$$w(t,x) = u(t, x - v_0 t)e^{(\mathrm{i}(v_0/2))(x-(v_0/2)t)} \tag{1.5}$$

also satisfies (1.1).

This article is concerned with questions related to special solutions of equation (1.1), called solitary wave solutions, which are fundamental in the dynamics of the equation. For $\omega_0 > 0$,

$$u(t,x) = e^{i\omega_0 t} Q_{\omega_0}(x) \tag{1.6}$$

is an H^1-solution of (1.1) if $Q_{\omega_0} : \mathbb{R} \to \mathbb{R}$ is an H^1-solution of

$$Q''_{\omega_0} + f(Q^2_{\omega_0})Q_{\omega_0} = \omega_0 Q_{\omega_0}, \quad Q_{\omega_0} > 0. \tag{1.7}$$

By the symmetries of the equation, for any $v_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}$, and $\gamma_0 \in \mathbb{R}$,

$$u(t,x) = Q_{\omega_0}(x - x_0 - v_0 t)e^{i((1/2)v_0 x - (1/4)v_0^2 t + \omega_0 t + \gamma_0)} \tag{1.8}$$

is also a solution of (1.1).

Recall that a necessary and sufficient condition for existence of nontrivial solutions of (1.7) is known; there exists a solution of (1.7) in H^1 if and only if

$$r_0 = \inf\{r > 0 \text{ such that } F(r) = \omega_0 r\} \quad \text{exists and satisfies } f(r_0) > \omega_0 \tag{1.9}$$

(see Section 2.1 for more details). Note that (1.9) also implies that there exists a unique positive solution Q_ω of (1.7) for all ω in a neighborhood of ω_0; moreover, by standard ordinary differential equation (ODE) theory, the map $\omega \mapsto Q_\omega \in H^1$ is C^1 locally around ω_0.

A first question concerning the solitary wave solutions of (1.1) is whether or not they are stable by perturbation of the initial data in the energy space, that is, whether or not the following property is satisfied.

Definition 1 (Stability of solitary waves)
A solitary wave solution of the form (1.8) is H^1-stable if for all $\epsilon > 0$, there exists $\delta > 0$ such that if

$$\|u(0) - Q_{\omega_0}(\cdot - x_0)e^{i((1/2)v_0 x + \gamma_0)}\|_{H^1} \leq \delta,$$

then for all $t \in \mathbb{R}$, there exist $x(t), \gamma(t) \in \mathbb{R}$ such that the solution $u(t)$ of (1.1) satisfies

$$\left\|u(t,\cdot) - Q_{\omega_0}\big(\cdot - x(t)\big)e^{i((1/2)v_0 x + \gamma(t))}\right\|_{H^1} \leq \epsilon.$$

By the invariances of the NLS equation, whether or not this property is satisfied does not depend on v_0, x_0, or γ_0. This question for solitary waves of the NLS equation has been addressed by several authors (not only in the one-space dimension but also in the case of NLS equations set in $\mathbb{R}^d$ for any $d \geqslant 1$). Let us review the known results.

In 1982, Cazenave and Lions [3] proved the stability of these solutions when they are minimizers, in a certain sense, of the energy functional and when a compactness condition on minimizing sequences holds. Their approach makes use of the concentration compactness method of P.-L. Lions [9]. The condition obtained on f is sharp for the case of power nonlinearities $f(s^2) \equiv s^{p-1}$. (Stability requires $1 < p < 5$.)

Later, in 1986, by a different approach based on the expansion of the conservation laws around the solitary wave, Weinstein [22] proved the stability in H^1 of a solitary wave solution in the case where Q_{ω_0} is a ground state under the nondegeneracy condition

$$\frac{d}{d\omega} \int_{\mathbb{R}} Q_\omega^2(x)\,dx \Big|_{\omega=\omega_0} > 0 \tag{1.10}$$

plus some assumptions on the spectrum of the linearized operator around Q_{ω_0}. These assumptions are checked in [22] for subcritical power nonlinearities for $d = 1$ and $d = 3$ and can also be checked under less-restrictive conditions. We refer to Section 2.1 for more information in the case where $d = 1$. See also Section 2.3 for details on Weinstein's proof of the H^1-stability.

Conversely, it is also known from a work of Grillakis, Shatah, and Strauss [8] that if

$$\frac{d}{d\omega} \int_{\mathbb{R}} Q_\omega^2(x)\,dx \Big|_{\omega=\omega_0} < 0,$$

then the solitary wave Q_{ω_0} is unstable in H^1.

We consider now the problem of stability of the *sum* of decoupled solitary waves. Known results on the question of stability of multisolitary wave solutions are based on asymptotic stability. (This notion means that the solution converges (in some sense) as $t \to +\infty$ to the sum of several solitary waves.) A first result is this direction was given by Perelman [18], following Buslaev and Perelman [2] on asymptotic stability of a single solitary wave for the NLS equation. In [18], Perelman proves that in the one-dimensional case, the sum of several solitary waves is stable and asymptotically stable under a set of conditions on the initial conditions, the solitary waves, and the nonlinearity; the nonlinearity has to be flat at zero ($|f(s^2)|s \leqslant s^q$ for $q \geqslant 9$), and the initial condition has to be close to the sum of two solitary waves in a weighted space. (The norm is related to $\|u\|_{H^1} + \|xu\|_{L^2} + \|\hat{u}\|_{L^1}$.) It is also required that the relative velocities be large but without a precise control of how large. The other assumptions concern the spectrum of the linearized operator around the solitary waves and cannot be checked easily; in particular, these spectral assumptions are not simple consequences of (1.10) as in Weinstein's article [22].

For $d \geqslant 3$, the question of asymptotic completeness of K solitary waves for nonlinear Schrödinger equations was considered simultaneously by Perelman [19] and Rodnianski, Schlag, and Soffer [20], who prove similar results, both using dispersive estimates first due to Cuccagna [4]. Both results require large velocities, flatness of $f(r)$ for r near zero, and some spectral assumptions (assumptions on the generalized null space of the linearized operator, nonexistence of nonzero eigenvalues, and nonresonance conditions). The closeness of the initial data to the sum of solitary waves is assumed in different norms. In [19], the initial data is in H^1 and is close to the sum of solitary waves in the norm $\|u\|_{L^1} + \|\hat{u}\|_{L^m}$ for some $m > 2$. In [20], closeness is required in the norm $\sum_{k=1}^{s} \|\nabla^k u\|_{L^1 \cap L^2}$ for some $s > d/2$ integer.

As we can see, known stability and asymptotic stability results rely on spectral assumptions and on dispersive estimates in spaces strictly included in the energy space.

In this article, we propose a different approach for the NLS equation, following [15] concerning the case of the subcritical generalized Korteweg–de Vries (gKdV) equations:

$$u_t + u_{xxx} + (u^p)_x = 0 \tag{1.11}$$

for $p = 2, 3, 4$. The first point that one can make from [15] is that the problem of stability of the sum of K decoupled solitons has to be considered independently from the problem of asymptotic stability. Indeed, these two properties rely on different tools; the stability is related to conserved quantities and monotonicity properties in time, whereas the asymptotic stability is related to local virial properties.

Recall that Weinstein's stability proof for a single solitary wave [22] is based only on conserved quantities, the L^2-norm and energy. (The proof is thus the same for the gKdV equations and for the NLS equations.) Recall also that a result of stability of K-soliton solutions for the KdV equation (i.e., the integrable case $p = 2$) in H^K was proved by Maddocks and Sachs [10], where H^K-regularity is really needed since the proof makes use of $K + 1$ conserved quantities of the equation.

In [15], we could prove the first result of stability of K-solitons in the energy space for the subcritical gKdV equations (without the use of integrability theory) by using the two H^1 conserved quantities in addition to $K - 1$ monotonicity properties. The monotonicity properties are related to L^2-quantities and allow us to decouple the different solitary waves from an energetic point of view. They were introduced in the subcritical case by Martel and Merle in [13] (see Section 3.1 for more information in the case of the gKdV equation). Note that in [15], the proof of stability applies equally well to a general subcritical $f(u)$ instead of u^p in (1.11) as in [22]. Therefore, under the condition that solitons of the gKdV equations are nonlinearly stable, considered independently, we prove that their sum is stable if they are sufficiently decoupled.

In [15], after the stability is proved, a completely different argument is used to prove the asymptotic stability of the sum of several solitons for the generalized KdV equations. The argument is mainly based on virial identities (see also [12], [13]).

Note that the tools developed for this approach of the stability problem for the gKdV equations have been used further by Martel [11] in order to construct asymptotic multisoliton solutions of the gKdV equations in the critical and subcritical cases ($1 < p \leqslant 5$). Finally, note also that similar results can be proved by similar tools for the generalized Benjamin-Bona-Mahony (BBM) equations (see El Dika [5] and El Dika and Martel [6]).

Turning back to nonlinear Schrödinger equations, we introduce in this article a property of monotonicity, related to momentum, which is similar to the L^2 monotonicity property for the gKdV equation used in [15]. This new property, together with an expansion of a functional related to the invariant quantities for a solution close to the sum of K solitary waves, allows us to follow the same strategy as for the gKdV equations in [15]. The main result in the one-dimensional case is the following theorem.

THEOREM 1 (Stability of the sum of K solitary waves in one dimension)
Assume that $f : \mathbb{R} \to \mathbb{R}$ is a function of class C^1 such that $f(0) = 0$ and the following hold.

(A1) Flatness at zero:

$$\textit{there exists } C > 0 \textit{ such that for all } r \in [0,1], \quad f'(r) \leqslant Cr.$$

Let $K \in \mathbb{N}$, and for all $k \in \{1, \ldots, K\}$, let $\omega_k^0 > 0$ be such that there exists $Q_{\omega_k^0} \in H^1(\mathbb{R})$, a positive solution of (1.7) satisfying

(A2) Nonlinear stability of each wave:

$$\frac{d}{d\omega} \int_{\mathbb{R}} Q_\omega^2(x)\, dx \Big|_{\omega=\omega_k^0} > 0.$$

For all $k \in \{1, \ldots, K\}$, let $x_k^0 \in \mathbb{R}$, let $\gamma_k^0 \in \mathbb{R}$, and let $v_k \in \mathbb{R}$ with $v_1 < v_2 < \cdots < v_K$. Assume further that for all $k \in \{1, \ldots, K-1\}$,

(A3) Condition on relative speeds:

$$(v_{k+1} - v_k)^2 > 4\,|\omega_{k+1}^0 - \omega_k^0|.$$

There exist $L_0 > 0$, $\theta_0 > 0$, $A_0 > 0$, and $\alpha_0 > 0$ such that for any $u_0 \in H^1(\mathbb{R})$, $L > L_0$ and $0 < \alpha < \alpha_0$ if

$$\Big\| u_0 - \sum_{k=1}^K Q_{\omega_k^0}(\cdot - x_k^0) e^{\mathrm{i}((1/2)v_k x + \gamma_k^0)} \Big\|_{H^1} \leqslant \alpha, \tag{1.12}$$

and if for all $k \in \{1, \ldots, K-1\}$,

$$x_{k+1}^0 - x_k^0 > L, \tag{1.13}$$

then the solution $u(t)$ of (1.1) is globally defined in H^1 for $t \geqslant 0$, and there exist C^1-functions $x_1(t), \ldots, x_K(t) \in \mathbb{R}$ and $\gamma_1(t), \ldots, \gamma_K(t) \in \mathbb{R}$ such that for all $t \geqslant 0$,

$$\Big\| u(t) - \sum_{k=1}^K Q_{\omega_k^0}\big(\cdot - x_k(t)\big) e^{\mathrm{i}((1/2)v_k x + \gamma_k(t))} \Big\|_{H^1} \leqslant A_0(\alpha + e^{-\theta_0 L}). \tag{1.14}$$

Moreover, for all $t \geqslant 0$,

$$|\dot{x}_k(t) - v_k| + \left|\dot{\gamma}_k(t) - \left(\omega_k^0 - \frac{v_k^2}{4}\right)\right| \leqslant A_0(\alpha + e^{-\theta_0 L}). \tag{1.15}$$

Comments on Theorem 1

(1) *Stability result in* H^1. Our first comment is that in contrast with previously existing results for the nonlinear Schrödinger equation, Theorem 1 is a stability result in the energy space. Moreover, no spectral assumption on the linearized operator is required, except the natural assumption that the various solitary waves are independently nonlinearly stable. This is due to the fact that our proof of stability is not a consequence of an asymptotic stability result.

(2) *Assumption on the nonlinearity.* We now comment on assumption (A1). The assumption on f which we really use in the proof of Theorem 1 is the following:

$$\text{There exists } C > 0 \text{ such that for all } s \in [0, 1], \quad f(s^2)s^2 - F(s^2) \leqslant Cs^6,$$

which is a consequence of (A1). The reason why we need such an assumption in this article is technically clear in our method (see the proof of Proposition 3.1); however, we do not claim that it is necessary for the result to hold.

Recall that in the case of a pure power nonlinearity $f(s^2) \equiv s^{p-1}$, the critical exponent for the stability of the solitary waves is $p = 5$, which means that the stability condition (A2) holds on the solitary waves if and only if $1 < p < 5$. But for $f(s^2) \equiv s^{p-1}$, assumption (A1) requires $p \geqslant 5$, which means that Theorem 1 does not apply to the pure power case for any p.

However, there are important explicit examples of nonlinearities f to which Theorem 1 applies. Let us give one such class of examples constructed from the pure power case.

Let $1 < p < 5$ and $q \geqslant 5$. Consider $f(s^2) = s^{p-1}$, for $s > s_0$, $f(s^2) = s^{q-1}$, for $0 \leqslant s < s_0/2$ and f increasing and of class C^1. For $s_0 > 0$ small, equation (1.1) is a perturbation of the pure power subcritical Schrödinger equation, and (A1) holds since $q \geqslant 5$. Thus, since (A2) is true for any solitary wave for $f(s^2) = s^{p-1}$, and since such a condition depends continuously on f, it follows that for small s_0, the solitary waves for $\omega > \omega_0 > 0$ are stable in the sense of Weinstein (i.e., (A2) is true; see [21], [22]). Therefore, provided that assumption (A3) on the speeds is satisfied, Theorem 1 applies to this case.

(3) *Integrable case*. For $f(s^2) \equiv s^2$, by the previous comment, Theorem 1 does not apply because (A1) breaks down. It is, however, a very important special case since (1.1) then becomes a completely integrable equation. In particular, following Zakharov and Shabat [23], by integrability theory, one can exhibit special solutions of (1.1) which are multisolitary wave solutions. It is in particular possible to construct

multisolitary waves with solitary waves of different speeds with arbitrary sizes. More surprisingly, one can also construct multisolitary waves for which the various solitary waves remain parallel for all time or separate as $\log t$ in large time (see [23, pp. 66–67]). One may expect that a further investigation of these solutions would provide more insight on the set of assumptions of Theorem 1.

(4) *Assumption (A2).* Note that by the equation of Q_ω, the function $S_\omega \in H^1$ defined by $S_\omega = \frac{\partial}{\partial\omega} Q_\omega$ satisfies $\mathscr{L}^+_\omega S_\omega = -Q_\omega$, where

$$\mathscr{L}^+_\omega = -\partial_x^2 + \omega - \big(f(Q_\omega^2) + 2Q_\omega^2 f'(Q_\omega^2)\big).$$

Therefore, assumption (A2) is equivalent to

$$(S_{\omega_0}, Q_{\omega_0}) > 0, \tag{1.16}$$

where (,) denotes the scalar product in L^2. Moreover, it turns out that (A2) implies

$$\inf\left\{\frac{(\mathscr{L}^+_{\omega_0} v, v)}{(v, v)};\ v \in H^1(\mathbb{R}), (v, Q_{\omega_0}) = (v, Q'_{\omega_0}) = 0\right\} > 0 \tag{1.17}$$

(see Lemma 2.2). Property (1.17) is what is really used in the proof of stability in [22] and in the present article.

(5) *Assumption (A3).* Assumption (A3) means that if ω_k^0 and ω_{k+1}^0 are different, then the relative speed of the corresponding solitary waves, that is, $v_{k+1} - v_k$, has to be sufficiently large. A remarkable fact is that any speeds $v_k < v_{k+1}$ are possible if $\omega_k^0 = \omega_{k+1}^0$. Note that condition (A3) is invariant by the Galilean transform (1.5).

Case of dimensions 2 *and* 3

Now we turn to the NLS equation set in $\mathbb{R}^d$,

$$\begin{cases} i\,\partial_t u = -\Delta u - f(|u|^2)u, & (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ u(0) = u_0, \end{cases} \tag{1.18}$$

with f of class C^1 satisfying

$$f(0) = 0 \qquad \text{and} \qquad \forall s \geqslant 1, \quad |f'(s^2)| < Cs^{p-2}, \quad \text{for some } p < \frac{d+2}{d-2}. \tag{1.19}$$

It is well known that this equation has properties similar to the ones described above in the one-dimensional case $d = 1$; local H^1 well-posedness (see [7]), conservation laws, and symmetries (translation and phase invariance and Galilean invariance). There may also exist traveling wave solutions; if, for $\omega_0 > 0$, Q_{ω_0} is the solution of the elliptic problem

$$\Delta Q_{\omega_0} + f(Q_{\omega_0}^2)Q_{\omega_0} = \omega_0 Q_{\omega_0}, \quad Q_{\omega_0} > 0, \tag{1.20}$$

then

$$u(t,x) = Q_{\omega_0}(x - x_0 - v_0 t)e^{\mathrm{i}((1/2)v_0.x-(1/4)|v_0|^2 t+\omega_0 t+\gamma_0)}$$

is the solution of (1.18), where $v_0 \in \mathbb{R}^d$, $x_0 \in \mathbb{R}^d$, $\gamma_0 \in \mathbb{R}$, where $|v_0|^2 = \sum_{j=1}^d v_{0,j}^2$, and where $v_0 \cdot x$ is the scalar product in $\mathbb{R}^d$.

As was mentioned above, the stability problem for one traveling wave solution is solved in a similar way for $d \geqslant 2$ as for $d = 1$ (see Weinstein [22]). From [22], a natural assumption for nonlinear stability is the existence of $\lambda > 0$ such that for any real-valued function $\eta \in H^1$,

(A2′)

$$(\eta, Q_\omega) = (\eta, \nabla Q_\omega) = 0 \Rightarrow \int \{|\nabla\eta|^2 + \omega|\eta|^2 - \big(f(Q_\omega^2) + 2Q_\omega^2 f'(Q_\omega^2)\big)|\eta|^2\} \geqslant \lambda\|\eta\|_{H^1}^2.$$

Note that this condition is equivalent to subcriticality in the pure power case.

It turns out that the proof of Theorem 1 given in this article cannot be extended in general to higher dimensions $d \geqslant 2$. However, the method still applies to $d = 2$ and $d = 3$ for some nonlinearities and with a suitable condition on the relative speeds of the solitary waves. We claim the following result.

THEOREM 2 (Stability of the sum of K solitary waves in dimensions two and three) *Let $d = 2$ or 3. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a function of class C^1 such that (1.19) holds, and assume for some constant $C > 0$ that*

(A1′)

$$\text{for all } r \geqslant 0, \quad f'(r) \leqslant Cr.$$

Let $K \in \mathbb{N}$, and for all $k \in \{1, \dots, K\}$, let $\omega_k^0 > 0$ be such that $Q_{\omega_k^0} > 0$ is a solution of (1.20) satisfying (A2′). For all $k \in \{1, \dots, K\}$, let $x_k^0 \in \mathbb{R}^d$, let $\gamma_k \in \mathbb{R}$, and let $v_k \in \mathbb{R}^d$, satisfying,

$$\text{for all } k \neq k', \quad v_k \neq v_{k'}. \tag{1.21}$$

There exist $\omega_0 = \omega_0(v_1, \dots, v_K) > 0$, $T_0 > 0$, $\theta_0 > 0$, $A_0 > 0$, and $\alpha_0 > 0$ such that if,

(A3′)

$$\text{for all } k \neq k', \quad |\omega_k^0 - \omega_{k'}^0| < \omega_0,$$

and if, for any $u_0 \in H^1(\mathbb{R}^d)$, $T > T_0$, and $0 < \alpha < \alpha_0$,

$$\Big\| u_0 - \sum_{k=1}^K Q_{\omega_k^0}(\cdot - x_k^0 - v_k T)e^{\mathrm{i}((1/2)v_k\cdot x+\gamma_k^0)} \Big\|_{H^1} \leq \alpha, \tag{1.22}$$

then the solution $u(t)$ of (1.18) is globally defined in H^1, and there exist $x_1(t), \dots, x_K(t) \in \mathbb{R}^d$ and $\gamma_1(t), \dots, \gamma_K(t) \in \mathbb{R}$ such that, for all $t \geqslant T$,

$$\Big\| u(t) - \sum_{k=1}^{K} Q_{\omega_k^0}\big(\,\cdot - x_k(t)\big) e^{\mathrm{i}((1/2)v_k \cdot x + \gamma_k(t))} \Big\|_{H^1} \leq A_0(\alpha + e^{-\theta_0 T}). \tag{1.23}$$

Moreover, for all $t \geqslant T$,

$$|\dot{x}_k(t) - v_k| + \Big| \dot{\gamma}_k(t) - \Big(\omega_k^0 - \frac{|v_k|^2}{4}\Big) \Big| \leqslant A_0(\alpha + e^{-\theta_0 T}). \tag{1.24}$$

Comments on Theorem 2

(1) *First result for $d = 2$.* To our knowledge, it is the first result of stability of the sum of solitary waves for semilinear Schrödinger equations in space dimension 2, which is in some sense a critical dimension for this problem. Moreover, as in Theorem 1, Theorem 2 holds in the energy space and without spectral assumptions. Therefore, for $d = 3$, it is different from the existing results described above (see [19], [20]).

(2) *Condition (A1′).* Condition (A1′) is similar to condition (A1) in Theorem 1. We impose, in addition, that this condition is satisfied for any r and not only for r close to zero. The fact that it is satisfied for r close to zero is essential in our proof, which consists in fact in going back to the one-dimensional case. It removes the possibility to apply Theorem 2 to the case of pure power nonlinearities. Again, we do not claim optimality.

(3) *Assumption (A3′).* Assumption (A3′) is less precise than condition (A3) imposed in the one-dimensional case since the constant ω_0 is not given explicitly in terms of the (v_k). Indeed, from the proof, the condition needed depends in an intricate way on the geometry of the vectors (v_k). In simple cases, however, it can be chosen explicitly; for example, when the vectors $\{v_j\}$ are colinear, the condition is the same as in the one-dimensional case.

(4) *Elliptic problem.* The existence and uniqueness problem for (1.20) is not completely solved as in the one-dimensional case. In case of existence, it is expected that condition (A2′) is equivalent to condition (A2) also for $d = 2, 3$.

(5) Note that in Theorem 2, we assume that T is large enough. Since the vectors v_k are all different, this is a way to assume that the solitary waves are sufficiently decoupled since they move on different lines. This replaces assumption (1.13) in Theorem 1.

The article is organized as follows. Sections 2, 3, and 4 concern the case $d = 1$. In Section 2, we recall some well-known properties of solitary waves, that is, of solutions of (1.7). In Section 3, we present the new monotonicity result for the nonlinear Schrödinger equations for $d = 1$, and in Section 4, we prove Theorem 1. Finally, Section 5 concerns the case where $d = 2$ and 3 and contains a sketch of the proof of Theorem 2.

The proof of some technical results are left to Appendices A, B, and C.

2. Properties of the solitary waves in one dimension

In this section, we recall some standard properties of solitary waves for equation (1.1). In Section 2.3, we recall the proof of the stability of the solitary waves for the sake of completeness. Note that it is slightly different from the proof due to Weinstein [22].

2.1. Existence of solitary waves

The problem of existence and uniqueness of solutions of the following elliptic problem in one space dimension,

$$Q_\omega'' + f(|Q_\omega|^2)Q_\omega = \omega Q_\omega, \qquad Q_\omega(x_0) > 0 \quad \text{for some } x_0 \in \mathbb{R}, \qquad Q_\omega \in H^1(\mathbb{R}), \tag{2.1}$$

is completely understood. Indeed, by Berestycki and Lions [1, Sec. 6], we have the following result.

LEMMA 2.1 (See [1, Th. 5])
A necessary and sufficient condition for the existence of a solution Q_ω of problem (2.1) is that

$$r_0 = \inf\{r > 0 \text{ such that } F(r) = \omega r\} \quad \text{exists and satisfies } f(r_0) > \omega. \tag{2.2}$$

Moreover, if (2.2) is satisfied, then

(1) *problem (2.1) has a unique solution Q_ω up to translation;*

(2) *$Q_\omega(0) = \sqrt{r_0}$; $Q_\omega(x) = Q_\omega(-x)$ for all $x \in \mathbb{R}$; $Q_\omega(x) > 0$ for all $x \in \mathbb{R}$; $Q_\omega'(x) < 0$ for all $x > 0$;*

(3) *there exists $C > 0$ such that*

$$|Q_\omega(x)| + |Q_\omega'(x)| + |Q_\omega''(x)| \leqslant Ce^{-(\sqrt{\omega}/2)|x|} \quad \text{for all } x \in \mathbb{R}. \tag{2.3}$$

Note that since $f(0) = 0$, we have, for all $\omega > 0$, $F(r) < \omega r$ in a neighborhood of zero. Therefore, condition (2.2) means that there exists $r > 0$ such that $F(r) = \omega r$ and that at the first such point r_0, the nondegeneracy condition $F'(r_0) = f(r_0) > \omega$ holds.

By C^1-regularity of f, if this condition is satisfied for some $\omega_0 > 0$, then it is also satisfied in a neighborhood of ω_0. Moreover, by standard ODE arguments, the map $\omega \in (\omega_0 - \overline{\omega}, \omega_0 + \overline{\omega}) \mapsto Q_\omega \in H^1$ is of class C^1.

Solitary waves of interest for this article are nonlinearly stable solitary waves. This means, following Weinstein's condition [22], that we assume (A2):

$$\frac{d}{d\omega}\int Q_\omega^2(x)\,dx\Big|_{\omega=\omega_0} > 0. \tag{2.4}$$

Note that the condition in Cazenave and Lions [3] implies $\frac{d}{d\omega}\int Q_\omega^2(x)\,dx\big|_{\omega=\omega_0} \geqslant 0$; thus, (2.4) is a nondegeneracy condition. Weinstein's stability proof [22] is based on an analysis of the linearized operator around Q_{ω_0}. Define, for a real-valued $v \in H^1$,

$$\mathscr{L}_\omega^+ v = -v_{xx} + \omega v - \big(f(Q_\omega^2) + 2Q_\omega^2 f'(Q_\omega^2)\big)v, \qquad \mathscr{L}_\omega^- v = -v_{xx} + \omega v - f(Q_\omega^2)v. \tag{2.5}$$

As we have seen in the introduction, by the equation of Q_ω, the function $S_\omega \in H^1$ defined by $S_\omega = \frac{\partial}{\partial\omega} Q_\omega$ satisfies $\mathscr{L}_\omega^+ S_\omega = -Q_\omega$, and condition (2.4) is equivalent to

$$\frac{1}{2}\frac{d}{d\omega}\int Q_\omega^2(x)\,dx\Big|_{\omega=\omega_0} = (S_{\omega_0}, Q_{\omega_0}) = -(\mathscr{L}_{\omega_0}^+ S_{\omega_0}, S_{\omega_0}) > 0. \tag{2.6}$$

(Recall that (,) denotes the scalar product in $L^2(\mathbb{R})$.) The next lemma relates condition (2.4) to positivity properties of $\mathscr{L}_\omega^+$.

LEMMA 2.2 (Weinstein [21])
If ω_0 satisfies (2.2) and (2.4), then we have the following.
(1) *There exists $\lambda^+ > 0$ such that for any real-valued $v \in H^1$,*

$$(v, Q_{\omega_0}) = (v, Q'_{\omega_0}) = 0 \quad \textit{implies } (\mathscr{L}_{\omega_0}^+ v, v) \geqslant \lambda^+ \|v\|_{H^1}^2. \tag{2.7}$$

(2) *There exists $\lambda^- > 0$ such that for any real-valued $v \in H^1$,*

$$(v, Q_{\omega_0}) = 0 \quad \textit{implies } (\mathscr{L}_{\omega_0}^- v, v) \geqslant \lambda^- \|v\|_{H^1}^2. \tag{2.8}$$

Proof
The proof of this result was given by Weinstein [21] for the power case $f(s^2) = s^{p-1}$. Under the assumptions on f in the present article, exactly the same arguments apply to prove Lemma 2.2. □

We also recall some variational properties of Q_{ω_0}. Let

$$\mathscr{F}_{\omega_0}(z) = E(z) + \omega_0 \int |z|^2. \tag{2.9}$$

Then we have the following result.

LEMMA 2.3
For $\eta \in H^1$ small, we have

$$\mathscr{F}_{\omega_0}(Q_{\omega_0}+\eta) = \mathscr{F}_{\omega_0}(Q_{\omega_0}) + (\mathscr{L}^+_{\omega_0}\operatorname{Re}\eta, \operatorname{Re}\eta) + (\mathscr{L}^-_{\omega_0}\operatorname{Im}\eta, \operatorname{Im}\eta) + \|\eta\|^2_{H^1}\beta(\|\eta\|_{H^1})$$

with $\beta(\epsilon) \to 0$ as $\epsilon \to 0$. In particular, for ω close to ω_0,

$$\mathscr{F}_{\omega_0}(Q_\omega) = \mathscr{F}_{\omega_0}(Q_{\omega_0}) + (\omega-\omega_0)^2(\mathscr{L}^+_{\omega_0}S_{\omega_0}, S_{\omega_0}) + |\omega-\omega_0|^2\beta(|\omega-\omega_0|),$$

and

$$|\mathscr{F}_{\omega_0}(Q_\omega) - \mathscr{F}_{\omega_0}(Q_{\omega_0})| \leqslant C|\omega-\omega_0|^2.$$

Proof
See, for example, Weinstein [22, Sec. 2, (2.5)]. □

Observe that since $(\mathscr{L}^+_{\omega_0}S_{\omega_0}, S_{\omega_0}) = -(Q_{\omega_0}, S_{\omega_0}) < 0$ by assumption (A2), $\mathscr{F}_{\omega_0}(Q_\omega)$ is strictly smaller than $\mathscr{F}_{\omega_0}(Q_{\omega_0})$ for ω close but different from ω_0.

2.2. Decomposition of a solution close to the sum of K solitary waves
Let $v_1 < \cdots < v_K$, and let $\omega_1^0, \ldots, \omega_K^0 \in \mathbb{R}$ be such that (1.9) and (A2) hold. Set

$$\theta_1 = \frac{1}{4}\min\big(v_2-v_1, \ldots, v_K-v_{K-1}, \sqrt{\omega_1^0}, \ldots, \sqrt{\omega_K^0}\big) > 0. \tag{2.10}$$

For $\alpha, L > 0$, we consider the H^1-neighborhood of size α of the sum of K solitary waves with parameters $\{v_k, \omega_k^0\}$, sorted by increasing speeds and located at distances larger than L:

$$\mathscr{U}(\alpha, L) = \Big\{u \in H^1;\ \inf_{\substack{y_k > y_{k-1}+L\\ \delta_k\in\mathbb{R}}} \Big\|u(t,\cdot) - \sum_{k=1}^K Q_{\omega_k^0}(\cdot - y_k)e^{\mathrm{i}((1/2)v_kx+\delta_k)}\Big\|_{H^1} < \alpha\Big\}.$$

The following lemma is a standard result.

LEMMA 2.4
There exist $L_1, \alpha_1, C_1 > 0$, and for any $k \in \{1, \ldots, K\}$, there exist unique C^1-functions $(\omega_k, x_k, \gamma_k) : \mathscr{U}(\alpha_1, L_1) \to (0, +\infty) \times \mathbb{R} \times \mathbb{R}$ such that if $u \in \mathscr{U}(\alpha_1, L_1)$ and if one defines

$$\varepsilon(x) = u(x) - \sum_{k=1}^K Q_{\omega_k}(\cdot - x_k)e^{\mathrm{i}((1/2)v_kx+\gamma_k)}, \tag{2.11}$$

then for all $k = 1, \dots, K$,

$$\mathrm{Re}\int Q_{\omega_k}(\cdot - x_k)e^{\mathrm{i}((1/2)v_k x+\gamma_k)}\overline{\varepsilon}(x)\,dx = \mathrm{Im}\int Q_{\omega_k}(\cdot - x_k)e^{\mathrm{i}((1/2)v_k x+\gamma_k)}\overline{\varepsilon}(x)\,dx = \mathrm{Re}\int Q'_{\omega_k}(\cdot - x_k)e^{\mathrm{i}((1/2)v_k x+\gamma_k)}\overline{\varepsilon}(x)\,dx = 0.$$

Moreover, if $u \in \mathcal{U}(\alpha, L)$ *for* $0 < \alpha < \alpha_1$, $0 < L_1 < L$, *then*

$$\|\varepsilon\|_{H^1} + \sum_{k=1}^{K} |\omega_k - \omega_k^0| \leqslant C_1\alpha, \qquad x_k - x_{k-1} > L - C_1\alpha > \frac{L}{2}. \tag{2.12}$$

We refer to Appendix A for the proof of this result. Note that it applies to time-independent functions.

A consequence of this decomposition for fixed u is the following result on a solution $u(t)$ of (1.1) which is close to the sum of sufficiently decoupled solitary waves on some time interval $[0, t_0]$.

COROLLARY 3

There exist $L_1, \alpha_1, C_1 > 0$ *such that the following is true. If for* $L > L_1$, $0 < \alpha < \alpha_1$, *and* $t_0 > 0$,

$$u(t) \in \mathcal{U}(\alpha, L) \quad \text{for all } t \in [0, t_0], \tag{2.13}$$

then there exist unique C^1*-functions* $\omega_k : [0, t_0] \to (0, +\infty)$, $x_k, \gamma_k : [0, t_0] \to \mathbb{R}$ *such that if we set*

$$\varepsilon(t, x) = u(t, x) - R(t, x), \tag{2.14}$$

where

$$R(t, x) = \sum_{k=1}^{K} R_k(t, x), \qquad R_k(t, x) = Q_{\omega_k(t)}\big(x - x_k(t)\big)e^{\mathrm{i}((1/2)v_k x+\gamma_k(t))}, \tag{2.15}$$

then $\varepsilon(t)$ *satisfies, for all* $k = 1, \dots, K$ *and all* $t \in [0, t_0]$,

$$\mathrm{Re}\int R_k(t)\overline{\varepsilon}(t) = \mathrm{Im}\int R_k(t)\overline{\varepsilon}(t) = \mathrm{Re}\int \partial_x R_k(t)\overline{\varepsilon}(t) = 0. \tag{2.16}$$

Moreover, for all $t \in [0, t_0]$ *and for all* $k = 1, \dots, K$,

$$\|\varepsilon(t)\|_{H^1} + \sum_{k=1}^{K} |\omega_k(t) - \omega_k^0| \leqslant C_1\alpha, \qquad x_k(t) - x_{k-1}(t) > \frac{L}{2}, \tag{2.17}$$

$$|\dot{\omega}_k(t)| + |\dot{x}_k(t) - v_k|^2 + \left|\dot{\gamma}_k(t) - \left(\omega_k(t) - \frac{v_k^2}{4}\right)\right|^2 \leqslant C_1 \int e^{-\theta_1|x-x_k(t)|}\varepsilon^2(t,x)\,dx + C_1 e^{-\theta_1(L+\theta_1 t)}. \tag{2.18}$$

Note that in Corollary 3 as in Lemma 2.4, we do not change the v_k. In particular, in this article, v_k are constant fixed in Theorem 1 and do not depend on t. The proof of Corollary 3 is given in Appendix A.

2.3. Stability of a solitary wave

We repeat the proof of the stability of a solitary wave following Weinstein [22] but without using the Galilean transformation and using modulation in the scaling parameter ω. This gives us the opportunity to introduce the functional $\mathscr{G}(t)$ and to prove the result directly, which is fundamental for the stability problem of the sum of decoupled solitary waves later in the article.

Let ω_0 satisfy (1.9) and (1.10). Let $u(t)$ be a solution of (1.1) satisfying, for some $x_0 \in \mathbb{R}$, $v_0 \in \mathbb{R}$, and $\gamma_0 \in \mathbb{R}$,

$$\|u_0 - Q_{\omega_0}(x - x_0)e^{\mathrm{i}((1/2)v_0 x + \gamma_0)}\|_{H^1} < \alpha$$

for α small.

(1) *Decomposition of the solution.* We argue on a time interval $[0, t^*]$, so that for all $t \in [0, t^*]$, $u(t)$ is close in H^1 to $Q_{\omega(t)}(x - x(t))e^{\mathrm{i}((1/2)v_0 x + \gamma(t))}$ for some $\omega(t)$, $x(t)$, and $\gamma(t)$. We can modify the parameters $\omega(t)$, $x(t)$, and $\gamma(t)$ such that

$$\varepsilon(t,x) = u(t,x) - R_0(t,x),$$

where

$$R_0(t,x) = Q_{\omega(t)}\big(x - x(t)\big)e^{\mathrm{i}((1/2)v_0 x + \gamma(t))}$$

satisfies the orthogonality conditions

$$\mathrm{Re}\big(\varepsilon(t), R_0(t)\big) = \mathrm{Im}\big(\varepsilon(t), R_0(t)\big) = \mathrm{Re}\big(\varepsilon(t), \partial_x R_0(t)\big) = 0. \tag{2.19}$$

This is standard in the case of a single solitary wave (see Section 2.2 with $K = 1$). This choice of orthogonality conditions is well adapted to the positivity properties on $\mathscr{L}_\omega^+$ and $\mathscr{L}_\omega^-$ (see Lemma 2.2), and thus it is suitable to apply an energy method.

Note that as in Lemma 2.4, we have

$$\|\varepsilon(0)\|_{H^1} + |\omega(0) - \omega_0| \leqslant C\alpha. \tag{2.20}$$

(2) *Introduction of a functional adapted to the stability problem.* Following Weinstein [22], we prove the stability property by using the conservation in time of a functional related to the three invariant quantities. We introduce, for $z \in H^1$,

$$\mathscr{G}(z) = E(z) + \Big(\omega(0) + \frac{v_0^2}{4}\Big) \int |z|^2 - v_0 \operatorname{Im} \int \partial_x z\, \overline{z}. \tag{2.21}$$

The introduction of $\mathscr{G}(z)$ is quite natural, and it comes from the following.

CLAIM 1
Let $z(x) = z_0(x - x_0)e^{\mathrm{i}((1/2)v_0 x + \gamma_0)}$. Then

$$\mathscr{G}(z) = E(z_0) + \omega(0) \int |z_0|^2 = \mathscr{F}_{\omega(0)}(z_0).$$

Proof
We have $\int |z|^2 = \int |z_0|^2$, $\int F(|z|^2) = \int F(|z_0|^2)$,

$$\int |\partial_x z|^2 = \int |\partial_x z_0|^2 + \frac{v_0^2}{4} \int |z_0|^2 + v_0 \operatorname{Im} \int \partial_x z_0\, \overline{z}_0,$$

and

$$v_0 \operatorname{Im} \int \partial_x z\, \overline{z} = v_0 \operatorname{Im} \int \partial_x z_0\, \overline{z}_0 + \frac{v_0^2}{2} \int |z_0|^2,$$

and thus, the result follows. □

By expanding $u(t) = R_0(t) + \varepsilon(t)$ in the definition of $\mathscr{G}(u(t))$, we obtain the following formula.

LEMMA 2.5
The following holds:

$$\mathscr{G}\big(u(t)\big) = \mathscr{F}_{\omega(0)}(Q_{\omega(0)}) + H_0\big(\varepsilon(t), \varepsilon(t)\big) + \|\varepsilon(t)\|_{H^1}^2 \beta\big(\|\varepsilon(t)\|_{H^1}\big) + O\big(|\omega(t) - \omega(0)|^2\big)$$

with $\beta(\epsilon) \to 0$ as $\epsilon \to 0$, where

$$H_0(\varepsilon, \varepsilon) = \int |\partial_x \varepsilon|^2 - \int \big\{ f(|R_0|^2)|\varepsilon|^2 + 2 f'(|R_0|^2)[\operatorname{Re}(\overline{R}_0 \varepsilon)]^2 \big\}$$
$$+ \Big(\omega(t) + \frac{v_0^2}{4}\Big) \int |\varepsilon|^2 - v_0 \operatorname{Im} \int \partial_x \varepsilon\, \overline{\varepsilon}.$$

Proof
First, we consider the term $\int |\partial_x u|^2$ coming from the energy:

$$\int |\partial_x u|^2 = \int |\partial_x R_0|^2 + 2\,\mathrm{Re}\int \partial_x \overline{R}_0\, \partial_x \varepsilon + \int |\partial_x \varepsilon|^2$$
$$= \int |\partial_x R_0|^2 - 2\,\mathrm{Re}\int \partial_x^2 \overline{R}_0\, \varepsilon + \int |\partial_x \varepsilon|^2.$$

Second, we consider the term $\int F(|u|^2)$ by expanding F near $|R_0|^2$:

$$\begin{aligned} F(|R_0+\varepsilon|^2) &= F\big(|R_0|^2 + 2\,\mathrm{Re}(\overline{R}_0\varepsilon) + |\varepsilon|^2\big) \\ &= F(|R_0|^2) + F'(|R_0|^2)\big(2\mathrm{Re}(\overline{R}_0\varepsilon) + |\varepsilon|^2\big) \\ &\quad + \frac{1}{2}F''(|R_0|^2)[2\,\mathrm{Re}(\overline{R}_0\varepsilon)]^2 + |\varepsilon(t)|^2\beta\big(\|\varepsilon(t)\|_{H^1}\big) \\ &= F(|R_0|^2) + 2f(|R_0|^2)\mathrm{Re}(\overline{R}_0\varepsilon) + f(|R_0|^2)|\varepsilon|^2 \\ &\quad + 2f'(|R_0|^2)[\mathrm{Re}(\overline{R}_0\varepsilon)]^2 + |\varepsilon(t)|^2\beta\big(\|\varepsilon(t)\|_{H^1}\big). \end{aligned}$$

Summing up the two expressions, we obtain

$$\begin{aligned} E\big(u(t)\big) &= E\big(R_0(t)\big) - 2\,\mathrm{Re}\int \big(\partial_x^2\overline{R}_0 + f(|R_0|^2)\overline{R}_0\big)\varepsilon + \int |\partial_x\varepsilon|^2 \\ &\quad - \int \big\{f(|R_0|^2)|\varepsilon|^2 + 2f'(|R_0|^2)[\mathrm{Re}(\overline{R}_0\varepsilon)]^2\big\} + \|\varepsilon(t)\|_{H^1}^2\beta\big(\|\varepsilon(t)\|_{H^1}\big). \end{aligned}$$

For the term $\int |u|^2$, we have

$$\int |u|^2 = \int |R_0|^2 + 2\,\mathrm{Re}\int \overline{R}_0\,\varepsilon + \int |\varepsilon|^2.$$

Finally, for the term $\mathrm{Im}\int \partial_x u\,\overline{u}$, we have

$$\mathrm{Im}\int \partial_x u\,\overline{u} = \mathrm{Im}\int \partial_x R_0\,\overline{R}_0 - 2\,\mathrm{Im}\int \partial_x\overline{R}_0\,\varepsilon + \mathrm{Im}\int \partial_x\varepsilon\,\overline{\varepsilon}.$$

Therefore, if we set $H_0(\varepsilon,\varepsilon)$ as in the statement of the lemma, we obtain

$$\mathscr{G}\big(u(t)\big) = \mathscr{G}\big(R_0(t)\big) + \Gamma + H_0\big(\varepsilon(t),\varepsilon(t)\big) - \big(\omega(t)-\omega(0)\big)\int |\varepsilon|^2 + \|\varepsilon(t)\|_{H^1}^2\beta\big(\|\varepsilon(t)\|_{H^1}\big),$$

where

$$\begin{aligned} \Gamma &= -2\,\mathrm{Re}\int \big(\partial_x^2\overline{R}_0 + f(|R_0|^2)\overline{R}_0\big)\,\varepsilon + 2\Big(\omega(0) + \frac{v_0^2}{4}\Big)\mathrm{Re}\int \overline{R}_0\,\varepsilon + 2v_0\,\mathrm{Im}\int \partial_x\overline{R}_0\,\varepsilon \\ &= 2\,\mathrm{Re}\int \Big[-\partial_x^2 R_0 - f(|R_0|^2)R_0 + \Big(\omega(0) + \frac{v_0^2}{4}\Big)R_0 + \mathrm{i}v_0\partial_x R_0\Big]\overline{\varepsilon}. \end{aligned}$$

We have

$$|\omega(t)-\omega(0)|\int|\varepsilon|^2\leqslant\frac{1}{2}|\omega(t)-\omega(0)|^2+\frac{1}{2}\Big(\int|\varepsilon|^2\Big)^2,$$

which takes care of this term.

Moreover, $\mathscr{G}(R_0(t))=\mathscr{F}_{\omega(0)}(Q_{\omega(t)})$ by Claim 1, and $|\mathscr{F}_{\omega(0)}(Q_{\omega(t)})-\mathscr{F}_{\omega(0)}(Q_{\omega(0)})|\leqslant C|\omega(t)-\omega(0)|^2$ by Lemma 2.3.

Finally, let us prove that Γ is zero. The proof of this would complete the proof of Lemma 2.5. Recall that

$$R_0(t,x)=Q_{\omega(t)}\big(x-x(t)\big)e^{\mathrm{i}((1/2)v_0x+\gamma(t))},$$

so that

$$\partial_x R_0(t,x)=\Big(Q'_{\omega(t)}\big(x-x(t)\big)+\frac{\mathrm{i}}{2}v_0Q_{\omega(t)}\big(x-x(t)\big)\Big)e^{\mathrm{i}((1/2)v_0x+\gamma(t))},$$

and, by the equation of $Q_{\omega(t)}$,

$$\begin{aligned}\partial_x^2R_0(t,x)&=\Big(Q''_{\omega(t)}\big(x-x(t)\big)+\mathrm{i}v_0Q'_{\omega(t)}\big(x-x(t)\big)-\frac{v_0^2}{4}Q_{\omega(t)}\big(x-x(t)\big)\Big)e^{\mathrm{i}((1/2)v_0x+\gamma(t))}\\&=\omega(t)R_0-f(|R_0|^2)R_0+\mathrm{i}v_0\partial_xR_0+\frac{v_0^2}{4}R_0.\end{aligned}$$

Therefore,

$$\Gamma=2\big(\omega(0)-\omega(t)\big)\operatorname{Re}\int R_0\,\overline{\varepsilon},$$

but this is zero by the orthogonality condition chosen on ε in (2.19). Thus Lemma 2.5 is proved. □

We now recall that the quadratic form H_0 is positive for ε satisfying the chosen orthogonality conditions.

LEMMA 2.6
There exists $\lambda_1>0$ such that if $\varepsilon(t)\in H^1$ satisfies

$$\operatorname{Re}\big(\varepsilon(t),R_0(t)\big)=\operatorname{Im}\big(\varepsilon(t),R_0(t)\big)=\operatorname{Re}\big(\varepsilon(t),\partial_xR_0(t)\big)=0,$$

then

$$H_0\big(\varepsilon(t),\varepsilon(t)\big)\geqslant\lambda_1\|\varepsilon(t)\|_{H^1}^2.$$

Remark. The constant $\lambda_1>0$ can be chosen independent of $\omega(t)$ for $\omega(t)$ close to ω_0.

We refer to Appendix B for the proof of Lemma 2.6.

(3) *Control of* $\|\varepsilon(t)\|_{H^1}$. Since $\mathcal{G}(u(t))$ is the sum of three conserved quantities, we have

$$\mathcal{G}\big(u(t)\big) = \mathcal{G}\big(u(0)\big).$$

Thus, by Lemma 2.5, it follows that

$$H_0\big(\varepsilon(t), \varepsilon(t)\big) \leqslant H_0\big(\varepsilon(0), \varepsilon(0)\big) + C|\omega(t) - \omega(0)|^2 + C\|\varepsilon(0)\|_{H^1}^2 \beta\big(\|\varepsilon(0)\|_{H^1}\big) + C\|\varepsilon(t)\|_{H^1}^2 \beta\big(\|\varepsilon(t)\|_{H^1}\big).$$

By Lemma 2.6, and since $H_0(\varepsilon(0), \varepsilon(0)) \leqslant C\|\varepsilon(0)\|_{H^1}^2$, we obtain

$$\lambda_1\|\varepsilon(t)\|_{H^1}^2 \leqslant H_0\big(\varepsilon(t), \varepsilon(t)\big) \leqslant C|\omega(t) - \omega(0)|^2 + C\|\varepsilon(0)\|_{H^1}^2 + C\|\varepsilon(t)\|_{H^1}^2 \beta\big(\|\varepsilon(t)\|_{H^1}\big),$$

which gives

$$\|\varepsilon(t)\|_{H^1}^2 \leqslant C|\omega(t) - \omega(0)|^2 + C\|\varepsilon(0)\|_{H^1}^2 \tag{2.22}$$

for $\|\varepsilon(t)\|_{H^1}$ small enough.

Thus it remains to estimate $|\omega(t) - \omega(0)|^2$ to conclude the proof.

(4) *Control of* $|\omega(t) - \omega(0)|$. We prove that $|\omega(t) - \omega(0)|$ is quadratic in $\varepsilon(t)$. Note that by the conservation of $\int |u(t)|^2$ and the orthogonality condition $\operatorname{Re} \int \overline{R}_0 \varepsilon = 0$, we have

$$\int Q_{\omega(t)}^2 - \int Q_{\omega(0)}^2 = -\int |\varepsilon(t)|^2 + \int |\varepsilon(0)|^2. \tag{2.23}$$

Recall that we assume that

$$\frac{d}{d\omega} \int Q_\omega^2(x)\, dx \Big|_{\omega=\omega_0} > 0$$

and that $\omega(t)$, $\omega(0)$ are close to ω_0. Thus,

$$\begin{aligned}\big(\omega(t) - \omega(0)\big)&\Big(\frac{d}{d\omega} \int Q_\omega^2(x)\, dx \Big|_{\omega=\omega_0}\Big) \\ &= \int Q_{\omega(t)}^2 - \int Q_{\omega(0)}^2 + \beta\big(\omega(t) - \omega(0)\big)\big(\omega(t) - \omega(0)\big)^2\end{aligned}$$

with $\beta(\epsilon) \to 0$ as $\epsilon \to 0$, which implies that for some constant $C = C(\omega_0)$,

$$|\omega(t) - \omega(0)| \leqslant C\Big|\int Q_{\omega(t)}^2 - \int Q_{\omega(0)}^2\Big|.$$

Therefore, by (2.23), we obtain

$$|\omega(t) - \omega(0)| \leqslant C\|\varepsilon(t)\|_{L^2}^2 + C\|\varepsilon(0)\|_{L^2}^2. \tag{2.24}$$

(5) *Conclusion.* Putting together (2.22), (2.24), and (2.20), we obtain, for some constant $C > 0$,

$$\|\varepsilon(t)\|_{H^1}^2 + |\omega(t) - \omega(0)| \leqslant C\|\varepsilon(0)\|_{H^1}^2 \leqslant C\alpha$$

for $\|\varepsilon(t)\|_{H^1}$ and $|\omega(t) - \omega(0)|$ small enough. Thus, for α small enough,

$$\begin{aligned}
&\big\|u(t) - Q_{\omega_0}\big(x - x(t)\big)e^{\mathrm{i}((1/2)v_0+\gamma(t))}\big\|_{H^1} \\
&\quad \leqslant \|u(t) - R_0(t)\|_{H^1} + \big\|R_0(t) - Q_{\omega_0}\big(x - x(t)\big)e^{\mathrm{i}((1/2)v_0+\gamma(t))}\big\|_{H^1} \\
&\quad \leqslant \|\varepsilon(t)\|_{H^1} + C|\omega(t) - \omega_0| \leqslant \|\varepsilon(t)\|_{H^1} + C|\omega(t) - \omega(0)| + C|\omega(0) - \omega_0| \leqslant C\alpha.
\end{aligned}$$

This completes the proof of stability of a single solitary wave.

3. Monotonicity property for the NLS equations

3.1. Monotone localized functional

Since we consider several solitary waves with different speeds $v_1, \ldots, v_K$, we cannot use the Galilean transform to make the solitary waves be all standing waves (i.e., for all k, $v_k = 0$). We have seen in Section 2.3, in the case of a single solitary wave, that if we do not make use of the Galilean transform, then the relevant functional to prove the stability result is $\mathscr{G}(u(t))$, defined by

$$\mathscr{G}\big(u(t)\big) = E\big(u(t)\big) + \Big(\omega(0) + \frac{v_0^2}{4}\Big)\int |u(t)|^2 - v_0 \operatorname{Im}\int \partial_x u(t)\,\overline{u}(t).$$

In the case of several solitary waves R_k, the strategy is to introduce a functional that, *locally around each solitary wave*, looks like the preceding functional with the parameters v_k and $\omega_k(0)$ of the solitary waves.

This idea is reminiscent of an article by the present authors (see [15]) on the generalized KdV equations. Let us recall the situation for the subcritical generalized KdV equations for $p = 2, 3, 4$:

$$u_t + u_{xxx} + (u^p)_x = 0. \tag{3.1}$$

In this case, the traveling waves are of the form $Q_{\omega_0}(x - x_0 - \omega_0 t)$. Observe that the speed of propagation of the wave is related to the scaling w_0. The suitable functional to study the stability of the one traveling wave is

$$E\big(u(t)\big) + \omega(0)\int u^2(t).$$

Therefore, in the case of K traveling waves $Q_{\omega_k}(x - x_k(t))$, we have introduced a functional

$$\mathscr{E}\big(u(t)\big) = E\big(u(t)\big) + \int \omega(t,x)u^2(t),$$

where locally around the kth traveling wave, $\omega(t,x)$ takes the value ω_k. Of course, this is not a conserved quantity in time; however, we have proved that $\mathscr{E}(u(t))$ is in some suitable sense decreasing in time, which allows us to conclude as in the case of the 1-soliton. Recall that the monotonicity in time of the functional $\int \omega(t,x)u^2(t)$ is related to the fact that $\omega(t,x)$ for fixed t is a nondecreasing function of x (since the traveling waves are sorted by increasing speeds). Indeed, for the gKdV equation, the Kato identity leads to

$$\frac{d}{dt}\int g(x-\omega t)u^2(t)\,dx = -\int \Big\{\Big(3u_x^2+\omega u^2-\frac{2p}{p+1}u^{p+1}\Big)g'(x-\omega t)+u^2 g'''(x-\omega t)\Big\}\,dx. \qquad (3.2)$$

If g''' is small (this is the case for a slowly varying g), and if u is small where g' is large (i.e., g varies only far away from the centers of the traveling waves), then we can control the positive part of the right-hand side of (3.2), which proves that this functional is almost decreasing in time. We refer to [15] for a precise result on the generalized KdV equations.

We turn back to the problem for the NLS equations. In the formula of $\mathscr{G}(u(t))$, two parameters appear: $\omega(0)$ and v_0. Both have to be adapted to suitable values around each solitary wave.

The analogue of Kato's identity in the case of the Schrödinger equation is the following virial formula.

LEMMA 3.1
Let $z(t)$ be a solution of (1.1). Let $g : x \in \mathbb{R} \mapsto g(x)$ be a C^3 real-valued function such that g' and g''' are bounded. Then for all $t \in \mathbb{R}$,

$$\frac{1}{2}\frac{d}{dt}\mathrm{Im}\int \partial_x z\,\overline{z}\,g(x) = \int |\partial_x z|^2 g'(x) - \frac{1}{4}\int |z|^2 g'''(x) - \frac{1}{2}\int \big\{f(|z|^2)|z|^2 - F(|z|^2)\big\}g'(x).$$

Proof
It is a standard identity. By elementary calculations, we have

$$\begin{aligned}\frac{d}{dt}\mathrm{Im}\int \partial_x z\,\overline{z}\,g &= \mathrm{Im}\int \partial_x z\,\partial_t\overline{z}\,g + \mathrm{Im}\int \partial_x\partial_t z\,\overline{z}\,g\\ &= \mathrm{Im}\int \partial_x z\,\partial_t\overline{z}\,g - \mathrm{Im}\int \partial_t z\,\partial_x\overline{z}\,g - \mathrm{Im}\int \partial_t z\,\overline{z}\,g'\\ &= 2\,\mathrm{Im}\int \partial_x z\,\partial_t\overline{z}\,g + \mathrm{Im}\int \partial_t\overline{z}\,z\,g' = \mathrm{Im}\int \partial_t\overline{z}(2\partial_x z\,g + z\,g'),\end{aligned}$$

but $\partial_t \overline{z} = -\mathrm{i}(\partial_x^2 \overline{z} + f(|z|^2)\overline{z})$, so that

$$\frac{d}{dt}\mathrm{Im}\int \partial_x z\,\overline{z}\,g = \mathrm{Im}\Big[-\mathrm{i}\int \big(\partial_x^2\overline{z} + f(|z|^2)\overline{z}\big)(2\partial_x z\,g + z\,g')\Big].$$

We have

$$\begin{aligned}&\mathrm{Re}\int \partial_x^2\overline{z}(2\partial_x z\,g + z\,g')\\ &\quad = -2\int |\partial_x z|^2 g' - \mathrm{Re}\int \partial_x \overline{z}\,z\,g'' = -2\int |\partial_x z|^2 g' + \frac{1}{2}\int |z|^2 g'''.\end{aligned}$$

Thus, finally, we obtain

$$\frac{d}{dt}\mathrm{Im}\int \partial_x z\,\overline{z}\,g = 2\Big[\int |\partial_x z|^2 g' - \frac{1}{4}\int |z|^2 g''' - \frac{1}{2}\int \big\{f(|z|^2)|z|^2 - F(|z|^2)\big\}g'\Big].$$

This proves Lemma 3.1. □

Compared to the case of the gKdV equation, the term g''' in Lemma 3.1 causes additional difficulties here. Indeed, for the gKdV equation, the term g''' in (3.2) is controlled by $\omega\int u^2 g'$ for functions g such that $|g'''| < g'$ (see [15]). In the case of the Schrödinger equation, there is no such L^2-term. To get around this problem, we take a cutoff function of the form $g(x/\ell(t))$ whose support depends on time. It turns out that a support $\ell(t)$ of size $\sqrt{t}$ is suitable. (In fact, any choice t^β for $\beta \in (1/3, 1)$ would provide the same results.) We deduce from Lemma 3.1 the following result.

LEMMA 3.2
Let $z(t)$ be a solution of (1.1). Let $g : x \in \mathbb{R} \mapsto g(x)$ be a C^3 real-valued function such that $g' \geqslant 0$ and $x^2 g'$, g''' are bounded. Let $a > 0$. Then for all $t \geqslant 0$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\mathrm{Im}\int \partial_x z\,\overline{z}\,g\Big(\frac{x}{\sqrt{t+a}}\Big)\\ &\quad\geqslant \frac{3}{4\sqrt{t+a}}\int |\partial_x z|^2 g'\Big(\frac{x}{\sqrt{t+a}}\Big) - \frac{\|g'''\|_{L^\infty} + \|x^2 g'\|_{L^\infty}}{4(t+a)^{3/2}}\\ &\quad\times \int_{x/\sqrt{t+a}\in \mathrm{supp}\,g'} |z|^2 - \frac{1}{2\sqrt{t+a}}\int \big\{f(|z|^2)|z|^2 - F(|z|^2)\big\}g'\Big(\frac{x}{\sqrt{t+a}}\Big),\end{aligned}$$

where supp g' *denotes the support of* g'.

Observe that the second term in the right-hand side is now integrable in time since the L^2-norm of a solution is constant. Thus, in order to prove an almost-monotonicity result in the spirit of the one for the gKdV equations, one only has to consider the nonlinear term; this is done in Section 3.2 for a solution that is close to the sum of K solitary waves.

Proof
From Lemma 3.1, we have, by straightforward calculations,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\mathrm{Im}\int \partial_x z\,\overline{z}\,g\Big(\frac{x}{\sqrt{t+a}}\Big) &= \frac{1}{\sqrt{t+a}}\int |\partial_x z|^2 g'\Big(\frac{x}{\sqrt{t+a}}\Big) - \frac{1}{4(t+a)^{3/2}}\int |z|^2 g'''\Big(\frac{x}{\sqrt{t+a}}\Big)\\ &\quad - \frac{1}{2\sqrt{t+a}}\int \big\{f(|z|^2)|z|^2 - F(|z|^2)\big\} g'\Big(\frac{x}{\sqrt{t+a}}\Big)\\ &\quad - \frac{1}{4(t+a)}\mathrm{Im}\int (\partial_x z\,\overline{z})\Big(\frac{x}{\sqrt{t+a}}\Big) g'\Big(\frac{x}{\sqrt{t+a}}\Big),\end{aligned}$$

where the last term in the right-hand side is coming from the time dependence in the function g. By the Cauchy-Schwarz inequality,

$$\begin{aligned}&\Big|\frac{1}{2(t+a)}\mathrm{Im}\int (\partial_x z\,\overline{z})\Big(\frac{x}{\sqrt{t+a}}\Big) g'\Big(\frac{x}{\sqrt{t+a}}\Big)\Big|\\ &\quad\leqslant \frac{1}{4\sqrt{t+a}}\int |\partial_x z|^2 g'\Big(\frac{x}{\sqrt{t+a}}\Big) + \frac{1}{4(t+a)^{3/2}}\int |z|^2 \Big(\frac{x}{\sqrt{t+a}}\Big)^2 g'\Big(\frac{x}{\sqrt{t+a}}\Big),\end{aligned}$$

so that we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\mathrm{Im}\int \partial_x z\,\overline{z}\,g\Big(\frac{x}{\sqrt{t+a}}\Big) &\geqslant \frac{3}{4\sqrt{t+a}}\int |\partial_x z|^2 g'\Big(\frac{x}{\sqrt{t+a}}\Big)\\ &\quad - \frac{1}{4(t+a)^{3/2}}\int |z|^2 \Big\{g'''\Big(\frac{x}{\sqrt{t+a}}\Big) + \Big(\frac{x}{\sqrt{t+a}}\Big)^2 g'\Big(\frac{x}{\sqrt{t+a}}\Big)\Big\}\\ &\quad - \frac{1}{2\sqrt{t+a}}\int \big\{f(|z|^2)|z|^2 - F(|z|^2)\big\} g'\Big(\frac{x}{\sqrt{t+a}}\Big),\end{aligned}$$

and the conclusion of Lemma 3.2 follows. □

3.2. First monotonicity result
From Lemma 3.2, we deduce a monotonicity property for a solution $u(t)$ which is close to the sum of K solitary waves on some time interval $[0, t_0]$ as in Corollary 3.

First, we choose a suitable cutoff function. Let $\psi(x)$ be a C^3-function such that

$$\psi(x) = 0 \quad \text{for } x \leq -1, \qquad \psi(x) = 1 \quad \text{for } x > 1, \qquad \psi' \geqslant 0,$$

and for some constant $C > 0$,

$$\big(\psi'(x)\big)^2 \leqslant C\psi(x), \qquad \big(\psi''(x)\big)^2 \leqslant C\psi'(x), \qquad \text{for all } x \in \mathbb{R},$$
$$\psi'(x) \neq 0 \qquad \text{for } x \in (-1, 1).$$

(Consider $\psi(x) = (1/16)(1+x)^4$ for $x \geqslant -1$ close to -1, and consider $\psi(x) = 1 - (1/16)(1-x)^4$ for $x \leqslant 1$ close to 1.)

Let $\sigma_k \in \mathbb{R}$ be such that for all $k = 2, \ldots, K$,

$$v_{k-1} < \sigma_k < v_k,$$

and set

$$\overline{x}_k^0 = \frac{x_{k-1}(0) + x_k(0)}{2}.$$

For

$$a = \frac{L^2}{64},$$

let

$$\psi_k(t,x) = \psi\Big(\frac{x - \overline{x}_k^0 - \sigma_k t}{\sqrt{t+a}}\Big) \quad (k = 2, \ldots, K), \qquad \psi_1 \equiv 1, \qquad \psi_{K+1} \equiv 0. \quad (3.3)$$

We define, for $k = 2, \ldots, K$,

$$\mathscr{I}_k(t) = \frac{\sigma_k}{2} \int |u(t,x)|^2 \psi_k(t,x)\,dx - \operatorname{Im} \int \partial_x u(t,x)\,\overline{u}(t,x)\,\psi_k(t,x)\,dx. \quad (3.4)$$

From Lemma 3.2 applied on a Galilean transformation of the solution $u(t)$, we claim that for any $k \geqslant 2$, the variation in time of $\mathscr{I}_k(t)$ is controlled from above in the following sense.

PROPOSITION 3.1

Let $u(t)$ be a solution of (1.1) satisfying the assumptions of Corollary 3 on $[0, t_0]$. There exist $L_2, \alpha_2, \theta_2, C_2 > 0$ such that if $L > L_2$, $0 < \alpha < \alpha_2$, then for any $k = 2, \ldots, K$, $t \in [0, t_0]$,

$$\mathscr{I}_k(t) - \mathscr{I}_k(0) \leqslant \frac{C_2}{L} \sup_{t' \in [0,t]} \|\varepsilon(t')\|_{L^2}^2 + C_2 e^{-\theta_2 L}.$$

Before proving Proposition 3.1, we recall the following technical result from Merle [16].

LEMMA 3.3
Let $w = w(x) \in H^1(\mathbb{R})$, and let $h = h(x) \geqslant 0$ be a C^2 bounded function such that $\sqrt{h}$ is of class C^1 and satisfies $(h')^2 \leqslant Ch$. Then

$$\int |w|^6 h \leqslant 8\Big(\int_{\operatorname{supp} h} |w|^2\Big)^2 \Big[\int |w'|^2 h + \int |w|^2 \frac{(h')^2}{h}\Big], \tag{3.5}$$

where supp h *denotes the support of* h.

Proof
We repeat the proof from [16, Lem. 6]. First, we have

$$\int |w|^6 h \leqslant \|w^2\sqrt{h}\|_{L^\infty}^2 \int_{\operatorname{supp} h} |w|^2.$$

Next,

$$|w(x)|^2\sqrt{h(x)} \leqslant 2\int_{-\infty}^{x} |w||w'|\sqrt{h} + \frac{1}{2}\int_{-\infty}^{x} |w|^2 \frac{|h'|}{\sqrt{h}}.$$

Thus, using the Cauchy-Schwarz inequality,

$$\|w^2\sqrt{h}\|_{L^\infty} \leqslant 2\Big(\int |w'|^2 h\Big)^{1/2}\Big(\int_{\operatorname{supp} h} |w|^2\Big)^{1/2} + \frac{1}{2}\Big(\int |w|^2\frac{(h')^2}{h}\Big)^{1/2}\Big(\int_{\operatorname{supp} h} |w|^2\Big)^{1/2},$$

which completes the proof of Lemma 3.3. □

Proof of Proposition 3.1
Set

$$0 < \theta_2 = \frac{1}{16}\min\Big(\sigma_2 - v_1, v_2 - \sigma_2, \dots, \sigma_K - v_{K-1}, v_K - \sigma_K, \frac{\sqrt{\omega_1^0}}{4}, \dots, \frac{\sqrt{\omega_K^0}}{4}\Big) < \frac{\theta_1}{16}. \tag{3.6}$$

Fix $k = 2, \dots, K$. For $t \in [0, t_0]$, $x \in \mathbb{R}$, let

$$z(t,x) = z_k(t,x) = u(t, x + \overline{x}_k^0 + \sigma_k t)e^{-\mathrm{i}(\sigma_k/2)(x+\overline{x}_k^0+\sigma_k t/2)}. \tag{3.7}$$

The function $z(t,x)$ is also a solution of (1.1) by the translation and Galilean (see (1.5)) invariances of the NLS equation. Moreover, we have, by elementary calculations,

$$-\mathscr{I}_k(t) = \operatorname{Im}\int \partial_x z(t)\,\overline{z}(t)\,\psi\Big(\frac{x}{\sqrt{t+a}}\Big).$$

Thus, by Lemma 3.2 applied to $z(t)$, we have

$$\frac{1}{2}\frac{d}{dt}\mathscr{I}_k(t) \leqslant -\frac{3}{4\sqrt{t+a}}\int |\partial_x z|^2\psi'\Big(\frac{x}{\sqrt{t+a}}\Big) + \frac{C}{(t+a)^{3/2}}\int_{|x|<\sqrt{t+a}} |z|^2 + \frac{1}{2\sqrt{t+a}}\int_{|x|<\sqrt{t+a}}\big\{f(|z|^2)|z|^2 - F(|z|^2)\big\}\psi'\Big(\frac{x}{\sqrt{t+a}}\Big) \quad (3.8)$$

since supp $\psi' \subset [-1,1]$, and so $\psi'(x/\sqrt{t+a}\,) = 0$ if $|x| > \sqrt{t+a}$.

To treat the L^2-term and the nonlinear terms, we claim the following.

CLAIM 2
For L_2 large enough and α_2 small enough, for any $t \in [0, t_0]$,

$$\text{if } |x| < \sqrt{t+a}, \text{ then } |z(t,x)|^2 < 1, \quad (3.9)$$

and

$$\int_{|x|<\sqrt{t+a}} |z(t)|^2\,dx < Ce^{-\theta_2(L+\theta_2 t)} + 2\int |\varepsilon(t)|^2. \quad (3.10)$$

CLAIM 3
For L_2 large enough and α_2 small enough, for any $t \in [0, t_0]$,

$$\int_{|x|<\sqrt{t+a}}\big\{f(|z|^2)|z|^2 - F(|z|^2)\big\}\psi'\Big(\frac{x}{\sqrt{t+a}}\Big) \leqslant \int |\partial_x z|^2\psi'\Big(\frac{x}{\sqrt{t+a}}\Big) + \frac{C}{(t+a)}\int |\varepsilon|^2 + Ce^{-\theta_2(L+\theta_2 t)}.$$

We assume Claims 2 and 3, and we finish the proof of Proposition 3.1. After this, we prove Claims 2 and 3.

End of the proof of Proposition 3.1
Inserting Claim 3 and (3.10) into (3.8), we obtain, for all $t \in [0, t_0]$ and for all $t' \in [0, t]$,

$$\frac{d}{dt}\mathscr{I}_k(t') \leqslant \frac{C}{(t'+a)^{3/2}}\sup_{0<\tau<t}\int |\varepsilon(\tau)|^2 + Ce^{-\theta_2(L+\theta_2 t')}$$

for α small enough.

By integration between 0 and t, and since $\sqrt{a} = L/8$, we get

$$\mathscr{I}_k(t) - \mathscr{I}_k(0) \leqslant \frac{C}{L}\sup_{0<\tau<t}\int |\varepsilon(\tau)|^2 + Ce^{-\theta_2 L},$$

which completes the proof of Proposition 3.1, assuming Claims 2 and 3. □

Proof of Claim 2

By the decomposition of Corollary 3 and the definition of z (see (3.7)), it follows that

$$|z(t,x)| = |u(t, x+\overline{x}_k^0+\sigma_k t)| \leqslant |R(t, x+\overline{x}_k^0+\sigma_k t)| + |\varepsilon(t, x+\overline{x}_k^0+\sigma_k t)|. \tag{3.11}$$

We have $\|\varepsilon(t)\|_{L^\infty} \leqslant C\|\varepsilon(t)\|_{H^1} \leqslant C\alpha$, and so for α small enough, we have $\|\varepsilon(t)\|_{L^\infty}^2 \leqslant 1/2$. For the other term, we have, by (2.3) and (2.17),

$$|R(t, x+\overline{x}_k^0+\sigma_k t)| \leqslant \sum_{j=1}^{K} Q_{\omega_j(t)}\big(x+\overline{x}_k^0+\sigma_k t - x_j(t)\big) \leqslant C\sum_{j=1}^{K} e^{-\left(\sqrt{\omega_j^0}/4\right)|x+\overline{x}_k^0+\sigma_k t - x_j(t)|}. \tag{3.12}$$

For $|x| < \sqrt{t+a} \leqslant \sqrt{t}+\sqrt{a} = \sqrt{t}+L/8$, we have

$$|x+\overline{x}_k^0+\sigma_k t - x_j(t)| \geqslant |\overline{x}_k^0+\sigma_k t - x_j(t)| - |x| \geqslant |\overline{x}_k^0+\sigma_k t - x_j(t)| - \sqrt{t} - \frac{L}{8}.$$

If $j \geqslant k$, then $\dot{x}_j(t) \geqslant \dot{x}_k(t) \geqslant v_k - 8\theta_2$ for α small by (2.18), and since $x_j(0) \geqslant x_k(0)$, we have $x_j(t) \geqslant x_k(0) + v_k t - 8\theta_2 t$. Since $\sigma_k \leqslant v_k - 16\theta_2$, we have, by $\overline{x}_k^0 = (x_k(0)+x_{k-1}(0))/2$,

$$|\overline{x}_k^0+\sigma_k t - x_j(t)| = x_j(t) - \overline{x}_k^0 - \sigma_k t \geqslant \frac{x_k(0)-x_{k-1}(0)}{2} + 8\theta_2 t,$$

and so for $|x| < \sqrt{t+a}$,

$$|x+\overline{x}_k^0+\sigma_k t - x_j(t)| \geqslant \frac{x_k(0)-x_{k-1}(0)}{2} + 8\theta_2 t - \sqrt{t} - \frac{L}{8} \geqslant 8\theta_2 t - \sqrt{t} + \frac{L}{8}$$

since $(x_k(0)-x_{k-1}(0))/2 > L/4$ by (2.17). Choose $L > 4/\theta_2$; then

$$|x+\overline{x}_k^0+\sigma_k t - x_j(t)| \geqslant 8\theta_2 t - \sqrt{t} + \frac{L}{8} \geqslant 4\theta_2 t + \frac{L}{16} + \left(4\theta_2 t - \sqrt{t} + \frac{L}{16}\right) \geqslant 4\theta_2 t + \frac{L}{16}.$$

Doing the same for $1 \leqslant j \leqslant k-1$, we obtain, for $|x| < \sqrt{t+a}$,

$$\sum_{j=1}^{K} e^{-(\sqrt{\omega_j^0}/4)|x+\overline{x}_k^0+\sigma_k t - x_j(t)|} \leqslant Ce^{-16\theta_2(4\theta_2 t + L/16)} \leqslant Ce^{-\theta_2 L}.$$

Thus, for $|x| < \sqrt{t+a}$, for L large enough,

$$|R(t, x+\overline{x}_k^0+\sigma_k t)| \leqslant Ce^{-16\theta_2(4\theta_2 t + L/16)} \leqslant Ce^{-\theta_2 L} \leqslant \frac{1}{2}. \tag{3.13}$$

Therefore, for L large enough, for $|x| < \sqrt{t+a}$, we obtain $|z(t,x)|^2 \leqslant 1$, which proves (3.9).

The proof of (3.10) is the same. From (3.11), we have

$$|z(t,x)|^2 \leqslant 2|R(t, x+\overline{x}_k^0+\sigma_k t)|^2 + 2|\varepsilon(t, x+\overline{x}_k^0+\sigma_k t)|^2, \tag{3.14}$$

and (3.13) then yields (3.10). Thus, Claim 2 is proved. □

Proof of Claim 3

From (A1) (see the statement of Theorem 1), we claim that there exists $C > 0$ such that

$$\text{if } 0 \leqslant s < 1, \text{ then } f(s^2)s^2 - F(s^2) \leqslant Cs^6. \tag{3.15}$$

Indeed, we have $(f(r)r - F(r))' = f'(r)r \leqslant Cr^2$ by (A1), and so we obtain $f(r)r - F(r) \leqslant Cr^3$ for $0 < r < 1$.

Using (3.9) and (3.15), we obtain

$$\int \big\{f(|z|^2)|z|^2 - F(|z|^2)\big\}\psi'\Big(\frac{x}{\sqrt{t+a}}\Big) \leqslant C\int |z|^6\psi'\Big(\frac{x}{\sqrt{t+a}}\Big).$$

Applying Lemma 3.3, we have

$$\begin{aligned}&\int |z|^6\psi'\Big(\frac{x}{\sqrt{t+a}}\Big)\\ &\quad\leqslant 8\Big(\int_{|x|<\sqrt{t+a}} |z|^2\Big)^2\Big[\int |\partial_x z|^2\psi'\Big(\frac{x}{\sqrt{t+a}}\Big) + \frac{1}{t+a}\int |z|^2\frac{(\psi'')^2}{\psi'}\Big(\frac{x}{\sqrt{t+a}}\Big)\Big].\end{aligned}$$

By (3.10), $(\psi'')^2 \leqslant C\psi'$, and for α and $1/L$ small enough, we obtain

$$\begin{aligned}&\int \big\{f(|z|^2)|z|^2 - F(|z|^2)\big\}\psi'\Big(\frac{x}{\sqrt{t+a}}\Big)\\ &\quad\leqslant \frac{1}{2}\int |\partial_x z|^2\psi'\Big(\frac{x}{\sqrt{t+a}}\Big) + \frac{C}{t+a}\Big(\int_{|x|\leqslant\sqrt{t+a}} |z|^2\Big)^3\\ &\quad\leqslant \frac{1}{2}\int |\partial_x z|^2\psi'\Big(\frac{x}{\sqrt{t+a}}\Big) + \frac{C}{t+a}\Big(\int |\varepsilon|^2\Big)^3 + Ce^{-\theta_2(L+\theta_2 t)}.\end{aligned}$$

But for α small enough, we have $(\int|\varepsilon|^2)^3 \leqslant \int|\varepsilon|^2$, and thus Claim 3 is proved. □

3.3. Transformation into a localized monotonicity result

We establish in this section a monotonicity result that is directly suitable for the first part of the proof of the stability result.

In view of the definition of $\mathscr{G}(u(t))$ in (2.21), it is relevant to introduce and study the variation in time of a functional that, locally around the kth solitary wave, is equal to

$$\Big(\omega_k(0) + \frac{v_k^2}{4}\Big)\int |u(t)|^2 - v_k\operatorname{Im}\int \partial_x u(t)\,\overline{u}(t).$$

For values of $\{\sigma_k\}_{k=2,\dots,K}$ such that $v_{k-1} < \sigma_k < v_k$ to be chosen later, and for the functions ψ_k defined in (3.3), we set, for all $k \in \{1, \dots, K\}$,

$$\varphi_k(t,x) = \psi_k(t,x) - \psi_{k+1}(t,x). \tag{3.16}$$

Note that $\varphi_k \equiv 1$ around the solitary wave k, and note that $\varphi_k \equiv 0$ around the solitary waves j for $j \neq k$, in particular, $\varphi_k(x_j(t)) = 0$.

We set

$$\mathscr{J}(t) = \sum_{k=1}^{K} \Big\{ \Big(\omega_k(0) + \frac{v_k^2}{4}\Big) \int |u(t)|^2 \varphi_k(t) - v_k \operatorname{Im} \int \partial_x u(t)\, \overline{u}(t) \varphi_k(t) \Big\}, \tag{3.17}$$

which is the sum of the energy functional related to the speed of the traveling wave localized around each traveling wave. We claim the following result.

PROPOSITION 3.2
For $k = 2, \dots, K$, let

$$\sigma_k = 2\,\frac{\omega_k(0) - \omega_{k-1}(0)}{v_k - v_{k-1}} + \frac{v_k + v_{k-1}}{2}. \tag{3.18}$$

Assume (A3); then, for α and $1/L$ small enough,

$$v_{k-1} < \sigma_k < v_k. \tag{3.19}$$

Moreover, taking these values of σ_k in the definition of $\mathscr{I}_k(t)$ and $\mathscr{J}(t)$, the following is true:

$$\mathscr{J}(t) = \int \Big[\Big(\omega_1(0) + \frac{v_1^2}{4}\Big) |u(t)|^2 - v_1 \operatorname{Im}\big(\partial_x u(t)\, \overline{u}(t)\big) \Big] + \sum_{k=2}^{N} (v_k - v_{k-1}) \mathscr{I}_k(t).$$

Remark. If the parameters σ_k do not satisfy (3.19), then the monotonicity property on $\mathscr{I}_k(t)$ breaks down, and it is unclear whether the stability property holds. Note also that the set of σ_k which yields the monotonicity formula is open.

Proof of Proposition 3.2
Let us check the first assertion concerning the values of σ_k. We have, for $k = 2, \dots, K$, by assumption (A3) since $\omega_k(0)$ is close to ω_k^0 (see (2.17)),

$$\begin{aligned} \sigma_k - v_{k-1} &= 2\frac{\omega_k(0) - \omega_{k-1}(0)}{v_k - v_{k-1}} + \frac{v_k - v_{k-1}}{2} \\ &= \frac{4(\omega_k(0) - \omega_{k-1}(0)) - (v_k - v_{k-1})^2}{2(v_k - v_{k-1})} > 0, \end{aligned}$$

and similarly,

$$v_k - \sigma_k = -2\frac{\omega_k(0) - \omega_{k-1}(0)}{v_k - v_{k-1}} + \frac{v_k - v_{k-1}}{2} > 0$$

for α small enough. Thus $v_{k-1} < \sigma_k < v_k$.

We now justify the second assertion, concerning $\mathcal{J}$. By the Abel resummation argument, we have

$$\begin{aligned}
\mathcal{J}(t) &= \sum_{k=1}^{K}\Big(\omega_k(0) + \frac{v_k^2}{4}\Big)\int |u|^2\varphi_k - \sum_{k=1}^{K} v_k \mathrm{Im}\int \partial_x u\,\overline{u}\varphi_k \\
&= \sum_{k=1}^{K}\int \Big[\Big(\omega_k(0) + \frac{v_k^2}{4}\Big)|u|^2 - v_k\mathrm{Im}(\partial_x u\,\overline{u})\Big](\psi_k - \psi_{k+1}) \\
&= \int\Big[\Big(\omega_1(0) + \frac{v_1^2}{4}\Big)|u|^2 - v_1\mathrm{Im}(\partial_x u\,\overline{u})\Big]\psi_1 \\
&\quad - \int\Big[\Big(\omega_K(0) + \frac{v_K^2}{4}\Big)|u|^2 - v_K\mathrm{Im}(\partial_x u\,\overline{u})\Big]\psi_{K+1} \\
&\quad + \sum_{k=2}^{K}\int\Big[\Big(\omega_k(0) + \frac{v_k^2}{4} - \omega_{k-1}(0) - \frac{v_{k-1}^2}{4}\Big)|u|^2 - (v_k - v_{k-1})\mathrm{Im}(\partial_x u\,\overline{u})\Big]\psi_k.
\end{aligned}$$

But by the definition of σ_k, we have

$$\omega_k(0) + \frac{v_k^2}{4} - \omega_{k-1}(0) - \frac{v_{k-1}^2}{4} = \omega_k(0) - \omega_{k-1}(0) + \frac{1}{4}(v_k^2 - v_{k-1}^2) = \frac{1}{2}(v_k - v_{k-1})\sigma_k,$$

and we have chosen

$$\psi_1 \equiv 1, \qquad \psi_{K+1} \equiv 0.$$

Thus, we obtain

$$\mathcal{J}(t) = \int\Big[\Big(\omega_1(0) + \frac{v_1^2}{4}\Big)|u(t)|^2 - v_1\mathrm{Im}\big(\partial_x u(t)\,\overline{u}(t)\big)\Big] + \sum_{k=2}^{N}(v_k - v_{k-1})\mathcal{I}_k(t).$$

Thus, Proposition 3.2 is proved. □

In particular, as a corollary of Propositions 3.2 and 3.1, we obtain the following monotonicity result on $\mathcal{J}$.

COROLLARY 4
For the values of σ_k given in Proposition 3.2, for all $t \in [0, t_0]$,

$$\mathscr{J}(t) - \mathscr{J}(0) \leqslant \frac{C}{L} \sup_{t' \in [0,t]} \|\varepsilon(t')\|_{L^2}^2 + Ce^{-\theta_2 L}.$$

Proof
The first term in the expression of $\mathscr{J}(t)$ given in Proposition 3.2 is constant in time, and the next terms satisfy Lemma 3.1. (Recall that $v_k - v_{k-1} > 0$.) Thus Corollary 4 is proved. □

From now on, we fix the values of σ_k, as in Proposition 3.2.

3.4. Monotonicity results for different lines
In Section 3.3, we chose a set of values of σ_k related to (v_k) and $(\omega_k(0))$. For use in Section 4, we now introduce quantities $\mathscr{I}_k(t)$ for other values of σ_k. Indeed, we define σ_k^+ and σ_k^- as follows. For $k = 2, \dots, K$,

$$\sigma_k^+ = \frac{v_k + \sigma_k}{2}, \qquad \sigma_k^- = \frac{v_{k-1} + \sigma_k}{2}. \tag{3.20}$$

Note that $v_{k-1} < \sigma_k^- < \sigma_k < \sigma_k^+ < v_k$ for any $k = 2, \dots, K$, and thus, we can define

$$\psi_k^\pm(t, x) = \psi\Big(\frac{x - \overline{x}_k^0 - \sigma_k^\pm t}{\sqrt{t + a}}\Big) \quad (k = 2, \dots, N) \tag{3.21}$$

and

$$\mathscr{I}_k^\pm(t) = \frac{\sigma_k^\pm}{2} \int |u|^2 \psi_k^\pm(t) - \operatorname{Im} \int \partial_x u \, \overline{u} \, \psi_k^\pm(t). \tag{3.22}$$

By the same proof as that of Proposition 3.1, we have the following result.

COROLLARY 5
For $\theta_3 > 0$ and for all $t \in [0, t_0]$,

$$\mathscr{I}_k^\pm(t) - \mathscr{I}_k^\pm(0) \leqslant \frac{C}{L} \sup_{t' \in [0,t]} \|\varepsilon(t')\|_{H^1}^2 + Ce^{-\theta_3 L}. \tag{3.23}$$

4. Proof of the stability result

In this section, we prove the stability result, that is, Theorem 1.

We fix $\theta_0 = \min(\theta_2, \theta_3)$ where θ_2 and θ_3 are defined in Proposition 3.1 and Corollary 5. For $A_0, L, \alpha > 0$, we define

$$\mathcal{V}_{A_0}(\alpha, L) = \Big\{u \in H^1; \inf_{y_k > y_{k-1} + L\delta_k \in \mathbb{R}} \Big\| u(t,\cdot) - \sum_{k=1}^{K} Q_{\omega_k^0}(\cdot - y_k) e^{\mathrm{i}((1/2)v_k x + \delta_k)} \Big\|_{H^1} \leqslant A_0(\alpha + e^{-(\theta_0/2)L}) \Big\}.$$

Let ω_k^0, v_k, x_k^0, and γ_k^0 be defined as in the statement of Theorem 1. We claim that Theorem 1 is a consequence of the following proposition.

PROPOSITION 4.1 (Reduction of the problem)
There exist $A_0 > 2$, $L_0 > 0$, and $\alpha_0 > 0$ such that for all $u_0 \in H^1$, if

$$\Big\| u_0 - \sum_{k=1}^{K} Q_{\omega_k^0}(\cdot - x_k^0) e^{\mathrm{i}((1/2)v_k x + \gamma_k^0)} \Big\|_{H^1} \leqslant \alpha, \tag{4.1}$$

where $L > L_0$, $0 < \alpha < \alpha_0$, and $x_k^0 - x_{k-1}^0 > L$, and if for some $t^* > 0$,

$$\forall t \in [0, t^*], \quad u(t) \in \mathcal{V}_{A_0}(\alpha, L), \tag{4.2}$$

then

$$\forall t \in [0, t^*], \quad u(t) \in \mathcal{V}_{A_0/2}(\alpha, L). \tag{4.3}$$

Before proving Proposition 4.1, we check that it implies Theorem 1. Indeed, suppose that u_0 satisfies the assumptions of Theorem 1. Let $u(t)$ be the solution of (1.1); then, by continuity of $u(t)$ in H^1, there exists $\tau > 0$ such that for any $0 \leqslant t \leqslant \tau$, $u(t) \in \mathcal{V}_{A_0}(\alpha, L)$. Let

$$t^* = \sup\big\{t \geqslant 0,\ u(t') \in \mathcal{V}_{A_0}(\alpha, L),\ \forall t' \in [0, t]\big\}. \tag{4.4}$$

Assume for the sake of contradiction that t^* is not $+\infty$; then by Proposition 4.1, for all $t \in [0, t^*]$, $u(t) \in \mathcal{V}_{A_0/2}(\alpha, L)$. Since $u(t)$ is continuous in H^1, there exist $\tau' > 0$ such that for all $t \in [0, t^* + \tau']$, $u(t) \in \mathcal{V}_{2A_0/3}(\alpha, L)$, which contradicts the definition of t^*. Therefore, $t^* = +\infty$, and (1.14) in Theorem 1 follows. The estimates (1.15) follow from the proof of Proposition 4.1.

The rest of this section is thus devoted to the proof of Proposition 4.1. Note that in all of the proof, we consider $u(t)$ for $t \in [0, t^*]$.

Proof
Let $A_0 > 0$, $L_0 = L_0(A_0)$, and $\alpha_0 = \alpha_0(A_0) > 0$ be chosen later. Recall that $\theta_0 = \min(\theta_2, \theta_3)$ is fixed independently of A_0.

(1) *Decomposition of the solution around K solitary waves.* First, since for all $t \in [0, t^*]$, $u(t) \in \mathscr{V}_{A_0}(\alpha, L)$, by choosing $L_0 = L_0(A_0, \theta_0)$ and $\alpha_0 = \alpha_0(A_0) > 0$ small enough (so that $A_0(\alpha_0 + e^{-\theta_0 L})$ is small), we can apply Corollary 3 to $u(t)$ on the time interval $[0, t^*]$. It follows that there exist unique C^1-functions $\omega_k : [0, t^*] \to (0, +\infty)$, $x_k, \gamma_k : [0, t^*] \to \mathbb{R}$ such that if we set

$$\varepsilon(t, x) = u(t, x) - R(t, x), \tag{4.5}$$

where

$$R(t, x) = \sum_{k=1}^{K} R_k(t, x), \qquad R_k(t, x) = Q_{\omega_k(t)}\big(x - x_k(t)\big)e^{\mathrm{i}((1/2)v_k x + \gamma_k(t))}, \tag{4.6}$$

then $\varepsilon(t)$ satisfies, for all $k = 1, \ldots, K$ and for all $t \in [0, t^*]$,

$$\mathrm{Re}\big(\varepsilon(t), R_k(t)\big) = \mathrm{Im}\big(\varepsilon(t), R_k(t)\big) = \mathrm{Re}\big(\varepsilon(t), \partial_x R_k(t)\big) = 0 \tag{4.7}$$

and, for all $t \in [0, t^*]$,

$$\begin{aligned}\|\varepsilon(t)\|_{H^1} + \sum_{k=1}^{K} |\omega_k(t) - \omega_k^0| + |\dot{\omega}_k(t)| + |\dot{x}_k(t) - v_k| + \left|\dot{\gamma}_k(t) - \Big(\omega_k(t) - \frac{v_k^2}{4}\Big)\right| \\ \leqslant C_1 A_0(\alpha + e^{-(\theta_0/2)L}).\end{aligned} \tag{4.8}$$

Observe that in (4.8), the estimate depends on A_0. However, directly from assumption (4.1) on u_0 and Lemma 2.4 applied to u_0, we have

$$\|\varepsilon(0)\|_{H^1} + \sum_{k=1}^{K} |\omega_k(0) - \omega_k^0| \leqslant C_1\alpha, \tag{4.9}$$

where C_1 does not depend on A_0. This is a main point in the proof of Proposition 4.1.

(2) *Introduction of a functional adapted to the stability problem for K solitary waves.* We define

$$\mathscr{G}_K(t) = E\big(u(t)\big) + \mathscr{J}(t), \tag{4.10}$$

where $\mathscr{J}(t)$ is defined in (3.17) with the (σ_k) as in (3.18).

The analogue of Lemma 2.5 for the case of multisolitary wave solutions is the following result.

PROPOSITION 4.2 (Expansion of $\mathcal{G}_K$ with respect to parameters)
For all $t \in [0, t^*]$, *we have*

$$\mathcal{G}_K(t) = \sum_{k=1}^{K} \mathcal{F}_{\omega_k(0)}(Q_{\omega_k(0)}) + H_K\big(\varepsilon(t), \varepsilon(t)\big) + \sum_{k=1}^{K} O\big(|\omega_k(t)-\omega_k(0)|^2\big) + \|\varepsilon(t)\|_{H^1}^2 \beta\big(\|\varepsilon(t)\|_{H^1}\big) + O(e^{-\theta_0(L+\theta_0 t)}) \tag{4.11}$$

with $\beta(\epsilon) \to 0$ *as* $\epsilon \to 0$, *where*

$$H_K(\varepsilon, \varepsilon) = \int |\partial_x \varepsilon|^2 - \sum_{k=1}^{K} \int \big(f(|R_k|^2)|\varepsilon|^2 + 2f'(|R_k|^2)[\mathrm{Re}(\overline{R}_k \varepsilon)]^2\big) + \sum_{k=1}^{K} \Big\{\Big(\omega_k(t) + \frac{v_k^2}{4}\Big) \int |\varepsilon|^2 \varphi_k(t) - v_k \,\mathrm{Im} \int \partial_x \varepsilon \,\overline{\varepsilon} \varphi_k(t)\Big\}.$$

Remark. As one can expect from the single solitary wave case, the terms $O(|\omega_k(t) - \omega_k(0)|^2)$ in (4.11) have the wrong sign (i.e., they are nonpositive).

The proof of Proposition 4.2, similar to that of Lemma 2.5, is given in Appendix C.

The analogue of Lemma 2.6 in the multisolitary wave case is the following result.

LEMMA 4.1 (Coercivity of H_K)
There exists $\lambda_K > 0$ *such that*

$$H_K\big(\varepsilon(t), \varepsilon(t)\big) \geqslant \lambda_K \|\varepsilon(t)\|_{H^1}^2.$$

Proof
As in the proof of Lemma 2.6, the proof of Lemma 4.1 is based on Lemma 2.2, concerning the operators $\mathcal{L}_\omega^+$ and $\mathcal{L}_\omega^-$. It also requires localization arguments similar to those used in [15]. We refer to Appendix B for the proof. □

As in the proof of the stability result in Section 2.2, we now proceed in two steps; first, we control the size of $\varepsilon(t)$ in H^1, and second, we check that for any k, $|\omega_k(t) - \omega_k(0)|$ is quadratic in $|\varepsilon(t)|$. Both steps use the monotonicity properties of Section 3.

(3) *Energetic control of* $\|\varepsilon(t)\|_{H^1}$. We claim the following lemma from the conservation laws and the monotonicity of $\mathcal{J}(t)$.

LEMMA 4.2
For all $t \in [0, t^*]$,

$$\|\varepsilon(t)\|_{H^1}^2 + \sum_{k=2}^{K} |\mathscr{I}_k(t) - \mathscr{I}_k(0)| \leqslant \frac{C}{L} \sup_{t' \in [0,t]} \|\varepsilon(t')\|_{H^1}^2 + C\|\varepsilon(0)\|_{H^1}^2 + C\sum_{k=1}^{K} |\omega_k(t) - \omega_k(0)|^2 + Ce^{-\theta_0 L}. \tag{4.12}$$

Proof
First, we write (4.11) at $t > 0$ and at $t = 0$:

$$E\big(u(t)\big) + \mathscr{J}(t) = \sum_{k=1}^{K} \mathscr{F}_{\omega_k(0)}(Q_{\omega_k(0)}) + H_K\big(\varepsilon(t), \varepsilon(t)\big) + \sum_{k=1}^{K} O\big(|\omega_k(t) - \omega_k(0)|^2\big) + \|\varepsilon(t)\|_{H^1}^2 \beta\big(\|\varepsilon(t)\|_{H^1}\big) + O(e^{-\theta_0(L+\theta_0 t)}),$$

and

$$E\big(u(0)\big) + \mathscr{J}(0) = \sum_{k=1}^{K} \mathscr{F}_{\omega_k(0)}(Q_{\omega_k(0)}) + H_K\big(\varepsilon(0), \varepsilon(0)\big) + \|\varepsilon(t)\|_{H^1}^2 \beta\big(\|\varepsilon(t)\|_{H^1}\big) + O(e^{-\theta_0 L}).$$

We take the difference of these two equalities. Since $E(u(t)) = E(u(0))$ and $H_K(\varepsilon(0), \varepsilon(0)) \leqslant C\|\varepsilon(0)\|_{H^1}^2$, we obtain

$$H_K\big(\varepsilon(t), \varepsilon(t)\big) \leqslant \big(\mathscr{J}(t) - \mathscr{J}(0)\big) + C\|\varepsilon(0)\|_{H^1}^2 + C\sum_{k=1}^{K} |\omega_k(t) - \omega_k(0)|^2 + C\|\varepsilon(t)\|_{H^1}^2 \beta\big(\|\varepsilon(t)\|_{H^1}\big) + Ce^{-\theta_0 L}.$$

Using Proposition 3.2, the conservation of $\int |u(t)|^2$ and $\operatorname{Im}\int \partial_x u\,\overline{u}$, and Lemma 4.1, we obtain, for α small enough,

$$\frac{\lambda_K}{2}\|\varepsilon(t)\|_{H^1}^2 \leqslant \sum_{k=2}^{K} (v_k - v_{k-1})\big(\mathscr{I}_k(t) - \mathscr{I}_k(0)\big) + C\|\varepsilon(0)\|_{H^1}^2 + C\sum_{k=1}^{K} |\omega_k(t) - \omega_k(0)|^2 + Ce^{-\theta_0 L}, \tag{4.13}$$

where $v_k - v_{k-1} > \theta_0 > 0$.

By Proposition 3.1, we have, for all $k = 2, \dots, K$,

$$\mathscr{I}_k(t) - \mathscr{I}_k(0) \leqslant \frac{C}{L} \sup_{t' \in [0,t]} \|\varepsilon(t')\|_{H^1}^2 + Ce^{-\theta_0 L}, \tag{4.14}$$

and thus, inserting (4.14) into (4.13), we obtain

$$\frac{\lambda_K}{2} \|\varepsilon(t)\|_{H^1}^2 \leqslant \frac{C}{L} \sup_{t' \in [0,t]} \|\varepsilon(t')\|_{H^1}^2 + C\|\varepsilon(0)\|_{H^1}^2 + C \sum_{k=1}^{K} |\omega_k(t) - \omega_k(0)|^2 + Ce^{-\theta_0 L}.$$

Using this and (4.14) and (4.13) again, we finally obtain

$$\theta_0 \sum_{k=2}^{K} |\mathscr{I}_k(t) - \mathscr{I}_k(0)| \leqslant \frac{C}{L} \sup_{t' \in [0,t]} \|\varepsilon(t')\|_{H^1}^2 + C\|\varepsilon(0)\|_{H^1}^2 + C \sum_{k=1}^{K} |\omega_k(t) - \omega_k(0)|^2 + Ce^{-\theta_0 L}.$$

This proves Lemma 4.2. □

(4) *Quadratic control of* $|\omega_k(t) - \omega_k(0)|$. We claim the following result from the monotonicity of $\mathscr{I}_k(t)$ and $\mathscr{I}_k^{\pm}(t)$.

LEMMA 4.3
For all $t \in [0, t^*]$,

$$\sum_{k=1}^{K} |\omega_k(t) - \omega_k(0)| \leqslant C \sup_{t' \in [0,t]} \|\varepsilon(t')\|_{H^1}^2 + Ce^{-\theta_0 L}. \tag{4.15}$$

To prove Lemma 4.3 in the case where $K = 1$, one only has to use the L^2-norm conservation. In the case of several solitary waves, the idea is to use the monotonicity results of Proposition 3.1. Recall that in Step 1, we use $\mathscr{I}_k$ for the value of σ_k defined in Proposition 3.2. This is necessary to relate $\mathscr{J}(t)$ to the $\mathscr{I}_k(t)$ as in Proposition 3.2 and to prove Lemma 4.2. Now we use the monotonicity property for different channels (i.e., different but close values of σ_k). Since, in the formula of $\mathscr{I}_k$, there is a weight in front of the L^2-term related to σ_k, different values of σ_k allow us to isolate the local L^2-norm of the solution. (At the first order, we isolate $\omega_k(t)$.) This argument is new compared to the gKdV equations, for which the monotonicity property (related only to L^2-quantities) gives directly a similar result without having to consider several channels.

Consider $\mathscr{I}_k^{\pm}$, defined with the values of $\sigma_k^{\pm}$ introduced in Section 3.4. We claim the following result.

CLAIM 4
The following hold:

$$\left|\mathscr{I}_k^+(t) - \mathscr{I}_k(t) - \frac{v_k - \sigma_k}{4} \sum_{k'=k}^{K} \int Q^2_{\omega_{k'}(t)}\right| \leqslant C\|\varepsilon(t)\|^2_{H^1} + Ce^{-\theta_0 L}, \tag{4.16}$$

$$\left|\mathscr{I}_k(t) - \mathscr{I}_k^-(t) - \frac{\sigma_k - v_{k-1}}{4} \sum_{k'=k}^{K} \int Q^2_{\omega_{k'}(t)}\right| \leqslant C\|\varepsilon(t)\|^2_{H^1} + Ce^{-\theta_0 L}. \tag{4.17}$$

Considering only the differences $\mathscr{I}_k^+(t) - \mathscr{I}_k(t)$ and $\mathscr{I}_k(t) - \mathscr{I}_k^-(t)$ allows us to ignore the term related to momentum in $\mathscr{I}_k(t)$, which is not very handy since it contains a first order term in $\varepsilon(t)$ ($\operatorname{Im} \int \partial_x \overline{R}\, \varepsilon\, \psi_k$), which is not zero by the orthogonality conditions.

Proof
The proof of Claim 4 follows from explicit calculations. We prove (4.16) only; formula (4.17) can be proved in exactly the same way. We have

$$\begin{aligned}
&\mathscr{I}_k^+(t) - \mathscr{I}_k(t) \\
&\quad = \frac{\sigma_k^+}{2} \int |u|^2 \psi_k^+(t) - \operatorname{Im} \int \partial_x u\, \overline{u}\, \psi_k^+(t) - \frac{\sigma_k}{2} \int |u|^2 \psi_k(t) - \operatorname{Im} \int \partial_x u\, \overline{u}\, \psi_k(t) \\
&\quad = \frac{\sigma_k^+ - \sigma_k}{2} \int |u|^2 \psi_k^+(t) - \frac{\sigma_k}{2} \int |u|^2 \big(\psi_k(t) - \psi_k^+(t)\big) + \operatorname{Im} \int \partial_x u\, \overline{u}\, \big(\psi_k(t) - \psi_k^+(t)\big).
\end{aligned}$$

On the one hand,

$$\int |\psi_k(t,x) - \psi_k^+(t,x)| \big(|R(t,x)| + |\partial_x R(t,x)|\big)\, dx \leqslant e^{-\theta_0 L}$$

since $\psi_k^+ - \psi_k \equiv 0$ for $x > x_k(t) - 4\theta_0 t$ and $x < x_{k-1}(t) + 4\theta_0 t$. Therefore,

$$\left|\int |u|^2 \big(\psi_k(t) - \psi_k^+(t)\big)\right| + \left|\operatorname{Im} \int \partial_x u\, \overline{u}\, \big(\psi_k(t) - \psi_k^+(t)\big)\right| \leqslant C\|\varepsilon(t)\|^2_{H^1} + Ce^{-\theta_0 L}.$$

On the other hand, note that $\sigma_k^+ - \sigma_k = (v_k - \sigma_k)/2$, $\psi_k^+ \equiv 1$ for $x > x_k(t) - 4\theta_0 t$, and $\psi_k^+ \equiv 0$ for $x < x_{k-1}(t) + 4\theta_0 t$. Thus, by the orthogonality condition $\operatorname{Re} \int R_{k'}(t) \varepsilon(t) = 0$ and by (C.1) (see Appendix C), we have

$$\left|\int |u|^2 \psi_k^+(t) - \sum_{k'=k}^{K} \int Q^2_{\omega_{k'}(t)}\right| \leqslant C\|\varepsilon(t)\|^2_{H^1} + Ce^{-\theta_0 L}.$$

Thus, Claim 4 is proved. □

Let us now prove Lemma 4.3.

Proof of Lemma 4.3
From Lemma 4.2 and Corollary 5, we have ($L \geqslant 1$)

$$\sum_{k=1}^{K} |\mathscr{I}_k(t) - \mathscr{I}_k(0)| \leqslant C \sup_{t' \in [0,t]} \|\varepsilon(t')\|_{H^1}^2 + C \sum_{k=1}^{K} |\omega_k(t) - \omega_k(0)|^2 + Ce^{-\theta_0 L}, \tag{4.18}$$

and, for all $k = 1, \ldots, K$,

$$\mathscr{I}_k^{\pm}(t) - \mathscr{I}_k^{\pm}(0) \leqslant \frac{C}{L} \sup_{t' \in [0,t]} \|\varepsilon(t')\|_{H^1}^2 + Ce^{-\theta_3 L}. \tag{4.19}$$

Combining (4.16), (4.19), and (4.18), we obtain, for all $k = 2, \ldots, K$,

$$\begin{aligned}
&\sum_{k'=k}^{K} \Big(\int Q_{\omega_k(t)}^2 - \int Q_{\omega_k(0)}^2 \Big) \\
&\quad \leqslant \frac{v_k - \sigma_k}{4} \big[\big(\mathscr{I}_k^+(t) - \mathscr{I}_k(t) \big) - \big(\mathscr{I}_k^+(0) - \mathscr{I}_k(0) \big) \big] + C \sup_{t' \in [0,t]} \|\varepsilon(t')\|_{H^1}^2 + Ce^{-\theta_0 L} \\
&\quad \leqslant \frac{v_k - \sigma_k}{4} \big(\mathscr{I}_k^+(t) - \mathscr{I}_k^+(0) \big) + \frac{v_k - \sigma_k}{4} |\mathscr{I}_k(t) - \mathscr{I}_k(0)| \\
&\qquad + C \sup_{t' \in [0,t]} \|\varepsilon(t')\|_{H^1}^2 + Ce^{-\theta_0 L} \\
&\quad \leqslant C \sup_{t' \in [0,t]} \|\varepsilon(t')\|_{H^1}^2 + C \sum_{k=1}^{K} |\omega_k(t) - \omega_k(0)|^2 + Ce^{-\theta_0 L}.
\end{aligned}$$

Similarly, using (4.17) instead of (4.16), we have, for all $k = 2, \ldots, K$,

$$\begin{aligned}
&-\sum_{k'=k}^{K} \Big(\int Q_{\omega_k(t)}^2 - \int Q_{\omega_k(0)}^2 \Big) \\
&\quad \leqslant \frac{\sigma_{k-1} - v_k}{4} \big[\big(\mathscr{I}_k^-(t) - \mathscr{I}_k(t) \big) - \big(\mathscr{I}_k^-(0) - \mathscr{I}_k(0) \big) \big] + C \sup_{t' \in [0,t]} \|\varepsilon(t')\|_{H^1}^2 + Ce^{-\theta_0 L} \\
&\quad \leqslant \frac{\sigma_{k-1} - v_k}{4} \big(\mathscr{I}_k^-(t) - \mathscr{I}_k^-(0) \big) + \frac{\sigma_{k-1} - v_k}{4} |\mathscr{I}_k(t) - \mathscr{I}_k(0)| \\
&\qquad + C \sup_{t' \in [0,t]} \|\varepsilon(t')\|_{H^1}^2 + Ce^{-\theta_0 L} \\
&\quad \leqslant C \sup_{t' \in [0,t]} \|\varepsilon(t')\|_{H^1}^2 + C \sum_{k=1}^{K} |\omega_k(t) - \omega_k(0)|^2 + Ce^{-\theta_0 L}.
\end{aligned}$$

Therefore, we obtain, for all $k = 2, \dots, K$,

$$\Big|\sum_{k'=k}^{K}\Big(\int Q^2_{\omega_{k'}(t)} - \int Q^2_{\omega_{k'}(0)}\Big)\Big| \leqslant C \sup_{t'\in[0,t]} \|\varepsilon(t')\|^2_{H^1} + C\sum_{k=1}^{K}|\omega_k(t)-\omega_k(0)|^2 + Ce^{-\theta_0 L}. \tag{4.20}$$

By the L^2-norm conservation $\int \big|u(t)^2 = \int |u(0)|^2$ and the orthogonality conditions on ε, we also have directly (without using any monotonicity property)

$$\Big|\sum_{k'=1}^{K}\Big(\int Q^2_{\omega_{k'}(t)} - \int Q^2_{\omega_{k'}(0)}\Big)\Big| \leqslant C \sup_{t'\in[0,t]} \|\varepsilon(t')\|^2_{H^1} + Ce^{-\theta_0 L}, \tag{4.21}$$

which means that (4.20) is also true for $k = 1$.

Recall that we assume (A2):

$$\frac{d}{d\omega}\int Q^2_\omega(x)\,dx\Big|_{\omega=\omega_k^0} > 0,$$

and $\omega_k(t)$, $\omega_k(0)$ are close to ω_k^0 by (4.8). Thus, for any $k = 1, \dots, K$,

$$|\omega_k(t) - \omega_k(0)| \leqslant C\Big|\int Q^2_{\omega_k(t)} - \int Q^2_{\omega_k(0)}\Big|. \tag{4.22}$$

From (4.20) and (4.22), for $k = K$, we obtain directly

$$|\omega_K(t) - \omega_K(0)| \leqslant C \sup_{t'\in[0,t]} \|\varepsilon(t')\|^2_{H^1} + +C\sum_{k=1}^{K}|\omega_k(t) - \omega_k(0)|^2 + Ce^{-\theta_0 L}.$$

Then, by a backward induction argument on k, using (4.20) and (4.22), we obtain, for any $k = K-1, \dots, 1$,

$$|\omega_k(t) - \omega_k(0)| \leqslant C \sup_{t'\in[0,t]} \|\varepsilon(t')\|^2_{H^1} + C\sum_{k=1}^{K}|\omega_k(t) - \omega_k(0)|^2 + Ce^{-\theta_0 L}.$$

And thus, for any $k = 1, \dots, K$,

$$|\omega_k(t) - \omega_k(0)| \leqslant C \sup_{t'\in[0,t]} \|\varepsilon(t')\|^2_{H^1} + Ce^{-\theta_0 L}.$$

This proves Lemma 4.3. □

(5) *Conclusion of the proof of Proposition 4.1*. Combining the conclusions of Lemmas 4.2 and 4.3, we obtain, for all $t \in [0, t^*]$,

$$\|\varepsilon(t)\|^2_{H^1} \leqslant \frac{C}{L}\sup_{t'\in[0,t]} \|\varepsilon(t')\|^2_{H^1} + C\sup_{t'\in[0,t]}\Big[\beta\big(\|\varepsilon(t')\|_{H^1}\big)\|\varepsilon(t')\|^2_{H^1}\Big]$$
$$+ C\|\varepsilon(0)\|^2_{H^1} + Ce^{-\theta_0 L}.$$

For α_0 and $1/L_0$ small enough with $L \geqslant L_0$ and $C/L_0 \leqslant 1/4$, we have, for all $t \in [0, t^*]$,

$$\|\varepsilon(t)\|_{H^1}^2 \leqslant \frac{1}{2} \sup_{t' \in [0,t]} \|\varepsilon(t')\|_{H^1}^2 + C\|\varepsilon(0)\|_{H^1}^2 + Ce^{-\theta_0 L},$$

and so, for all $t \in [0, t^*]$,

$$\|\varepsilon(t)\|_{H^1}^2 \leqslant C\|\varepsilon(0)\|_{H^1}^2 + Ce^{-\theta_0 L}.$$

Using (4.15) again, we obtain, for all $t \in [0, t^*]$,

$$\|\varepsilon(t)\|_{H^1}^2 + \sum_{k=1}^{K} |\omega_k(t) - \omega_k(0)| \leqslant C\|\varepsilon(0)\|_{H^1}^2 + Ce^{-\theta_0 L}. \tag{4.23}$$

By (4.9), we thus obtain

$$\|\varepsilon(t)\|_{H^1}^2 + \sum_{k=1}^{K} |\omega_k(t) - \omega_k(0)| + \sum_{k=1}^{K} |\omega_k(0) - \omega_k^0| \leqslant C\alpha^2 + Ce^{-\theta_0 L}, \tag{4.24}$$

where C is independent of A_0.

To conclude the proof, we go back to $u(t)$:

$$\begin{aligned}
&\Big\| u(t) - \sum_{k=1}^{K} Q_{\omega_k^0}\big(x - x_k(t)\big)e^{\mathrm{i}((1/2)v_k + \gamma_k(t))} \Big\|_{H^1} \\
&\leqslant \Big\| u(t) - \sum_{k=1}^{K} R_k(t) \Big\|_{H^1} + \sum_{k=1}^{K} \big\| R_k(t) - Q_{\omega_k^0}\big(x - x(t)\big)e^{\mathrm{i}((1/2)v_k + \gamma_k(t))} \big\|_{H^1} \\
&\leqslant \|\varepsilon(t)\|_{H^1} + C\sum_{k=1}^{K} |\omega_k(t) - \omega_k^0| \leqslant \|\varepsilon(t)\|_{H^1} \\
&\quad + C\sum_{k=1}^{K} |\omega_k(t) - \omega_k(0)| + C\sum_{k=1}^{K} |\omega_k(0) - \omega_k^0| \\
&\leqslant C_1(\alpha + e^{-(\theta_0/2)L}).
\end{aligned}$$

Observe that the constant $C_1 > 0$ here does not depend on A_0. Thus, we can choose $A_0 = 2C_1$, and then for $\alpha_0 = \alpha_0(A_0) > 0$ small enough and for $L_0 = L_0(A_0)$ large enough, we obtain the conclusion of Proposition 4.1. □

5. The two- and three-dimensional cases: Proof of Theorem 2

In this section, we adapt to the two- and three-dimensional cases the arguments developed in Sections 3 and 4 for the one-dimensional case; that is, we prove Theorem 2 with the tools used in the proof of Theorem 1.

Let $d = 2$ or 3, and consider the nonlinear Schrödinger equation

$$\begin{cases} \mathrm{i}\,\partial_t u = -\Delta u - f(|u|^2)u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0) = u_0, \end{cases} \tag{5.1}$$

under the assumptions of Theorem 2 on f. Let $T > 0$, and for $\omega_k^0 > 0$, $v_k, x_k^0 \in \mathbb{R}^d$, $\gamma_k \in \mathbb{R}$, consider K solitary waves $R_k(t,x)$ of the form

$$R_k(t,x) = Q_{\omega_k^0}\big(x - x_k^0 - v_k(t+T)\big)e^{\mathrm{i}((1/2)v_k\cdot x + \gamma_k)}, \tag{5.2}$$

where $Q_{\omega_k^0} > 0$ is a solution of (1.20) and where the parameters (ω_k^0) and (v_k) satisfy the assumptions of Theorem 2. Recall that $Q_{\omega_k^0}$ enjoy exponential decay properties as in (2.3) (see [1]). We assume that (A2$'$) holds for all ω_k^0; that is, each solitary wave considered independently is nonlinearly stable. Moreover, we impose that for any $k \neq k'$, $v_k \neq v_{k'}$. Since all the (v_k) are different, the (x_k^0) being fixed, taking T large enough in (5.2) ensures that the various solitary waves are decoupled for all $t \geqslant 0$.

The proof of Theorem 2 proceeds as follows. First, we observe that without loss of generality, we can assume that the $(v_{k,1})_{k\in\{1,\dots,K\}}$ are all different, and similarly for $(v_{k,2})$ and $(v_{k,3})$. (We denote by $v_{k,j}$ for $j = 1,\dots,d$ the components of the vector v_k.) Second, we give monotonicity results for the case of dimension d which are very similar to the ones introduced in Section 3 for the one-dimensional case. This is where the restriction $d = 2$ or 3 appears. Then, we introduce a functional similar to $\mathcal{J}(t)$ in the case where $d \geqslant 2$. Finally, we conclude the proof as in the one-dimensional case by expanding conservation laws and using the monotonicity results.

5.1. Splitting of the solitary waves

First, we claim the following.

CLAIM 5

Let (v_k) be K vectors of $\mathbb{R}^d$ such that for any $k \neq k'$, $v_k \neq v_{k'}$. Then there exists $(e_1,\dots,e_d)$, an orthonormal basis of $\mathbb{R}^d$ such that for any $k \neq k'$, and for any $j = 1,\dots,d$, $(v_k, e_j) \neq (v_{k'}, e_j)$.

Proof

This is an elementary geometrical property of $\mathbb{R}^d$. For fixed $k \neq k'$, since $v_k \neq v_{k'}$, the set of unitary vectors ($|e_0| = 1$) such that $(e_0, v_k - v_{k'}) = 0$ is of measure zero on the sphere. Therefore, the set of unitary vectors e_0 such that for some $k, k' \in \{1,\dots,K\}$, $k \neq k'$, $(e_0, v_k - v_{k'}) = 0$ is also of measure zero. Thus, we can pick up a set of vectors $(e_1,\dots,e_d)$ satisfying the desired property. □

Without any restriction, we can assume that the direction e_1 given by Claim 5 is x_1 since equation (1.18) is invariant by rotation. Therefore, we restrict ourselves to the case where the following holds:

$$\text{for any } k \neq k' \text{ and for any } j = 1, \ldots, d, \quad v_{k,j} \neq v_{k',j}.$$

Without loss of generality, we can also assume that the (v_k) are sorted by increasing $v_{k,1}$; that is,

$$v_{1,1} < v_{2,1} < \cdots < v_{K,1}. \tag{5.3}$$

Then, for $j = 2, \ldots, d$, the $(v_{k,j})$ are not necessarily sorted by increasing values. However, since they are all different, there exists a one-to-one mapping $\phi_j : \{1, \ldots, K\} \to \{1, \ldots, K\}$ such that

$$v_{\phi_j(1),j} < \cdots < v_{\phi_j(K),j}. \tag{5.4}$$

5.2. Reduction of the proof of Theorem 2

As for the proof of Theorem 1, we first reduce the proof of Theorem 2. For $A_0, L, \alpha > 0$, we define

$$\begin{aligned}&\mathscr{V}_{A_0}(\alpha, L)\\ &= \Big\{u \in H^1 \text{ s.t. } \exists\, y_k \in \mathbb{R}^d, \delta_k \in \mathbb{R} \text{ s.t. } y_{k,1} > y_{k-1,1} + L,\ y_{\phi_j(k),j} > y_{\phi_j(k-1),j} + L,\\ &\qquad \Big\| u(t,\cdot) - \sum_{k=1}^{K} Q_{\omega_k^0}(\cdot - y_k) e^{\mathrm{i}((1/2)v_k.x + \delta_k)} \Big\|_{H^1} \leqslant A_0(\alpha + e^{-(\theta_0/2)L}) \Big\}.\end{aligned}$$

The following proposition implies Theorem 2 for $T_0 > 0$ large enough.

PROPOSITION 5.1 (Reduction of the problem)

There exists $A_0 > 2$, $L_0 > 0$, and $\alpha_0 > 0$ such that for all $u_0 \in H^1$, if

$$\Big\| u_0 - \sum_{k=1}^{K} Q_{\omega_k^0}(\cdot - y_k^0) e^{\mathrm{i}((1/2)v_k \cdot x + \gamma_k^0)} \Big\|_{H^1} \leqslant \alpha, \tag{5.5}$$

where $L > L_0$, $0 < \alpha < \alpha_0$, $y_{k,1}^0 > y_{k-1,1}^0 + L$, $y_{\phi_j(k),j}^0 > y_{\phi_j(k-1),j}^0 + L$, and if for some $t^* > 0$,

$$\forall t \in [0, t^*], \quad u(t) \in \mathscr{V}_{A_0}(\alpha, L), \tag{5.6}$$

then

$$\forall t \in [0, t^*], \quad u(t) \in \mathscr{V}_{A_0/2}(\alpha, L). \tag{5.7}$$

Therefore, we are reduced to prove Proposition 5.1.

5.3. Decomposition of the solution

Let $u(t)$ be a solution of (5.1) satisfying the assumptions of Proposition 5.1. Since for all $t \in [0, t^*]$, $u(t) \in \mathcal{V}_{A_0}(\alpha, L)$, by choosing $L_0 = L_0(A_0, \theta_0)$ and $\alpha_0 = \alpha_0(A_0) > 0$ small enough ($A_0(\alpha_0 + e^{-\theta_0 L})$ small), we apply a variant of Corollary 3 to $u(t)$ on the time interval $[0, t^*]$. As for the one-dimensional case, it follows that there exist unique C^1-functions $\omega_k : [0, t^*] \to (0, +\infty)$, $x_k : [0, t^*] \to \mathbb{R}^d$, and $\gamma_k : [0, t^*] \to \mathbb{R}$ such that if we set

$$\varepsilon(t, x) = u(t, x) - R(t, x), \tag{5.8}$$

where

$$R(t, x) = \sum_{k=1}^{K} R_k(t, x), \qquad R_k(t, x) = Q_{\omega_k(t)}\big(x - x_k(t)\big)e^{\mathrm{i}((1/2)v_k\cdot x + \gamma_k(t))}, \tag{5.9}$$

then $\varepsilon(t)$ satisfies, for all $k = 1, \dots, K$ and for all $t \in [0, t^*]$,

$$\operatorname{Re}\big(\varepsilon(t), R_k(t)\big) = \operatorname{Im}\big(\varepsilon(t), R_k(t)\big) = \operatorname{Re}\big(\varepsilon(t), \nabla R_k(t)\big) = 0 \tag{5.10}$$

and, for all $t \in [0, t^*]$,

$$\begin{aligned} &\|\varepsilon(t)\|_{H^1} + \sum_{k=1}^{K} |\omega_k(t) - \omega_k^0| + |\dot{\omega}_k(t)| + |\dot{x}_k(t) - v_k| + \left|\dot{\gamma}_k(t) - \left(\omega_k(t) - \frac{|v_k|^2}{4}\right)\right| \\ &\quad \leqslant C_1 A_0(\alpha + e^{-(\theta_0/2)L}). \end{aligned} \tag{5.11}$$

At $t = 0$, as in the one-dimensional case, we have

$$\|\varepsilon(0)\|_{H^1} + \sum_{k=1}^{K} |\omega_k(0) - \omega_k^0| \leqslant C_1 \alpha, \tag{5.12}$$

where C_1 does not depend on A_0.

5.4. Monotonicity results in the dimension d case

We now present the monotonicity results in dimension d using (5.3) and (5.4). For this purpose, we introduce quantities that are the natural analogues in higher dimensions of the ones introduced in dimension 1. As in the one-dimensional case, the interest of using such functionals is clear from the energy functional used to prove stability of a solitary wave (see Section 5.6).

Let

$$a = \frac{L^2}{64}.$$

By analogy with the quantities $\mathscr{I}_k(t)$ defined in the one-dimensional case, we introduce, for $k = 2, \ldots, K$ and for $t \in [0, t^*]$,

$$\mathscr{I}_{k,1}(t) = \int \Big\{ \frac{\sigma_{k,1}}{2} |u(x)|^2 - \operatorname{Im}\big(\partial_{x_1} u(x)\, \overline{u}(x)\big) \Big\} \psi\Big(\frac{x_1 - \sigma_{k,1} t}{\sqrt{t+a}} \Big)\, dx, \tag{5.13}$$

where $\sigma_{k,1}$ satisfies

$$v_{k-1,1} < \sigma_{k,1} < v_{k,1}.$$

Similarly, we introduce, for $k = 2, \ldots, K$, $j = 2, \ldots, d$, and $t \in [0, t^*]$,

$$\mathscr{I}_{k,j}(t) = \int \Big\{ \frac{\sigma_{k,j}}{2} |u(x)|^2 - \operatorname{Im}\big(\partial_{x_j} u(x)\, \overline{u}(x)\big) \Big\} \psi\Big(\frac{x_j - \sigma_{k,j} t}{\sqrt{t+a}} \Big)\, dx, \tag{5.14}$$

where $\sigma_{k,j}$ satisfies

$$v_{\phi_j(k-1),j} < \sigma_{k,j} < v_{\phi_j(k),j}.$$

As in the one-dimensional case, we claim the following result.

LEMMA 5.1
There exist L_2, α_2, and $\theta_2, C_2 > 0$ such that if $L > L_2$ and $0 < \alpha < \alpha_2$, then for $j = 1, \ldots, d$, for any $k = 2, \ldots, K$, and for any $t \in [0, t^*]$,

$$\mathscr{I}_{k,j}(t) - \mathscr{I}_{k,j}(0) \leqslant \frac{C_2}{L} \sup_{0<t'<t} \|\varepsilon(t')\|_{L^2}^2 + C_2 e^{-\theta_2 L}.$$

The identity that allows us to prove monotonicity-type results in the multidimensional case is the following.

CLAIM 6
Let $z(t)$ be a solution of (1.18); let $g : s \in \mathbb{R} \mapsto g(s)$ be a real-valued C^3-function. Then for $j = 1, \ldots, d$,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \operatorname{Im} \int \partial_{x_j} z(x)\, \overline{z}(x)\, g(x_j)\, dx = \int |\partial_{x_j} z(x)|^2 g'(x_j)\, dx - \frac{1}{4} \int |z(x)|^2 g'''(x_j)\, dx \\ - \frac{1}{2} \int \big\{ f\big(|z(x)|^2\big) |z(x)|^2 - F\big(|z(x)|^2\big) \big\} g'(x_j)\, dx. \end{aligned}$$

Observe that in the second member, the positive part contains only the partial derivative in the x_j-direction and not the full gradient.

Proof
The proof follows from direct calculations. As in the proof of Lemma 3.1, we have

$$\frac{d}{dt}\mathrm{Im}\int \partial_{x_j} z\, \overline{z}\, g = 2\mathrm{Im}\int \partial_{x_j} z\, \partial_t \overline{z}\, g + \mathrm{Im}\int \partial_t \overline{z}\, z\, g' = \mathrm{Im}\int \partial_t \overline{z}(2\partial_{x_j} z\, g + z\, g'),$$

and so

$$\frac{d}{dt}\mathrm{Im}\int \partial_{x_j} z\, \overline{z}\, g = \mathrm{Im}\Big[-\mathrm{i}\int \big(\Delta \overline{z} + f(|z|^2)\overline{z}\big)(2\partial_{x_j} z\, g + z\, g')\Big].$$

We have

$$\begin{aligned}\mathrm{Re}\int \partial_{x_j}^2 \overline{z}(2\partial_{x_j} z\, g + z\, g') &= -2\int |\partial_{x_j} z|^2 g' - \mathrm{Re}\int \partial_{x_j}\overline{z}\, z\, g'' \\ &= -2\int |\partial_{x_j} z|^2 g' + \frac{1}{2}\int |z|^2 g'''.\end{aligned}$$

And for $l = 1, \dots, d, l \neq j$, by integration by parts, since $g = g(x_j)$ does not depend on x_l,

$$\mathrm{Re}\int \partial_{x_l}^2 \overline{z}(2\partial_{x_j} z\, g + z\, g') = -\int \partial_{x_j}(|\partial_{x_l} z|^2)\, g - \int |\partial_{x_l} z|^2 g' = 0.$$

Thus, Claim 6 is proved. □

The analogue of Lemma 3.3 in dimensions 2 and 3 is the following result.

LEMMA 5.2
Let $d = 2$ or 3. Let $w = w(x) \in H^1(\mathbb{R}^d)$, and let $h = h(x_1) \geqslant 0$ be a C^1 bounded function such that the support of h is a bounded interval $[a, b]$, $h > 0$ on (a, b), $\sqrt{h}$ is of class C^1 and $(h')^2 \leqslant Ch$. Then, in dimension 2,

$$\begin{aligned}\int_{\mathbb{R}^2} |w(x)|^6 h(x_1)\, dx \leqslant C\Big(\int_{\operatorname{supp} h} |w(x)|^2\, dx\Big)\Big(\int_{\operatorname{supp} h} |\partial_{x_2} w(x)|^2\, dx\Big) \\ \times \int_{\mathbb{R}^2} \Big[|\partial_{x_1} w(x)|^2 h(x_1) + |w(x)|^2 \frac{(h'(x_1))^2}{h(x_1)}\Big]\, dx, \quad (5.15)\end{aligned}$$

and in dimension 3,

$$\begin{aligned}\int_{\mathbb{R}^3} |w(x)|^6 h(x_1)\, dx \leqslant C\prod_{j=2}^{3}\Big(\int_{\operatorname{supp} h} |\partial_{x_j} w(x)|^2\, dx\Big) \\ \times \int_{\mathbb{R}^3} \Big[|\partial_{x_1} w(x)|^2 h(x_1) + |w(x)|^2 \frac{(h'(x_1))^2}{h(x_1)}\Big]\, dx, \quad (5.16)\end{aligned}$$

where supp h *denotes the support of* h.

Proof
We prove the case where $d = 3$. First, we recall the Sobolev-type inequality

$$\int_{\mathbb{R}^3} |z(y)|^6 \, dy \leqslant C \prod_{j=1}^{3} \int_{\mathbb{R}^3} |\partial_{y_j} z|^2, \tag{5.17}$$

obtained from the Gagliado-Nirenberg inequality

$$\|v\|_{3/2} \leqslant \Big(\prod_{j=1}^{3} \int_{\mathbb{R}^3} |\partial_{y_j} v| \Big)^{1/3}$$

applied to $v = |z|^4$.

Next, let $a < b$ be such that $[a, b] = \operatorname{supp} h$, and let $c \in (a, b)$. For $x_1 \in (a, b)$, set

$$q(x_1) = \int_c^{x_1} \frac{ds}{\sqrt{h(s)}}.$$

The function q is increasing from (a, b) to $\mathbb{R}$. Moreover, since $\sqrt{h}$ is of class C^1 on $\mathbb{R}$, we have $q((a, b)) = \mathbb{R}$. (Indeed, $0 < \sqrt{h(s)} \leqslant C(b - s)$ for $s < b$ close to b.) Thus q is a one-to-one mapping from (a, b) to $\mathbb{R}$. Let $\zeta_1 = q^{-1}$. Let $\zeta(y) = (\zeta_1(y_1), y_2, y_3)$. By the change of variable $x_1 \mapsto y_1$ with $x_1 = \zeta_1(y_1)$ and $x_2 \mapsto y_2, x_3 \mapsto y_3$, estimate that (5.16) is equivalent to

$$\begin{aligned} \int_{\mathbb{R}^3} |z(y)|^6 g^{3/2}(y_1) \, dy &\leqslant C \prod_{j=2}^{3} \Big(\int_{\mathbb{R}^3} |\partial_{y_j} z(y)|^2 g^{1/2}(y_1) \, dy \Big) \\ &\quad \times \int_{\mathbb{R}^3} \Big[|\partial_{y_1} z(y)|^2 g^{1/2}(y_1) + |z(y)|^2 \frac{(g'(y_1))^2}{g^{3/2}(y_1)} \Big] \, dy, \end{aligned} \tag{5.18}$$

where $z(y) = w(\zeta(x))$, $g(y_1) = h(\zeta_1(x_1))$. But (5.18) follows directly from (5.17) applied to $zg^{1/4}$ instead of z.

The case where $d = 2$ is completely similar; using the inequality

$$\|v\|_{L^2} \leqslant \Big(\prod_{j=1}^{2} \int |\partial_{x_j} v| \Big)^{1/2}$$

applied to $v = |z|^3$, we have

$$\int_{\mathbb{R}^2} |z(y)|^6 \, dy \leqslant C \Big(\int_{\mathbb{R}^2} |z|^2 \Big) \Big(\prod_{j=1}^{2} \int_{\mathbb{R}^2} |\partial_{x_j} z|^2 \Big). \tag{5.19}$$

The rest is done in exactly the same way. □

Sketch of the proof of Lemma 5.1

Using Claim 6 and Lemma 5.2, the proof of Lemma 5.1 follows the same lines as the proof of Proposition 3.1. Setting

$$z(t,x) = z_k(t,x) = u\big(t,(x_1+\sigma_k t, x_2)\big)e^{-\mathrm{i}(\sigma_k/2)(x_1+\sigma_k t/2)},$$

we obtain, as in the proof of Proposition 3.1,

$$\frac{1}{2}\frac{d}{dt}\mathscr{I}_{k,1}(t) \leqslant -\frac{3}{4\sqrt{t+a}}\int |\partial_{x_1} z|^2 \psi'\Big(\frac{x_1}{\sqrt{t+a}}\Big) + \frac{C}{(t+a)^{3/2}}\int_{|x_1|<\sqrt{t+a}} |z|^2 + \frac{C}{\sqrt{t+a}}\int \big\{f(|z|^2)|z|^2 - F(|z|^2)\big\}\psi'\Big(\frac{x_1}{\sqrt{t+a}}\Big). \tag{5.20}$$

A difficulty is that in (5.20), only the partial derivative in x_1, and not the full gradient, appears on the right-hand side. A full gradient would have allowed us to control any supercritical nonlinear term in any dimension. In the present situation, we can only use Lemma 5.2, which is a variant of Lemma 3.3 for $d = 2$ and 3 but which does not seem to have an analogue for $d \geqslant 4$.

Note that since $f'(r) \leqslant Cr$ (assumption (A1′) in Theorem 2), we have, for all $r \geqslant 0$,

$$\big(f(r)r - F(r)\big)' = f'(r)r \leqslant Cr^2,$$

and so for all $r \geqslant 0$, $f(r)r - F(r) \leqslant Cr^3$. Thus

$$\int \big\{f(|z|^2)|z|^2 - F(|z|^2)\big\}\psi'\Big(\frac{x_1}{\sqrt{t+a}}\Big) \leqslant C\int_{\mathbb{R}^d} |z|^6\psi'\Big(\frac{x_1}{\sqrt{t+a}}\Big)\,dx_1\,dx_2.$$

Using Lemma 5.2 and the properties of ψ', the rest of the proof of Lemma 5.1 is exactly the same as that of Proposition 3.1. Note that here, we do not need (3.9) since assumption (A1′) is made for all $r \geqslant 0$. In fact, (3.9) is not true for $d \geqslant 2$. □

5.5. Introduction of a functional adapted to the stability problem

As in the one-dimensional case, we need to introduce a functional that locally in space is adapted to each solitary wave, that is, which is locally around each solitary wave equal to

$$\int \Big\{\Big(\omega_k(0) + \frac{|v_k|^2}{4}\Big)|u(t)|^2 - v_k\,\mathrm{Im}\big(\partial_x u(t)\,\overline{u}(t)\big)\Big\}. \tag{5.21}$$

Indeed, this quantity is the one that appears in dimension $d \geqslant 2$ when proving the stability of a single solitary wave.

We set

$$v_0 = \min(v_{2,1} - v_{1,1}, \dots, v_{K,1} - v_{K-1,1}) > 0. \tag{5.22}$$

We assume that for all $k \in \{1, \dots, K-1\}$,

$$|\omega_{k+1}^0 - \omega_k^0| < \frac{1}{4} v_0^2, \tag{5.23}$$

so that

$$|\omega_{k+1}(0) - \omega_k(0)| < \frac{1}{4} v_0^2 \tag{5.24}$$

is true by (5.11) for α and $1/L$ small enough. This is condition (A3′) in the statement of Theorem 2. It is similar to condition (A3) of Theorem 1. Note that condition (5.24) depends on the direction e_1 chosen in Claim 5 through the value of v_0. In particular, there exists an optimal choice of e_1, so that v_0 is largest possible. However, it would not necessarily give the optimal result in Theorem 2; another method could provide a less-restrictive condition on ω_k. Especially when there are a large number of solitary waves, it is not clear how to provide the best possible condition on ω_k^0. This is why we do not try to optimize the set of conditions. Nevertheless, it may be an interesting direction for further investigation.

We define

$$\sigma_{k,1} = 2\,\frac{\omega_{k,1}(0) - \omega_{k-1,1}(0)}{v_{k,1} - v_{k-1,1}} + \frac{v_{k,1} + v_{k-1,1}}{2},$$

and from (5.24), we easily check as in the proof of Proposition 3.2 that, for $k = 2, \dots, K$,

$$v_{k-1,1} < \sigma_{k,1} < v_{k,1}.$$

We set $\varphi_{1,1} \equiv 1$, and for $k = 2, \dots, K-1$,

$$\varphi_{k,1}(t, x_1) = \psi\Big(\frac{x_1 - \sigma_{k,1} t}{\sqrt{t+a}}\Big) - \psi\Big(\frac{x_1 - \sigma_{k+1,1} t}{\sqrt{t+a}}\Big), \qquad \varphi_{K,1}(t, x_1) = \psi\Big(\frac{x_1 - \sigma_{K,1} t}{\sqrt{t+a}}\Big).$$

For the x_j-direction, where $j = 2, \dots, d$, we set, for $k = 2, \dots, K$,

$$\sigma_{k,j} = \frac{v_{\phi_j(k),j} + v_{\phi_j(k-1),j}}{2}.$$

It is immediate that

$$v_{\phi_j(k-1),j} < \sigma_{k,j} < v_{\phi_j(k),j}.$$

We introduce $\varphi_{1,j} \equiv 1$, and for $k = 2, \dots, K-1$,

$$\varphi_{k,j}(t, x_j) = \psi\Big(\frac{x_j - \sigma_{k,j} t}{\sqrt{t+a}}\Big) - \psi\Big(\frac{x_j - \sigma_{k+1,j} t}{\sqrt{t+a}}\Big), \qquad \varphi_{K,j}(t, x_j) = \psi\Big(\frac{x_j - \sigma_{K,j} t}{\sqrt{t+a}}\Big).$$

We are now able to define quantities similar to $\mathscr{J}(t)$ in the x_1- and x_j-directions ($j = 2, \dots, d$). We set

$$\mathscr{J}_1(t) = \sum_{k=1}^{K} \Big\{ \int \Big[\Big(\omega_k(0) + \frac{v_{k,1}^2}{4}\Big)|u(t)|^2 - v_{k,1} \operatorname{Im}\big(\partial_{x_1} u(t)\, \overline{u}(t)\big) \Big] \varphi_{k,1}(t, x_1)\, dx \Big\}, \tag{5.25}$$

$$\mathscr{J}_j(t) = \sum_{k=1}^{K} \Big\{ \int \Big[\frac{v_{\phi_j(k),j}^2}{4}|u(t)|^2 - v_{\phi_j(k),j} \operatorname{Im}\big(\partial_{x_j} u(t)\, \overline{u}(t)\big) \Big] \varphi_{k,j}(t, x_j)\, dx \Big\}. \tag{5.26}$$

Note that the expressions of $\mathscr{J}_j(t)$ are similar to those of $\mathscr{J}(t)$ introduced in the one-dimensional case. In addition, the sum $\sum_{j=1}^{d} \mathscr{J}_j$ is locally around each solitary wave equal to (5.21).

Now we check that $\mathscr{J}_j$ satisfy monotonicity properties as $\mathscr{J}$. Indeed, we claim the following, from the Abel transformation, Lemma 5.1, and the fact that the L^2-norm and the momentum are invariant in time.

PROPOSITION 5.2
For $j = 1, \dots, d$, the following is true:

$$\mathscr{J}_j(t) - \mathscr{J}_j(0) \leqslant \frac{C}{L} \sup_{t' \in [0,t]} \|\varepsilon(t')\|_{L^2}^2 + Ce^{-\theta_2 L}.$$

Remark. Here we use, for the property of $\mathscr{J}_j(t)$ with $j = 2, \dots, d$, the fact that in dimension 1, if all the ω_k are equal, then the condition for the parameters reduces to the simple condition that the speeds v_k are different.

Proof of Proposition 5.2
The proof is completely similar to that of Proposition 3.2. Indeed, thanks to the choice of $\sigma_{k,j}$, we claim

$$\mathscr{J}_1(t) = \int \Big\{ \Big(\omega_1(0) + \frac{v_{1,1}^2}{4}\Big)|u(t)|^2 - v_{1,1} \operatorname{Im}\big(\partial_{x_1} u(t)\, \overline{u}(t)\big) \Big\} + \sum_{k=2}^{K} (v_{k,1} - v_{k-1,1}) \mathscr{I}_{k,1}(t),$$

and similarly, for $j = 2, \dots, d$,

$$\mathscr{I}_j(t) = \int \Big\{ \frac{v_{1,j}^2}{4} |u(t)|^2 - v_{1,j} \operatorname{Im}\big(\partial_{x_j} u(t) \overline{u}(t)\big) \Big\} + \sum_{k=2}^{K} (v_{\phi_j(k),j} - v_{\phi_j(k-1),j}) \mathscr{I}_{k,j}(t).$$

Thus, Proposition 5.2 follows directly from Lemma 5.1. □

5.6. Energetic control of $\varepsilon(t)$

The functional related to the stability of K solitary waves in the d dimensional case is

$$\mathscr{G}(t) = E\big(u(t)\big) + \sum_{j=1}^{d} \mathscr{I}_d(t).$$

We claim the following result, whose proof is completely similar to that of Proposition 4.2.

PROPOSITION 5.3

The following holds:

$$\mathscr{G}(t) = \sum_{k=1}^{K} \Big\{ E(Q_{\omega_k(0)}) + \omega_k(0) \int |Q_{\omega_k(0)}|^2 \Big\} + H\big(\varepsilon(t), \varepsilon(t)\big)$$
$$+ \sum_{k=1}^{K} O\big(|\omega_k(t) - \omega_k(0)|^2\big) + \|\varepsilon\|_{H^1}^2 \beta(\|\varepsilon\|_{H^1}) + O(e^{-\theta_0 t})$$

with $\beta(\epsilon) \to 0$ *as* $\epsilon \to 0$, *where*

$$H(\varepsilon, \varepsilon) = \int |\nabla \varepsilon|^2 - \sum_{k=1}^{K} \int \big(f(|R_k|^2)|\varepsilon|^2 + 2f'(|R_k|^2)[\operatorname{Re}(\overline{R}_k \varepsilon)]^2 \big)$$
$$+ \sum_{k=1}^{K} \int \Big\{ \Big(\omega_k(t) + \frac{v_{k,1}^2}{4} \Big) |\varepsilon|^2 - v_{k,1} \operatorname{Im}(\partial_{x_1} \varepsilon\, \overline{\varepsilon}) \Big\} \varphi_{k,1}(t)$$
$$+ \sum_{j=2}^{d} \sum_{k=1}^{K} \int \Big\{ \frac{v_{\phi_j(k),j}^2}{4} |\varepsilon|^2 - v_{\phi_j(k),j} \operatorname{Im}(\partial_{x_j} \varepsilon\, \overline{\varepsilon}) \Big\} \varphi_{k,j}(t). \tag{5.27}$$

Moreover, with the orthogonality conditions chosen on ε, that is, (5.10), we have, for $\lambda > 0$,

$$H(\varepsilon, \varepsilon) \geqslant \lambda \|\varepsilon\|_{H^1}^2. \tag{5.28}$$

The proof of (5.28) is the same as the proof of Lemma 4.1.

5.7. Proof of Proposition 5.1

The proof of Proposition 5.1 proceeds exactly along the lines of the proof of Proposition 4.1. We check that steps 3 and 4 in the proof of Proposition 4.1 apply identically to the case where $d \geqslant 2$.

First, we directly prove, from the conservation of $E(u(t))$, the monotonicity of $\mathscr{I}_j(t)$, and the expansion of $\mathscr{G}(t)$ in Proposition 5.3, that for all $t \in [0, t^*]$,

$$\|\varepsilon(t)\|_{H^1}^2 + \sum_{j=1}^{d} \sum_{k=2}^{K} |\mathscr{I}_{k,j}(t) - \mathscr{I}_{k,j}(0)| \leqslant \frac{C}{L} \sup_{t' \in [0,t]} \|\varepsilon(t')\|_{H^1}^2 + C\|\varepsilon(0)\|_{H^1}^2 + C \sum_{k=1}^{K} |\omega_k(t) - \omega_k(0)|^2 + Ce^{-\theta_0 L}. \quad (5.29)$$

Second, we estimate $|\omega_k(t) - \omega_k(0)|$ for all k. This is done exactly as for the one-dimensional case by considering different channels and different quantities $\mathscr{I}_{k,1}^{\pm}(t)$ to which to apply the monotonicity property. Note that we need only do this in one direction, for example, the direction x_1. This is enough to split the different solitary waves and to control their size. We obtain, for all $t \in [0, t^*]$,

$$\sum_{k=1}^{K} |\omega_k(t) - \omega_k(0)| \leqslant C \sup_{t' \in [0,t]} \|\varepsilon(t')\|_{H^1}^2 + Ce^{-\theta_0 L}. \quad (5.30)$$

This is sufficient to finish the proof of Proposition 5.1 exactly as that of Proposition 4.1, from (5.29) and (5.30). □

Appendices

A. Proofs of Lemma 2.4 and Corollary 3

Proof of Lemma 2.4

The proof is a standard application of the implicit function theorem. Let $\alpha > 0$, and let $L > 0$. Let $x_1^0, \dots, x_K^0$ be such that $x_k^0 > x_{k-1}^0 + L$ and $\gamma_1^0, \dots, \gamma_K^0 \in \mathbb{R}$. Denote by B_0 the H^1-ball of center $\sum_{k=1}^K Q_{\omega_k^0}(\cdot - x_k^0)e^{\mathrm{i}((1/2)v_k x + \gamma_k^0)}$ and of radius 10α. For any $u \in B_0$ and parameters $\omega_1, \dots, \omega_K; x_1, \dots, x_K; \gamma_1, \dots, \gamma_K$, let $q = (\omega_1, \dots, \omega_K; x_1, \dots, x_K; \gamma_1, \dots, \gamma_K; u)$, and define

$$\varepsilon(x) = u(x) - \sum_{k=1}^{K} Q_{\omega_k}(\cdot - x_k)e^{\mathrm{i}((1/2)v_k x + \gamma_k)}. \quad (A.1)$$

Define the following functions of q,

$$\rho_k^1(q) = \mathrm{Re} \int Q_{\omega_k}(x - x_k) e^{\mathrm{i}((1/2)v_k x + \gamma_k)} \overline{\varepsilon}(q; x)\, dx,$$

$$\rho_k^2(q) = \mathrm{Re} \int Q'_{\omega_k}(x - x_k) e^{\mathrm{i}((1/2)v_k x + \gamma_k)} \overline{\varepsilon}(q; x)\, dx,$$

$$\rho_k^3(q) = \mathrm{Im} \int Q_{\omega_k}(x - x_k) e^{\mathrm{i}((1/2)v_k x + \gamma_k)} \overline{\varepsilon}(q; x)\, dx,$$

for q close to

$$q_0 = \Big(\omega_1^0, \dots, \omega_K^0; x_1^0, \dots, x_K^0; \gamma_1^0, \dots, \gamma_K^0; \sum_{k=1}^{K} Q_{\omega_k^0}(\cdot - x_k^0) e^{\mathrm{i}((1/2)v_k x + \gamma_k^0)}\Big).$$

For $q = q_0$, we have $\varepsilon(q_0) \equiv 0$, and thus for $j = 1, 2, 3$, $\rho_k^j(q_0) = 0$. We check by applying the implicit function theorem that for any $u \in B_0$, one can choose in a unique way the coefficients $(\omega_1, \dots, \omega_K; x_1, \dots, x_K; \gamma_1, \dots, \gamma_K)$, so that q is close to q_0 and verifies $\rho_k^j(q) = 0$ for $j = 1, 2, 3$. In order to apply the implicit function theorem to this situation, we compute the derivatives of ρ_k^j for any k, j with respect to each $(\omega_k, x_k, \gamma_k)$. Note that

$$\frac{\partial \varepsilon}{\partial \omega_k}(q_0) = -\frac{\partial Q_\omega}{\partial \omega}\Big|_{|\omega=\omega_k^0}(\cdot - x_k^0) e^{\mathrm{i}((1/2)v_k x + \gamma_k^0)},$$

$$\frac{\partial \varepsilon}{\partial x_k}(q_0) = Q'_{\omega_k^0}(\cdot - x_k^0) e^{\mathrm{i}((1/2)v_k x + \gamma_k^0)},$$

$$\frac{\partial \varepsilon}{\partial \gamma_k}(q_0) = -\mathrm{i} Q_{\omega_k^0}(\cdot - x_k^0) e^{\mathrm{i}((1/2)v_k x + \gamma_k^0)},$$

and thus,

$$\frac{\partial \rho_{k'}^1}{\partial \omega_k}(q_0) = -\mathrm{Re} \int Q_{\omega_{k'}^0}(x - x_{k'}^0) e^{\mathrm{i}((1/2)v_{k'} x + \gamma_{k'}^0)} \frac{\partial Q_\omega}{\partial \omega}\Big|_{\omega=\omega_k^0}(x - x_k^0) e^{-\mathrm{i}((1/2)v_k x + \gamma_k^0)}\, dx,$$

$$\frac{\partial \rho_{k'}^1}{\partial x_k}(q_0) = \mathrm{Re} \int Q_{\omega_{k'}^0}(x - x_{k'}^0) e^{\mathrm{i}((1/2)v_{k'} x + \gamma_{k'}^0)} Q'_{\omega_k^0}(x - x_k^0) e^{-\mathrm{i}((1/2)v_k x + \gamma_k^0)}\, dx,$$

$$\frac{\partial \rho_{k'}^1}{\partial \gamma_k}(q_0) = -\mathrm{Im} \int Q_{\omega_{k'}^0}(x - x_{k'}^0) e^{\mathrm{i}((1/2)v_{k'} x + \gamma_{k'}^0)} Q_{\omega_k^0}(x - x_k^0) e^{-\mathrm{i}((1/2)v_k x + \gamma_k^0)}\, dx,$$

and similar formulas hold for $\frac{\partial \rho_{k'}^2}{\partial \omega_k}(q_0)$, $\frac{\partial \rho_{k'}^2}{\partial x_k}(q_0)$, $\frac{\partial \rho_{k'}^2}{\partial \gamma_k}(q_0)$, $\frac{\partial \rho_{k'}^3}{\partial \omega_k}(q_0)$, $\frac{\partial \rho_{k'}^3}{\partial x_k}(q_0)$, and $\frac{\partial \rho_{k'}^3}{\partial \gamma_k}(q_0)$.

Now, we finish the computations for $k' = k$. By assumption (A2), $\frac{\partial \rho_k^1}{\partial \omega_k}(q_0) = a_k < 0$; since Q_{ω_k} is even, $\frac{\partial \rho_k^1}{\partial x_k}(q_0) = 0$; and finally, since Q_{ω_k} is real, $\frac{\partial \rho_k^1}{\partial \gamma_k}(q_0) = 0$. Doing the same for ρ_k^2 and ρ_k^3, we obtain the following conclusions:

$$\begin{aligned}
&\frac{\partial\rho_k^1}{\partial\omega_k}(q_0)=a_k<0, \qquad \frac{\partial\rho_k^1}{\partial\omega_k}(q_0)=0, \qquad \frac{\partial\rho_k^1}{\partial\omega_k}(q_0)=0,\\
&\frac{\partial\rho_k^2}{\partial x_k}(q_0)=0, \qquad \frac{\partial\rho_k^2}{\partial x_k}(q_0)=b_k>0, \qquad \frac{\partial\rho_k^2}{\partial x_k}(q_0)=0,\\
&\frac{\partial\rho_k^3}{\partial\gamma_k}(q_0)=0, \qquad \frac{\partial\rho_k^3}{\partial\gamma_k}(q_0)=0, \qquad \frac{\partial\rho_k^3}{\partial\gamma_k}(q_0)=c_k>0.
\end{aligned} \tag{A.2}$$

For $k'\neq k$ and $j=1,2,3$, since the different Q_{ω_k} are exponentially decaying (see Lemma 2.1) and located at centers distant at least of L, we have

$$\left|\frac{\partial\rho_{k'}^j}{\partial\omega_k}(q_0)\right|+\left|\frac{\partial\rho_{k'}^j}{\partial x_k}(q_0)\right|+\left|\frac{\partial\rho_{k'}^j}{\partial\gamma_k}(q_0)\right|\leqslant Ce^{-\theta_1 L}. \tag{A.3}$$

These terms are arbitrarily small by choosing L large.

Therefore, by (A.2) and (A.3), the Jacobian of $\rho=(\rho_1^1,\dots,\rho_K^1;\rho_1^2,\dots,\rho_K^2;\rho_1^3,\dots,\rho_K^3)$ as a function of $(\omega_1,\dots,\omega_K;x_1,\dots,x_K;\gamma_1,\dots,\gamma_K)$ at the point q_0 is not zero. Thus we can apply the implicit function theorem to prove, for α small and $u\in B_0$, the existence and uniqueness of parameters $(\omega_1,\dots,\omega_K;x_1,\dots,x_K;\gamma_1,\dots,\gamma_K)$ such that $\rho(q)=0$. We obtain directly estimates (2.12) with constants that are independent of the ball B_0. This proves the result for $u\in B_0$. If we now take $u\in\mathscr{U}(\alpha,L)$, then u belongs to such a ball B_0, and the result follows. □

Proof of Corollary 3

Assume that $u(t)$ satisfies (2.13) on $[0,t_0]$. Then, applying Lemma 2.4 to $u(t)$ for any $t\in[0,t_0]$, and since the map $t\mapsto u(t)$ is continuous in H^1, we obtain for any $k=1,\dots,K$ the existence of continuous functions $\omega_k:[0,t_0]\to(0,+\infty)$, $x_k,\gamma_k:[0,t_0]\to\mathbb{R}$ such that (2.16) holds. Note in particular that

$$\begin{aligned}
\operatorname{Re}\bigl(\varepsilon(t),\partial_x R_k(t)\bigr)=\operatorname{Re}\int\Bigl\{&Q'_{\omega_k(t)}\bigl(x-x_k(t)\bigr)\\
&+\frac{\mathrm{i}v_k}{2}Q_{\omega_k(t)}\bigl(x-x_k(t)\bigr)\Bigr\}e^{\mathrm{i}((1/2)v_kx+\gamma_k(t))}\overline{\varepsilon}(t)=0.
\end{aligned}$$

Moreover, (2.17) is a consequence of (2.12). □

To prove that the functions (ω_k), (x_k), and (γ_k) are in fact of class C^1, we use regularization arguments and computations based on the equation of $\varepsilon(t)$. These computations also justify estimates (2.18). We refer to [14] for more details on standard regularization arguments needed for the proof, and we just give the equation of $\varepsilon(t)$ to justify formally estimates (2.18). It is straightforward to check that the equation of $\varepsilon(t)$ is

$$\begin{aligned} \mathrm{i}\,\partial_t\varepsilon + \widetilde{\mathscr{L}}_K\varepsilon = & -\mathrm{i}\sum_{k=1}^{K}\dot{\omega}_k(t)\frac{\partial Q_\omega}{\partial\omega}\Big|_{\omega=\omega_k(t)}\big(x-x_k(t)\big)e^{\mathrm{i}((1/2)v_kx+\gamma_k(t))} \\ & +\mathrm{i}\sum_{k=1}^{K}\big(\dot{x}_k(t)-v_k\big)Q'_{\omega_k(t)}\big(x-x_k(t)\big)e^{\mathrm{i}((1/2)v_kx+\gamma_k(t))} \\ & +\sum_{k=1}^{K}\Big(\dot{\gamma}_k(t)-\Big(\omega_k(t)-\frac{v_k^2}{4}\Big)\Big)Q_{\omega_k(t)}\big(x-x_k(t)\big)e^{\mathrm{i}((1/2)v_kx+\gamma_k(t))} \\ & +O(\|\varepsilon\|_{H^1}^2)+O(e^{-\theta_1(L+\theta_1 t)}), \end{aligned} \tag{A.4}$$

where

$$\widetilde{\mathscr{L}}_K\varepsilon = -\partial_x^2\varepsilon - \sum_{k=1}^{K}\Big\{f(Q_{\omega_k(t)}^2)\varepsilon + 2f'(Q_{\omega_k(t)}^2)\mathrm{Re}\big(Q_{\omega_k(t)}(x-x_k(t))e^{-\mathrm{i}((1/2)v_kx+\gamma_k(t))}\varepsilon\big)\Big\}. \tag{A.5}$$

From the equation of ε, it is straightforward by taking scalar products by $Q_{\omega_k(t)}$ and then by $Q'_{\omega_k(t)}$ to check that $|\dot{\omega}_k(t)|^2$, $|\dot{x}_k(t)-v_k|^2$, and $|\dot{\gamma}_k(t)-(\omega_k(t)-v_k^2/4)|^2$ are estimated by the second member in (2.18). In fact, looking more carefully, we see that a specific cancellation implies that $|\dot{\omega}_k(t)|$ is quadratic; that is, (2.18) holds for $|\dot{\omega}_k(t)|$.

B. Proof of positivity of quadratic forms

In this appendix, we prove Lemmas 2.6 and 4.1.

Proof of Lemma 2.6

Lemma 2.6 is a direct consequence of the following claim applied to $Q_{\omega(t)}$ and ε. □

CLAIM 7

Let $\omega_0 > 0$, v_0, x_0, and $\gamma_0 \in \mathbb{R}$. Assume that there exists a solution Q_{ω_0} of (1.7), and assume that ω_0 satisfies (1.10). Let

$$\begin{aligned} H_0(w,w) = & \int\Big\{|\partial_x w|^2 + \Big(\omega_0+\frac{v_0^2}{4}\Big)|w|^2 - v_0\,\mathrm{Im}(\partial_x w\,\overline{w})\Big\} \\ & -\int\big\{f\big(|Q_{\omega_0}(\cdot-x_0)|^2\big)|w|^2 \\ & +2f'\big(|Q_{\omega_0}(\cdot-x_0)|^2\big)\big[\mathrm{Re}\big(Q_{\omega_0}(\cdot-x_0)e^{-\mathrm{i}((1/2)v_0x+\gamma_0)}w\big)\big]^2\big\}. \end{aligned}$$

There exists $\lambda_0 > 0$ *such that if* $w \in H^1(\mathbb{R})$ *satisfies*

$$\operatorname{Re}\int Q_{\omega_0}(\cdot - x_0)e^{-\mathrm{i}((1/2)v_0x+\gamma_0)}w = \operatorname{Re}\int Q'_{\omega_0}(\cdot - x_0)e^{-\mathrm{i}((1/2)v_0x+\gamma_0)}w$$
$$= \operatorname{Im}\int Q_{\omega_0}(\cdot - x_0)e^{-\mathrm{i}((1/2)v_0x+\gamma_0)}w = 0,$$

then

$$H_0(w, w) \geqslant \lambda_0\|w\|^2_{H^1}.$$

Proof

Consider $\eta(x)$ such that

$$w(x) = \eta(x - x_0)e^{\mathrm{i}((1/2)v_0x+\gamma_0)}.$$

Note that $\int |w|^2 = \int |\eta|^2$,

$$\int |\partial_x w|^2 = \int |\partial_x \eta|^2 + \frac{v_0^2}{4}\int |\eta|^2 + v_0 \operatorname{Im}\int \partial_x\eta\,\overline{\eta}, \tag{B.1}$$

and

$$\operatorname{Im}\int \partial_x w\,\overline{w} = \operatorname{Im}\int \partial_x\eta\,\overline{\eta} + \frac{v_0}{2}\int |\eta|^2.$$

Thus,

$$H_0(w, w) = \int |\partial_x\eta|^2 + \omega_0\int |\eta|^2 - \int \left\{ f(|Q_{\omega_0}|^2)|\eta|^2 + 2f'(|Q_{\omega_0}|^2)Q^2_{\omega_0}\,[\operatorname{Re}\eta]^2 \right\}$$
$$= (\mathscr{L}^+_{\omega_0}\operatorname{Re}\eta, \operatorname{Re}\eta) + (\mathscr{L}^-_{\omega_0}\operatorname{Im}\eta, \operatorname{Im}\eta).$$

The orthogonality conditions on w imply directly (Q_{ω_0} being real valued)

$$(\operatorname{Re}\eta, Q_{\omega_0}) = (\operatorname{Re}\eta, Q'_{\omega_0}) = (\operatorname{Im}\eta, Q_{\omega_0}) = 0,$$

and so by Lemma 2.2, we obtain, for $\lambda > 0$,

$$H_0(w, w) \geqslant \lambda\|\eta\|^2_{H^1}.$$

Using (B.1), we also have

$$\int |\partial_x w|^2 \leqslant \frac{3}{2}\int |\partial_x\eta|^2 + \frac{3v_0^2}{4}\int |\eta|^2,$$

and so $H_0(w, w) \geqslant \lambda'\|w\|^2_{H^1}$, which completes the proof of Claim 7. □

Proof of Lemma 4.1
First, we give a localized version of Lemma 2.6. Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a C^2-function such that $\Phi(x) = \Phi(-x)$, $\Phi' \leqslant 0$ on $\mathbb{R}^+$ with

$$\Phi(x) = 1 \quad \text{on } [0, 1]; \qquad \Phi(x) = e^{-x} \quad \text{on } [2, +\infty);$$
$$e^{-x} \leqslant \Phi(x) \leqslant 3\, e^{-x} \quad \text{on } \mathbb{R}.$$

Let $B > 0$, and let $\Phi_B(x) = \Phi(x/B)$. Set

$$\begin{aligned} H_{\Phi_B}(w, w) = \int \Phi_B(\cdot - x_0)\Big\{|\partial_x w|^2 + \Big(\omega_0 + \frac{v_0^2}{4}\Big)|w|^2 - v_0 \mathrm{Im}(\partial_x w\, \overline{w})\Big\} \\ - \int \big\{ f\big(|Q_{\omega_0}(\cdot - x_0)|^2\big)|w|^2 + 2 f'\big(|Q_{\omega_0}(\cdot - x_0)|^2\big) \\ \times \big[\mathrm{Re}\big(Q_{\omega_0}(\cdot - x_0) e^{-\mathrm{i}((1/2)v_0 x + \gamma_0)} w\big)\big]^2 \big\}. \end{aligned}$$

□

CLAIM 8
Under the assumptions of Claim 7, there exists $B_0 > 2$ such that for all $B > B_0$, if $w \in H^1(\mathbb{R})$ satisfies

$$\begin{aligned} \mathrm{Re} \int Q_{\omega_0}(\cdot - x_0) e^{-\mathrm{i}((1/2)v_0 x + \gamma_0)} w &= \mathrm{Re} \int Q'_{\omega_0}(\cdot - x_0) e^{-\mathrm{i}((1/2)v_0 x + \gamma_0)} w \\ &= \mathrm{Im} \int Q_{\omega_0}(\cdot - x_0) e^{-\mathrm{i}((1/2)v_0 x + \gamma_0)} w = 0, \end{aligned}$$

then

$$H_{\Phi_B}(w, w) \geqslant \frac{\lambda_0}{4} \int \Phi_B(\cdot - x_0)\{|\partial_x w|^2 + |w|^2\}.$$

Proof
For the sake of simplicity, we assume that $x_0 = 0$ and $\gamma_0 = 0$. We set $z = w\sqrt{\Phi_B}$. Then, by simple calculations,

$$\int |\partial_x w|^2 \Phi_B = \int |\partial_x z|^2 + \frac{1}{4} \int |z|^2 \Big(\frac{\Phi_B'}{\Phi_B}\Big)^2 - 2\,\mathrm{Re} \int \partial_x z\, \overline{z}\, \frac{\Phi_B'}{\Phi_B}, \quad \int |w|^2 \Phi_B = \int |z|^2.$$

Since, by definition of Φ_B, we have $|\Phi_B'| \leqslant (C/B)\Phi_B$, we obtain

$$\int |\partial_x z|^2 - \frac{C}{B} \int (|\partial_x z|^2 + |z|^2) \leqslant \int |\partial_x w|^2 \Phi_B \leqslant \int |\partial_x z|^2 + \frac{C}{B} \int (|\partial_x z|^2 + |z|^2).$$

Moreover, since $(\partial_x w\,\overline{w})\Phi_B = \partial_x z\,\overline{z} - (\Phi_B'/(2\Phi_B))|z|^2$, we have

$$\operatorname{Im}(\partial_x w\,\overline{w})\Phi_B = \operatorname{Im}(\partial_x z\,\overline{z}).$$

We also have

$$\int \left\{ f(|Q_{\omega_0}|^2)|w|^2 + 2f'(|Q_{\omega_0}|^2)[\operatorname{Re}(Q_{\omega_0}e^{-\mathrm{i}((1/2)v_0x)}w)]^2 \right\} \\ = \int \left\{ f(|Q_{\omega_0}|^2)|z|^2 + 2f'(|Q_{\omega_0}|^2)[\operatorname{Re}(Q_{\omega_0}e^{-\mathrm{i}((1/2)v_0x)}z)]^2 \right\} \frac{1}{\Phi_B}.$$

Since $\Phi_B \equiv 1$ on $[-B, B]$ and $Q_{\omega_0}(x) \leqslant Ce^{-(\sqrt{\omega_0}/2)|x|}$, we have, for all $x \in \mathbb{R}$,

$$\left|\frac{1}{\Phi_B} - 1\right| Q_{\omega_0}(x) \leqslant e^{-(\sqrt{\omega_0}-2/B)|x|/2} \leqslant Ce^{-\sqrt{\omega_0}B/4} \leqslant \frac{1}{B}$$

for B large enough. Thus,

$$\int \left\{ f(|Q_{\omega_0}|^2)|w|^2 + 2f'(|Q_{\omega_0}|^2)[\operatorname{Re}(Q_{\omega_0}e^{-\mathrm{i}((1/2)v_0x)}w)]^2 \right\} \\ \leqslant \int \left\{ f(|Q_{\omega_0}|^2)|z|^2 + 2f'(|Q_{\omega_0}|^2)[\operatorname{Re}(Q_{\omega_0}e^{-\mathrm{i}((1/2)v_0x)}z)]^2 \right\} + \frac{C}{B}\int |z|^2.$$

Gathering these calculations, we obtain

$$H_{\Phi_B}(w, w) \geqslant H_0(z, z) - \frac{C}{B}\int (|\partial_x z|^2 + |z|^2).$$

Thanks to the orthogonality conditions on w, we verify easily using the property of Φ_B that

$$\left|\operatorname{Re}\int Q_{\omega_0}e^{-\mathrm{i}((1/2)v_0x)}z\right| \leqslant Ce^{-\sqrt{\omega_0}B/4} \leqslant \frac{1}{B}$$

for B large enough, and similarly with the two other quantities: $\operatorname{Re}\int Q'_{\omega_0}e^{-\mathrm{i}((1/2)v_0x)}z$ and $\operatorname{Im}\int Q_{\omega_0}e^{-\mathrm{i}((1/2)v_0x)}z$. By Claim 7, we obtain, for B large enough,

$$H_{\Phi_B}(w, w) \geqslant \left(\lambda_0 - \frac{C}{B}\right)\|z\|_{H^1}^2 \geqslant \frac{\lambda_0}{2}\|z\|_{H^1}^2 \geqslant \frac{\lambda_0}{2}\left(1 - \frac{C}{B}\right)\int (|w|^2 + |\partial_x w|^2)\Phi_B \\ \geqslant \frac{\lambda_0}{4}\int (|w|^2 + |\partial_x w|^2)\Phi_B,$$

which proves Claim 8. □

Now, we finish the proof of Lemma 4.1. Let $B > B_0$, and let $L > 0$. Since $\sum_{k=1}^{K} \varphi_k(t) \equiv 1$, we decompose $H_K(\varepsilon, \varepsilon)$ as follows:

$$\begin{aligned}
H_K(\varepsilon,\varepsilon) = &\sum_{k=1}^{K} \int \Phi_B\big(\cdot - x_k(t)\big)\Big\{|\partial_x \varepsilon|^2 + \Big(\omega_k(t) + \frac{v_k^2}{4}\Big)|\varepsilon|^2 - v_k \mathrm{Im}(\partial_x \varepsilon\, \overline{\varepsilon})\Big\} \\
&- \sum_{k=1}^{K} \int \big(f(|R_k|^2)|\varepsilon|^2 + 2f'(|R_k|^2)[\mathrm{Re}(\overline{R}_k \varepsilon)]^2\big) \\
&+ \sum_{k=1}^{K} \big(\varphi_k(t) - \Phi_B(\cdot - x_k(t))\big)\Big\{|\partial_x \varepsilon|^2 + \Big(\omega_k(t) + \frac{v_k^2}{4}\Big)|\varepsilon|^2 - v_k \mathrm{Im}(\partial_x \varepsilon \overline{\varepsilon})\Big\}.
\end{aligned}$$

By Claim 8, for any $k = 1, \ldots, K$, we have, for B large enough,

$$\begin{aligned}
&\int \Phi_B\big(\cdot - x_k(t)\big)\Big\{|\partial_x \varepsilon|^2 + \Big(\omega_k(t) + \frac{v_k^2}{4}\Big)|\varepsilon|^2 - v_k \mathrm{Im}(\partial_x \varepsilon\, \overline{\varepsilon})\Big\} \\
&\quad - \int \big(f(|R_k|^2)|\varepsilon|^2 + 2f'(|R_k|^2)[\mathrm{Re}(\overline{R}_k \varepsilon)]^2\big) \geqslant \lambda_k \int \Phi_B\big(\cdot - x_k(t)\big)(|\partial_x \varepsilon|^2 + |\varepsilon|^2).
\end{aligned}$$

Moreover, by the properties of Φ_B and $\varphi_k(t)$, for L large enough, we have

$$\varphi_k(t) - \Phi_B\big(\cdot - x_k(t)\big) \geqslant -e^{-L/(4B)},$$

and, for $\delta_k = \delta_k(\omega_k, v_k) > 0$,

$$|\partial_x \varepsilon|^2 + \Big(\omega_k(0) + \frac{v_k^2}{4}\Big)|\varepsilon|^2 - v_k \mathrm{Im}(\partial_x \varepsilon\, \overline{\varepsilon}) \geqslant \delta_k(|\partial_x \varepsilon|^2 + |\varepsilon|^2) \geqslant 0,$$

and so

$$\begin{aligned}
&\int \big(\varphi_k(t) - \Phi_B(\cdot - x_k(t))\big)\Big\{|\partial_x \varepsilon|^2 + \Big(\omega_k(0) + \frac{v_k^2}{4}\Big)|\varepsilon|^2 - v_k \mathrm{Im}(\partial_x \varepsilon\, \overline{\varepsilon})\Big\} \\
&\quad \geqslant \delta_k \int \big(\varphi_k(t) - \Phi_B(\cdot - x_k(t))\big)(|\partial_x \varepsilon|^2 + |\varepsilon|^2) - Ce^{-L/(4B)} \int (|\partial_x \varepsilon|^2 + |\varepsilon|^2).
\end{aligned}$$

Thus, putting everything together, we obtain, with $\lambda_k' = \min(\lambda_k, \delta_k)$,

$$H_K(\varepsilon,\varepsilon) \geqslant \lambda_k' \int \Big(\sum_{k=1}^{K} \varphi_k\Big)(|\partial_x \varepsilon|^2 + |\varepsilon|^2) - Ce^{-L/(4B)} \int (|\partial_x \varepsilon|^2 + |\varepsilon|^2),$$

and since $\sum_{k=1}^{K} \varphi_k(t) \equiv 1$, we obtain the result by taking L large enough.

C. Proof of Proposition 4.2

The proof is similar to that of Lemma 2.5 by expanding

$$u(t) = R(t) + \varepsilon(t) = \sum_{k=1}^{K} R_k(t) + \varepsilon(t)$$

in the expression of $\mathscr{G}_K$. Note from (3.3) and (3.16) that

$$1 = \psi_1 \equiv \sum_{k=1}^{K} \varphi_k,$$

so that

$$\mathscr{G}_K(t) = \sum_{k=1}^{K} \int \Big\{ |\partial_x u|^2 - F(|u|^2) - \Big(\omega_k(0) + \frac{v_k^2}{4}\Big)|u|^2 - v_k \mathrm{Im}(\partial_x u\, \overline{u}) \Big\} \varphi_k(t).$$

Since $\varphi_k = 1$ in a large neighborhood around the solitary wave k and equals zero outside, we see in this expression that $\mathscr{G}_K$ is the sum of the contributions of the K solitary waves.

Expanding $u(t) = R(t) + \varepsilon(t)$ in the expression of $E(u(t))$, we obtain, as in the proof of Proposition 4.2,

$$\begin{aligned} E\big(u(t)\big) = E\big(R(t)\big) - 2\,\mathrm{Re} \int \big(\partial_x^2 \overline{R} + f(|R|^2)\overline{R}\,\big)\varepsilon + \int |\partial_x \varepsilon|^2 \\ - \int \big\{ f(|R|^2)|\varepsilon|^2 + 2 f'(|R|^2)[\mathrm{Re}(\overline{R}\varepsilon)]^2 \big\} + \|\varepsilon(t)\|_{H^1}^2 \beta\big(\|\varepsilon(t)\|_{H^1}\big). \end{aligned}$$

Note that the centers of $R_k(t)$ and $R_{k-1}(t)$ are located at the distance $x_k(t) - x_{k-1}(t) > L + \theta_0 t$, and since the R_k are exponentially decaying (see (2.3) in Lemma 2.1), we have, for $k \neq k'$,

$$\int |R_k R_{k'}| + \int |\partial_x R_k R_{k'}| + \int |\partial_x R_k \partial_x R_{k'}| < C e^{-\theta_0 (L + \theta_0 t)}. \tag{C.1}$$

Also, note that $|F(s)| < Cs^2$, and note that $|f(s)| < Cs$ in a neighborhood of zero. Thus,

$$\begin{aligned} E\big(u(t)\big) = \sum_{k=1}^{K} \Big\{ E\big(R_k(t)\big) - 2\,\mathrm{Re} \int \big(\partial_x^2 \overline{R}_k + f(|R_k|^2)\overline{R}_k\big)\varepsilon \Big\} + O(e^{-\theta_0 (L+\theta_0 t)}) \\ + \int |\partial_x \varepsilon|^2 - \sum_{k=1}^{K} \Big\{ \int \big\{ f(|R_k|^2)|\varepsilon|^2 + 2 f'(|R_k|^2)[\mathrm{Re}(\overline{R}_k \varepsilon)]^2 \big\} \Big\} \\ + \|\varepsilon(t)\|_{H^1}^2 \beta\big(\|\varepsilon(t)\|_{H^1}\big). \end{aligned}$$

We turn now to $\mathscr{J}(t)$. Recall that

$$\mathscr{J}(t) = \sum_{k=1}^{K} \Big\{ \Big(\omega_k(0) + \frac{v_k^2}{4}\Big) \int |u(t)|^2 \varphi_k(t) - v_k \operatorname{Im} \int \partial_x u(t)\, \overline{u}(t) \varphi_k(t) \Big\}. \tag{C.2}$$

For the first term, we have

$$\int |u(t)|^2 \varphi_k(t) = \int |R(t)|^2 \varphi_k(t) + \int |\varepsilon(t)|^2 \varphi_k(t) + 2 \operatorname{Re} \int R(t)\varepsilon(t)\varphi_k(t).$$

By the properties of φ_k and R,

$$\int |R(t)|^2 \varphi_k(t) = \int |R_k(t)|^2 + O(e^{-\theta_0(L+\theta_0 t)}),$$

and

$$\operatorname{Re} \int R(t)\varepsilon(t)\varphi_k(t) = \operatorname{Re} \int R_k(t)\varepsilon(t) + O(e^{-\theta_0(L+\theta_0 t)}) = O(e^{-\theta_0(L+\theta_0 t)}),$$

by the orthogonality conditions on $\varepsilon(t)$.

For the second term in (C.2), we have, by similar arguments and integration by parts,

$$\operatorname{Im} \int \partial_x u\, \overline{u}\, \varphi_k(t) = \operatorname{Im} \int \partial_x R_k\, \overline{R}_k - \operatorname{Im} \int \overline{R}_k\, \varepsilon\, \varphi_k'(t) - 2 \operatorname{Im} \int \partial_x \overline{R}_k\, \varepsilon$$
$$+ \operatorname{Im} \int \partial_x \varepsilon\, \overline{\varepsilon}\, \varphi_k(t) + O(e^{-\theta_0(L+\theta_0 t)}).$$

By the properties of R_k and φ_k', we have $\big|\int \overline{R}_k\, \varepsilon\, \varphi_k'(t)\big| \leqslant C e^{-\theta_0(L+\theta_0 t)}$, and so

$$\operatorname{Im}\int \partial_x u\, \overline{u}\, \varphi_k(t) = \operatorname{Im}\int \partial_x R_k\, \overline{R}_k - 2 \operatorname{Im} \int \partial_x \overline{R}_k\, \varepsilon + \operatorname{Im}\int \partial_x \varepsilon\, \overline{\varepsilon}\, \varphi_k(t) + O(e^{-\theta_0(L+\theta_0 t)}).$$

Gathering these calculations, we obtain, finally, for $\mathscr{J}(t)$,

$$\mathscr{J}(t) = \sum_{k=1}^{K} \Big(\omega_k(0) + \frac{v_k^2}{4}\Big) \Big\{ \int |R_k(t)|^2 + 2 \operatorname{Re} \int R_k(t)\varepsilon(t) + \int |\varepsilon(t)|^2 \varphi_k(t) \Big\}$$
$$- v_k \Big\{ \operatorname{Im} \int \partial_x R_k\, \overline{R}_k - 2 \operatorname{Im} \int \partial_x \overline{R}_k\, \varepsilon + \operatorname{Im} \int \partial_x \varepsilon\, \overline{\varepsilon} \varphi_k(t) \Big\} + O(e^{-\theta_0(L+\theta_0 t)}).$$

By the equation of R_k and the orthogonality conditions (4.7), as in the proof of Lemma 2.5, we have

$$-2 \operatorname{Re} \int \big(\partial_x^2 \overline{R}_k + f(|R_k|^2)\overline{R}_k\big)\varepsilon + 2\Big(\omega_k(0) + \frac{v_k^2}{4}\Big) \operatorname{Re} \int \overline{R}_k\, \varepsilon + 2 v_k \operatorname{Im} \int \partial_x \overline{R}_k\, \varepsilon = 0,$$

which means that the terms of order 1 in ε all disappear when we sum $E(u(t))$ and $\mathscr{J}(t)$.

Therefore, with the definition of $H_K(\varepsilon, \varepsilon)$ given in the proposition, and

$$\mathscr{G}\big(R_k(t)\big) = E\big(R_k(t)\big) + \Big(\omega_k(0) + \frac{v_k^2}{4}\Big) \int |R_k|^2 - v_k \operatorname{Im} \int \partial_x R_k \overline{R}_k = \mathscr{F}_{\omega_k(0)}(Q_{\omega_k(t)}),$$

we obtain

$$\mathscr{G}_K(t) = \sum_{k=1}^{K} \mathscr{F}_{\omega_k(0)}(Q_{\omega_k(t)}) + H_K\big(\varepsilon(t), \varepsilon(t)\big) + \|\varepsilon(t)\|_{H^1}^2 \beta\big(\|\varepsilon(t)\|_{H^1}\big) + O(e^{-\theta_0(L+\theta_0 t)}),$$

and we get (4.11) by using Lemma 2.3. □

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Martel
Département de Mathématiques, Université de Versailles-Saint-Quentin-en-Yvelines, 45 avenue des Etats-Unis, F-78035 Versailles CEDEX, France; martel@math.uvsq.fr

Merle
Département de Mathématiques, Université de Cergy-Pontoise, 2 avenue Adolphe-Chauvin, F-95302 Cergy-Pontoise CEDEX, France; frank.merle@math.u-cergy.fr

Tsai
Department of Mathematics, University of British Columbia, 1984 Mathematics Road, Vancouver, British Columbia, Canada V6T1Z2; ttsai@math.ubc.ca