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STABLE DIRECTIONS FOR EXCITED STATES OF NONLINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT

We consider nonlinear Schrödinger equations in $\mathbb{R}^3$. Assume that the linear Hamiltonians have two bound states. For certain finite codimension subset in the space of initial data, we construct solutions converging to the excited states in both non-resonant and resonant cases. In the resonant case, the linearized operators around the excited states are non-self adjoint perturbations to some linear Hamiltonians with embedded eigenvalues. Although self-adjoint perturbation turns embedded eigenvalues into resonances, this class of non-self adjoint perturbations turn an embedded eigenvalue into two eigenvalues with the distance to the continuous spectrum given to the leading order by the Fermi golden rule.

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1. INTRODUCTION

Consider the nonlinear Schrödinger equation

$$i\partial_t \psi = (-\Delta + V)\psi + \lambda|\psi|^2\psi, \quad \psi(t=0) = \psi_0, \tag{1.1}$$

where V is a smooth localized real potential, $\lambda = \pm 1$ and $\psi = \psi(t, x) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ is a wave function. The goal of this paper is to study the asymptotic dynamics of the solution for initial data ψ_0 near some *nonlinear excited state*.

For any solution $\psi(t) \in H^1(\mathbb{R}^3)$ the L^2-norm and the Hamiltonian

$$\mathcal{H}[\psi] = \int \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}V|\psi|^2 + \frac{1}{4}\lambda|\psi|^4\, dx \tag{1.2}$$

are constant for all t. The global well-posedness for small solutions in $H^1(\mathbb{R}^3)$ can be proved using these conserved quantities and a continuity argument.

We assume that the linear Hamiltonian $H_0 := -\Delta + V$ has two simple eigenvalues $e_0 < e_1 < 0$ with normalized eigen-functions ϕ_0, ϕ_1. The nonlinear bound states to the Schrödinger equation (1.1) are solutions to the equation

$$(-\Delta + V)Q + \lambda|Q|^2 Q = EQ. \tag{1.3}$$

They are critical points to the Hamiltonian $\mathcal{H}[\psi]$ defined in Eq. (1.2) subject to the constraint that the L^2-norm of ψ is fixed. We may obtain two families of such bound states by standard bifurcation theory, corresponding to the two eigenvalues of the linear Hamiltonian. For any E sufficiently close to e_0 so that $E - e_0$ and λ have the same sign, there is a unique positive solution $Q = Q_E$ to Eq. (1.3) which decays exponentially as $x \to \infty$. See Lemma 2.1 of Ref. [24]. We call this family the *nonlinear ground states* and we refer to it as $\{Q_E\}_E$. Similarly, there is a *nonlinear excited state* family $\{Q_{1,E_1}\}_{E_1}$ for E_1 near e_1. We will abbreviate them as Q and Q_1. From the same Lemma 2.1 of Ref. [24], these solutions are small and we have $\|Q_E\| \sim |E - e_0|^{1/2}$ and $\|Q_{1,E_1}\| \sim |E_1 - e_1|^{1/2}$.

It is well-known that the family of nonlinear ground states is stable in the sense that if

$$\inf_{\Theta, E} \| \psi(t) - Q_E\, e^{i\Theta} \|_{L^2}$$

is small for $t = 0$, it remains so for all t, see Ref. [16]. Let $\|\cdot\|_{L^2_{\text{loc}}}$ denote a local L^2 norm, for example the L^2-norm in a ball with large radius. One expects that this difference actually approaches zero in local L^2 norm, i.e.,

$$\lim_{t\to\infty} \inf_{\Theta, E} \| \psi(t) - Q_E\, e^{i\Theta} \|_{L^2_{\text{loc}}} = 0. \tag{1.4}$$

If $-\Delta + V$ has only one bound state, it is proved in Refs. [12,20] that the evolution will eventually settle down to some ground state Q_{E_∞} with E_∞ close to E. Suppose now that $-\Delta + V$ has two bound states: a ground state ϕ_0 with eigenvalue e_0 and an excited state ϕ_1 with eigenvalue e_1. It is proved in Ref. [23] that the evolution with initial data ψ_0 near some Q_E will eventually settle down to some ground state Q_{E_∞} with E_∞ close to E. See also Refs. [2–4] for the one dimensional case, Refs. [5,6] for its extension to higher dimensions, and Ref. [21] for real-valued nonlinear Klein–Gorden equations.

If the initial data is not restricted to near the ground states, the problem becomes much more delicate due to the presence of the excited states. On physical ground, quantum mechanics tells us that excited states are unstable and all perturbations should result in a release of radiation and the relaxation of the excited states to the ground states. Since bound states are periodic orbits, this picture differs from the classical one where periodic orbits are in general stable.

There were extensive linear analysis for bound states of nonlinear Schrödinger and wave equations, see, e.g., Refs. [7,8,17–19,25,26]. A special case of Theorem 3.5 of Ref. [8], page 330, states that

Theorem A. *Let $H_1 = -\Delta + V - E_1$. The matrix operator*

$$JH_1 = \begin{bmatrix} 0 & H_1 \\ -H_1 & 0 \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

is structurally stable if and only if $e_0 > 2e_1$.

The precise meaning of structural stability was given in Ref. [8]. Roughly speaking, it means that the operator remains stable under small perturbations. Theorem A will not be directly used in this paper.

As we will see later, the linearized operator around an excited state is a perturbation of JH_1. Thus, two different situations occur:

1. Non-resonant case: $e_0 > 2e_1$. $(e_{01} < |e_1|)$.
2. Resonant case: $e_0 < 2e_1$. $(e_{01} > |e_1|)$.

Here $e_{01} = e_1 - e_0 > 0$. In the resonant case, Theorem A says the linearized operator is in general unstable, which agrees with the physical picture. In the non-resonant case, however, the linearized operator becomes stable. The difference here is closely related to the fact that $2e_1 - e_0$ lies in the continuum spectrum of H_0 only in the resonant case.

In the resonant case, the unstable picture is confirmed for most data near excited states in our work.[24] We prove that, as long as the ground state component in $\psi_0 - Q_1$ is larger than $\|\psi_0\|^2$ times the size of the dispersive part corresponding to the continuous spectrum, the solution will move away from the excited states and relax and stabilize to ground states locally. Since $\|\psi_0\|^2$ is small, this assumption allows the dispersive part to be much larger than the ground state component.

There is a small set of data where Ref. [24] does not apply, namely, those data with ground state component in $\psi_0 - Q_1$ smaller than $\|\psi_0\|^2$ times the size of the dispersive part. The aim of this paper is to show that this restriction is almost optimal: we will construct within this small set of initial data a "hypersurface" whose corresponding solutions converge to *excited states*.

This does not contradict with the physical intuition since this hypersurface in certain sense has zero measure and cannot be observed in experiments. These solutions, however, show that linear instability does not imply all solutions to be unstable. In the language of dynamical systems, *the excited states are one parameter family of hyperbolic fixed points and this hypersurface is contained in the stable manifold of the fixed points*. We believe that this surface is the whole stable manifold.

We will also construct solutions converging to excited states in the non-resonant case, where it is expected since the linearized operator is stable. We now state our assumptions on the potential V:

Assumption A0. $H_0 := -\Delta + V$ acting on $L^2(\mathbb{R}^3)$ has two simple eigenvalues $e_0 < e_1 < 0$, with normalized eigenvectors ϕ_0 and ϕ_1.

Assumption A1. The bottom of the continuous spectrum to $-\Delta + V$, 0, is not a generalized eigenvalue, i.e., not an eigenvalue nor a resonance. There is a small $\sigma > 0$ such that

$$|\nabla^\alpha V(x)| \le C\langle x\rangle^{-5-\sigma}, \quad \text{for} \quad |\alpha| \le 2.$$

Also, the functions $(x \cdot \nabla)^k V$, for $k = 0, 1, 2, 3$, are $-\Delta$ bounded with a $-\Delta$-bound < 1:

$$\|(x \cdot \nabla)^k V\phi\|_2 \le \sigma_0\| - \Delta\phi\|_2 + C\|\phi\|_2, \quad \sigma_0 < 1, \ \ k = 0, 1, 2, 3.$$

Assumption A1 contains some standard conditions to assure that most tools in linear Schrödinger operators apply. In particular, it satisfies the assumptions of Ref. [27] so that the wave operator $W_{H_0} = \lim_{t\to\infty} e^{itH_0}e^{it\Delta}$ satisfies the $W^{k,p}$ estimates for $k \le 2$. These conditions are certainly not optimal.

Let $e_{01} = e_1 - e_0$ be the spectral gap of the ground state. In the resonant case $2e_{01} > |e_0|$ so that $2e_1 - e_0$ lies in the continuum spectrum of H_0, we further assume

Assumption A2. For some $s_0 > 0$,

$$\gamma_0 \equiv \inf_{|s|<s_0} \lim_{\sigma\to 0+} \mathrm{Im}\left(\phi_0\phi_1^2,\ \frac{1}{H_0 + e_0 - 2e_1 + s - \sigma i}\, \mathbf{P}_{\mathrm{c}}{}^{H_0}\phi_0\phi_1^2\right) > 0. \tag{1.5}$$

Note that $\gamma_0 \ge 0$ since the expression above is quadratic. This assumption is generically true.

Let $Q_1 = Q_{1,E_1}$ be a nonlinear excited state with $\|Q_{1,E_1}\|_2$ small. Since (Q_1, E_1) satisfies Eq. (1.3), the function $\psi(t,x) = Q_1(x)e^{-iE_1 t}$ is an exact solution of Eq. (1.1). If we consider solutions $\psi(t,x)$ of Eq. (1.1) of the form

$$\psi(t,x) = [Q_1(x) + h(t,x)]\, e^{-iE_1 t}$$

with $h(t,x)$ small in a suitable sense, then $h(t,x)$ satisfies

$$\partial_t h = \mathcal{L}_1 h + \text{nonlinear terms},$$

where $\mathcal{L}_1$, the *linearized operator around the nonlinear excited state solution* $Q_1(x)e^{-iE_1 t}$, is defined by

$$\mathcal{L}_1 h = -i\{(-\Delta + V - E_1 + 2\lambda Q_1^2)\, h + \lambda Q_1^2\, \bar{h}\}. \tag{1.6}$$

Theorem 1.1. *Suppose $H_0 = -\Delta + V$ satisfies Assumptions* A0–A1. *Suppose either*

(NR) $e_0 > 2e_1$, *or*
(R) $e_0 < 2e_1$, *and the Assumption* A2 *for γ_0 holds.*

Then there are $n_0 > 0$ and $\varepsilon_0(n) > 0$ defined for $n \in (0, n_0]$ such that the following holds. Let $Q_1 := Q_{1,E_1}$ be a nonlinear excited state with $\|Q_1\|_{L^2} = n \le n_0$, and let $\mathcal{L}_1$ be the corresponding linearized operator. For any $\xi_\infty \in \mathbf{H}_{\mathrm{c}}(\mathcal{L}_1) \cap (W^{2,1} \cap H^2)(\mathbb{R}^3)$ with $\|\xi_\infty\|_{W^{2,1}\cap H^2} = \varepsilon$, $0 < \varepsilon \le \varepsilon_0(n)$, there is a solution $\psi(t,x)$ of Eq. (1.1) *and a real function $\theta(t) = O(t^{-1})$ for $t > 0$ so that*

$$\|\psi(t) - \psi_{as}(t)\|_{H^2} \le C\varepsilon^2(1+t)^{-7/4},$$

where $C = C(n)$ *and*

$$\psi_{as}(t) = Q_1\, e^{-iE_1 t + i\theta(t)} + e^{-iE_1 t} e^{t\mathcal{L}_1}\xi_\infty.$$

To prove this theorem, a detailed spectral analysis of the linearized operator $\mathcal{L}_1$ is required. We shall classify the spectrum of $\mathcal{L}_1$ completely in both non-resonant and resonant cases, see Theorems 2.1 and 2.2. It is well-known that the continuous spectrum Σ_c of $\mathcal{L}_1$ is the same as that of JH_1, i.e., $\Sigma_c = \{si : s \in \mathbb{R}, |s| \ge |E_1|\}$. The point spectrum of $\mathcal{L}_1$ is more subtle. By definition, $H_1\phi_1 = -(E_1 - e_1)\phi_1$ and $H_1\phi_0 = -(E_1 - e_0)\phi_0$, and thus the matrix operator JH_1 has 4 eigenvalues $\pm i(E_1 - e_1)$ and $\pm i(E_1 - e_0)$. In the non-resonant case, the eigenvalues of $\mathcal{L}_1$ are purely imaginary and are small perturbations of these eigenvalues. In the resonant case, the eigenvalues $\pm i(E_1 - e_0)$ are embedded inside the continuum spectrum Σ_c. In general perturbation theory for embedded eigenvalues, they turn into resonances under self-adjoint perturbations. The operator $\mathcal{L}_1$ is however not a self-adjoint perturbation of H_1. In this case, we shall prove that *the embedded eigenvalues* $\pm i(E_1 - e_0)$ *split into four eigenvalues* $\pm\omega_*$ *and* $\pm\bar{\omega}_*$ *with the real part given approximately by the Fermi golden rule* (see Ref. [15], Chap. XII.6):

$$n^4\, \mathrm{Im}\left(\lambda\phi_0\phi_1^2,\ \frac{1}{-\Delta + V + e_0 - 2e_1 - 0i}\mathbf{P}_c\lambda\phi_1^2\phi_0\right).$$

Here $n \ll 1$ is the size of Q_1, see Eq. (2.45). In particular, $e^{t\mathcal{L}_1}$ is *exponentially unstable* with the decay rate (or the blow-up rate) given approximately by the Fermi golden rule. In other words, *although self-adjoint perturbation turns embedded eigenvalues into resonances*, *the non-self adjoint perturbations given by* $\mathcal{L}_1$ *turns an embedded eigenvalue into two eigenvalues with the shifts in the real axis given to the leading order by the Fermi golden rule*. The dynamics of self-adjoint perturbation of embedded eigenvalues were studied in Ref. [22].

In the appendix we will prove the existence of solutions vanishing locally as $t \to \infty$, independent of the number of bound states of H_0. Although it is probably known to experts, we are unable to find a reference and hence include it for completeness.

Proposition 1.2. *Suppose* $H_0 = -\Delta + V$ *satisfies Assumption* A1. *There is a small constant* $\varepsilon_0 > 0$ *such that the following holds. For any* $\xi_\infty \in \mathbf{H}_c(H_0) \cap (W^{2,1} \cap H^2)(\mathbb{R}^3)$ *with* $0 < \|\xi_\infty\|_{W^{2,1}\cap H^2} = \varepsilon \le \varepsilon_0$, *there is a solution* $\psi(t,x)$ *of*

Eq. (1.1) *of the form*

$$\psi(t) = e^{-itH_0}\xi_\infty + g(t), \quad (t \geq 0),$$

with $\|g(t)\|_{H^2} \leq C\varepsilon^2(1+t)^{-2}$.

2. LINEAR ANALYSIS FOR EXCITED STATES

As mentioned in §1, there is a family $\{Q_{1,E_1}\}_{E_1}$ of nonlinear excited states with the frequency E_1 as the parameter. They satisfy

$$(-\Delta + V)Q_1 + \lambda|Q_1|^2 Q_1 = E_1 Q_1. \tag{2.1}$$

Let $Q_1 = Q_{1,E_1}$ be a fixed nonlinear excited state with $n = \|Q_{1,E_1}\|_2 \leq n_0 \ll 1$. The linearized operator around the nonlinear bound state solution $Q_1(x)e^{-iE_1 t}$ is defined in Eq. (1.6)

$$\mathcal{L}_1 h = -i\left\{(-\Delta + V - E_1 + 2\lambda Q_1^2)\, h + \lambda Q_1^2 \bar{h}\right\}.$$

We will study the spectral properties of $\mathcal{L}_1$ in this section. Its properties are best understood in the complexification of $L^2(\mathbb{R}^3, \mathbb{C})$.

Definition 2.1. Identify $\mathbb{C}$ with $\mathbb{R}^2$ and $L^2 = L^2(\mathbb{R}^3, \mathbb{C})$ with $L^2(\mathbb{R}^3, \mathbb{R}^2)$. Denote by $\mathbb{C}L^2 = L^2(\mathbb{R}^3, \mathbb{C}^2)$ the complexification of $L^2(\mathbb{R}^3, \mathbb{R}^2)$. $\mathbb{C}L^2$ consists of 2-dimensional vectors whose components are in L^2. We have the natural embedding

$$\mathbf{j} : f \in L^2 \longrightarrow \begin{bmatrix} \operatorname{Re} f \\ \operatorname{Im} f \end{bmatrix} \in \mathbb{C}L^2.$$

We equip $\mathbb{C}L^2$ with the natural inner product: For $f, g \in \mathbb{C}L^2$, $f = \left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right]$, $g = \left[\begin{smallmatrix} g_1 \\ g_2 \end{smallmatrix}\right]$, we define

$$(f, g) = \int_{\mathbb{R}^3} \bar{f} \cdot g \, d^3x = \int_{\mathbb{R}^3} (\bar{f}_1 g_1 + \bar{f}_2 g_2)\, d^3x. \tag{2.2}$$

Denote by **RE** the operator first taking the real part of functions in $\mathbb{C}L^2$ and then pulling back to L^2:

$$\mathbf{RE} : \mathbb{C}L^2 \longrightarrow L^2, \quad \mathbf{RE}\begin{bmatrix} f \\ g \end{bmatrix} = (\operatorname{Re} f) + i(\operatorname{Re} g).$$

We have $\mathbf{RE} \circ \mathbf{j} = \mathbf{id}_{L^2}$.

Recall the matrix operator JH_1 defined in Theorem A. Since $H_1\phi_1 = -(E_1 - e_1)\phi_1$ and $H_1\phi_0 = -(E_1 - e_0)\phi_0$, the matrix operator JH_1 has 4 eigenvalues $\pm i(E_1 - e_1)$ and $\pm i(E_1 - e_0)$ with corresponding eigenvectors

$$\begin{bmatrix} \phi_1 \\ -i\phi_1 \end{bmatrix}, \quad \begin{bmatrix} \phi_1 \\ i\phi_1 \end{bmatrix}, \quad \begin{bmatrix} \phi_0 \\ -i\phi_0 \end{bmatrix}, \quad \begin{bmatrix} \phi_0 \\ i\phi_0 \end{bmatrix}. \tag{2.3}$$

Notice that

$$E_1 - e_1 = O(n^2), \qquad E_1 - e_0 = e_{01} + O(n^2). \tag{2.4}$$

The continuous spectrum of JH_1 is

$$\Sigma_c = \left\{ si : s \in \mathbb{R}, |s| \geq |E_1| \right\}, \tag{2.5}$$

which consists of two rays on the imaginary axis.

The operator $\mathcal{L}_1$ in its matrix form

$$\begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}, \quad \text{with} \quad \begin{cases} L_- = -\Delta + V - E_1 + \lambda Q_1^2 \\ L_+ = -\Delta + V - E_1 + 3\lambda Q_1^2 \end{cases} \tag{2.6}$$

is a perturbation of JH_1. By Weyl's lemma, the continuous spectrum of $\mathcal{L}_1$ is also Σ_c. The eigenvalues are more complicated. In both cases ($e_{01} < |e_1|$ and $e_{01} > |e_1|$) they are near 0 and $\pm ie_{01}$. As we shall see, in both cases 0 is an eigenvalue of $\mathcal{L}_1$. The main difference between the two cases are the eigenvalues near ie_{01} and $-ie_{01}$. If $e_{01} < |e_1|$, then ie_{01} lies outside the continuous spectrum and $\mathcal{L}_1$ has an eigenvalue near ie_{01} which is purely imaginary. On the other hand, if $e_{01} > |e_1|$, then ie_{01} lies inside the continuous spectrum. It splits under our perturbation and the eigenvalues of $\mathcal{L}_1$ near $\pm ie_{01}$ have non-zero real parts.

We shall show that $L^2(\mathbb{R}^3, \mathbb{C})$, as a real vector space, can be decomposed as the direct sum of three invariant subspaces

$$L^2(\mathbb{R}^3, \mathbb{C}) = S(\mathcal{L}_1) \oplus \mathbf{E}_1(\mathcal{L}_1) \oplus \mathbf{H}_c(\mathcal{L}_1). \tag{2.7}$$

Here $S(\mathcal{L}_1)$ is the generalized null space, $\mathbf{E}_1(\mathcal{L}_1)$ is the eigenspace associated to nonzero generalized eigenvalues (they become eigenvalues for the complexified space $\mathbb{C}\mathbf{E}_1(\mathcal{L}_1)$, see below), and $\mathbf{H}_c(\mathcal{L}_1)$ corresponds to the continuous spectrum. Both $S(\mathcal{L}_1)$ and $\mathbf{E}_1(\mathcal{L}_1)$ are finite dimensional.

Recall the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

They are self-adjoint and

$$\sigma_1 \mathcal{L}_1 = \mathcal{L}_1^* \sigma_1, \qquad \sigma_3 \mathcal{L}_1 = -\mathcal{L}_1 \sigma_3, \tag{2.8}$$

where $\mathcal{L}_1^* = \left[\begin{smallmatrix} 0 & -L_+ \\ L_- & 0 \end{smallmatrix}\right]$.

Let $R_1 = \partial_{E_1} Q_{1,E_1}$. Direct differentiation of Eq. (2.1) with respect to E_1 gives $L_+ R_1 = Q_1$. Since $L_- Q_1 = 0$ and $L_+ R_1 = Q_1$, we have $\mathcal{L}_1 \left[\begin{smallmatrix} 0 \\ Q_1 \end{smallmatrix}\right] = 0$ and $\mathcal{L}_1 \left[\begin{smallmatrix} R_1 \\ 0 \end{smallmatrix}\right] = -\left[\begin{smallmatrix} 0 \\ Q_1 \end{smallmatrix}\right]$. We will show $\dim_{\mathbb{R}} S(\mathcal{L}_1) = 2$, hence

$$S(\mathcal{L}_1) = \operatorname*{span}_{\mathbb{R}} \left\{ \begin{bmatrix} 0 \\ Q_1 \end{bmatrix}, \begin{bmatrix} R_1 \\ 0 \end{bmatrix} \right\}. \tag{2.9}$$

$\mathbf{H}_c(\mathcal{L}_1)$ can be characterized as

$$\mathbf{H}_c(\mathcal{L}_1) = \left\{ \psi \in L^2 : (\sigma_1 \psi, f) = 0, \forall f \in S(\mathcal{L}_1) \oplus \mathbf{E}_1(\mathcal{L}_1) \right\}. \tag{2.10}$$

We will use Eq. (2.10) as a working definition of $\mathbf{H}_c(\mathcal{L}_1)$. After we have proved the spectrum of $\mathcal{L}_1$ and the resolvent estimates, we will use the wave operator of $\mathcal{L}_1$ (see Refs. [5,27,28]) to show that Eq. (2.10) agrees with the usual definition of the continuous spectrum subspace. See §2.5.

The space $\mathbf{E}_1(\mathcal{L}_1)$, however, has very different properties in the two cases, resonant or nonresonant, due to whether $\pm i(E_1 - e_0)$ are embedded eigenvalues of JH_1. We will consider $\mathbf{E}_1 = \mathbf{E}_1(\mathcal{L}_1)$ as a subspace of $L^2(\mathbb{R}^3, \mathbb{R}^2)$ and denote by $\mathbb{C}\mathbf{E}_1 \subset \mathbb{C}L^2$ the complexification of $\mathbf{E}_1$. We will show that $\mathbb{C}\mathbf{E}_1$ is a direct sum of eigenspaces of $\mathcal{L}_1$ in $\mathbb{C}L^2$. We also have

$$(\sigma_1 f, g) = 0, \quad \forall f \in S(\mathcal{L}_1), \quad \forall g \in \mathbf{E}_1(\mathcal{L}_1). \tag{2.11}$$

We have the following two theorems for the two cases.

Theorem 2.1 (Non-resonant case). *Suppose $e_0 > 2e_1$, and the Assumptions* A0–A1 *hold. Let $Q_1 = Q_{1,E_1}$ be a nonlinear excited state with $\|Q_1\|_{L^2} = n$ sufficiently small, and let $\mathcal{L}_1$ be defined as in Eq.* (1.6).

(1) *The eigenvalues of $\mathcal{L}_1$ are* 0 *and $\pm\omega_*$. The multiplicity of* 0 *is two. The other eigenvalues are simple. Here $\omega_* = i\kappa$, κ is real, $\kappa = e_{01} + O(n^2)$. There is no embedded eigenvalue. The bottoms of the continuous spectrum are not eigenvalue nor resonance.*

(2) *The space $L^2 = L^2(\mathbb{R}^3, \mathbb{C})$, as a real vector space, can be decomposed as in Eq.* (2.7). *Here $S(\mathcal{L}_1)$ and $\mathbf{H}_c(\mathcal{L}_1)$ are given in Eqs.* (2.9) *and* (2.10), *respectively; $\mathbf{E}_1(\mathcal{L}_1)$ is the space corresponding to the perturbation of the eigenvalues $\pm i(E_1 - e_0)$ of JH_1. We have the orthogonality relation* (2.11).

(3) *Let $\mathbb{C}\mathbf{E}_1$ denotes the complexification of $\mathbf{E}_1 = \mathbf{E}_1(\mathcal{L}_1)$. $\mathbb{C}\mathbf{E}_1$ is* 2*-complex-dimensional. $\mathbf{E}_1$ is* 2*-real-dimensional. We have*

$$\begin{aligned} \mathbb{C}\mathbf{E}_1 &= \operatorname*{span}_{\mathbb{C}} \{\Phi, \overline{\Phi}\}, \\ \mathbf{E}_1 &= \operatorname*{span}_{\mathbb{R}} \left\{ \begin{bmatrix} u \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ v \end{bmatrix} \right\}. \end{aligned} \tag{2.12}$$

Here $\Phi = \left[\begin{smallmatrix} u \\ -iv \end{smallmatrix}\right]$ is an eigenfunction of $\mathcal{L}_1$ with eigenvalue ω_*. u and v are real-valued L^2-functions satisfying $L_+ u = -\kappa v$, $L_- v = -\kappa u$ and $(u, v) = 1$. u and v

are perturbations of ϕ_0. $\bar{\Phi} = \left[\begin{smallmatrix} u \\ iv \end{smallmatrix}\right]$ *is another eigenfunction with eigenvalue* $-\omega_*$. *We have* $\mathcal{L}_1\Phi = \omega_*\Phi$, $\mathcal{L}_1\bar{\Phi} = -\omega_*\bar{\Phi}$.

(4) *For any function* $\zeta \in \mathbf{E}_1(\mathcal{L}_1)$, *there is a unique* $\alpha \in \mathbb{C}$ *so that*

$$\zeta = \mathbf{RE}\ \alpha\Phi.$$

We have $\mathcal{L}_1\zeta = \mathbf{RE}\ \omega_*\alpha\Phi$ *and* $e^{t\mathcal{L}_1}\zeta = \mathbf{RE}\ e^{t\omega_*}\alpha\Phi$.

(5) *We have the orthogonality relations in Eqs.* (2.10) *and* (2.11). *Hence any* $\psi \in L^2$ *can be decomposed as* (*see Eq.* (2.7))

$$\psi = a\begin{bmatrix} R_1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ Q_1 \end{bmatrix} + c\begin{bmatrix} u \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ v \end{bmatrix} + \eta, \tag{2.13}$$

with $\eta \in \mathbf{H}_c(\mathcal{L}_1)$,

$$\begin{aligned} a &= (Q_1, R_1)^{-1}(Q_1,\ \mathrm{Re}\,\psi), \qquad & c &= (u, v)^{-1}(v,\ \mathrm{Re}\,\psi), \\ b &= (Q_1, R_1)^{-1}(R_1,\ \mathrm{Im}\,\psi), \qquad & d &= (u, v)^{-1}(u,\ \mathrm{Im}\,\psi). \end{aligned} \tag{2.14}$$

(6) *Let* $M_1 \equiv \mathbf{E}_1(\mathcal{L}_1) \oplus \mathbf{H}_c(\mathcal{L}_1)$. *We have*

$$M_1 \equiv \mathbf{E}_1(\mathcal{L}_1) \oplus \mathbf{H}_c(\mathcal{L}_1) = \begin{bmatrix} Q_1^\perp \\ R_1^\perp \end{bmatrix}. \tag{2.15}$$

There is a constant $C > 1$ *such that*, *for all* $\phi \in M_1$ *and all* $t \in \mathbb{R}$, *we have*

$$C^{-1}\|\phi\|_{H^k} \le \|e^{t\mathcal{L}_1}\phi\|_{H^k} \le C\|\phi\|_{H^k}, \quad (k = 1, 2). \tag{2.16}$$

(7) *Decay estimates: For all* $\eta \in \mathbf{H}_c(\mathcal{L}_1)$, *for all* $p \in [2, \infty]$, *one has*

$$\|e^{t\mathcal{L}_1}\eta\|_{L^p} \le C|t|^{-3(1/2-1/p)}\|\eta\|_{L^{p'}}.$$

Theorem 2.2 (Resonant case). *Suppose* $e_0 < 2e_1$, *and the Assumptions* A0–A2 *hold. Let* $Q_1 = Q_{1,}\ E_1$ *be a nonlinear excited state with* $\|Q_1\|_{L^2} = n$ *sufficiently small, and let* $\mathcal{L}_1$ *be defined as in Eq.* (1.6).

(1) *The eigenvalues of* $\mathcal{L}_1$ *are* 0, $\pm\omega_*$ *and* $\pm\bar{\omega}_*$. *The multiplicity of* 0 *is two. The other eigenvalues are simple. Here* $\omega_* = i\kappa + \gamma$, $\kappa, \gamma > 0$, $\kappa = e_{01} + O(n^2)$, *and* $\frac{3}{4}\lambda^2\gamma_0 n^4 \le \gamma \le Cn^4$. ($\gamma_0$ *is given in Eq.* (1.5)). *There is no embedded eigenvalue. The bottoms of the continuous spectrum are not eigenvalue nor resonance.*

There is an ω_**-eigenvector* Φ, $\mathcal{L}_1\Phi = \omega_*\Phi$, *which is of order one in* L^2 and $\Phi - \left[\begin{smallmatrix} \phi_0 \\ -i\phi_0 \end{smallmatrix}\right]$ *is locally small in the sense that*

$$\left|\left(\phi, \Phi - \begin{bmatrix} \phi_0 \\ -i\phi_0 \end{bmatrix}\right)\right| \le C_r n^2 \|\langle x\rangle^r \phi\|_{L^2}, \tag{2.17}$$

for any ϕ, *for any* $r > 3$. *However,* Φ *is not a perturbation of* $\begin{bmatrix} \phi_0 \\ -i\phi_0 \end{bmatrix}$ *in* $\mathbb{C}L^2$. *In fact,* $\Phi = \begin{bmatrix} u \\ v \end{bmatrix}$ *with* $u - \phi_0$ *and* $v + i\phi_0$ *of order one in* L^2,

$$u = \phi_0 - \frac{1}{-\Delta + V - E_1 - \kappa + \gamma i} \mathbf{P}_c(H_0)\lambda\phi_0 Q_1^2 + O(n^2) \quad in \quad L^2,$$

and $v = -L_+ u/\omega_*$. *Note* $-E_1 - \kappa = e_0 - 2e_1 + O(n^2)$.

(2) *The space* $L^2 = L^2(\mathbb{R}^3, \mathbb{C})$, *as a real vector space, can be decomposed as in Eq.* (2.7). *Here* $S(\mathcal{L}_1)$ *and* $\mathbf{H}_c(\mathcal{L}_1)$ *are given in Eqs.* (2.9) *and* (2.10), *respectively;* $\mathbf{E}_1(\mathcal{L}_1)$ *is the space corresponding to the perturbation of the eigenvalues* $\pm i(E_1 - e_0)$ *of* JH_1. *We have the orthogonality relation* (2.11).

(3) *Let* $\mathbb{C}\mathbf{E}_1$ *denotes the complexification of* $\mathbf{E}_1 = \mathbf{E}_1(\mathcal{L}_1)$. $\mathbb{C}\mathbf{E}_1$ *is* 4-*complex-dimensional.* $\mathbf{E}_1$ *is* 4-*real-dimensional. If we write* $\Phi = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u_1 + u_2 i \\ v_1 + v_2 i \end{bmatrix}$ with u_1, u_2, v_1, v_2 *real-valued* L^2 *functions, we have*

$$\begin{aligned} \mathbb{C}\mathbf{E}_1 &= \operatorname*{span}_{\mathbb{C}}\{\Phi, \bar{\Phi}, \sigma_3\Phi, \sigma_3\bar{\Phi}\}, \\ \mathbf{E}_1 &= \operatorname*{span}_{\mathbb{R}}\left\{\begin{bmatrix} u_1 \\ 0 \end{bmatrix}, \begin{bmatrix} u_2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ v_1 \end{bmatrix}, \begin{bmatrix} 0 \\ v_2 \end{bmatrix}\right\}. \end{aligned} \tag{2.18}$$

Recall $\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. *The other eigenvectors are* $\bar{\Phi}$, $\sigma_3\Phi$ *and* $\sigma_3\bar{\Phi}$,

$$\begin{aligned} &\mathcal{L}_1\Phi = \omega_*\Phi, \qquad \mathcal{L}_1\bar{\Phi} = \bar{\omega}_*\bar{\Phi}, \\ &\mathcal{L}_1\sigma_3\Phi = -\omega_*(\sigma_3\Phi), \qquad \mathcal{L}_1\sigma_3\bar{\Phi} = -\bar{\omega}_*(\sigma_3\bar{\Phi}). \end{aligned} \tag{2.19}$$

(4) *For any function* $\zeta \in \mathbf{E}_1(\mathcal{L}_1)$, *there is a unique pair* $(\alpha, \beta) \in \mathbb{C}^2$ *so that*

$$\zeta = \mathbf{RE}\{\alpha\Phi + \beta\sigma_3\Phi\}. \tag{2.20}$$

We have $\mathcal{L}_1\zeta = \mathbf{RE}\{\omega_*\alpha\Phi - \omega_*\beta\sigma_3\Phi\}$ *and* $e^{t\mathcal{L}_1}\zeta = \mathbf{RE}\{e^{t\omega_*}\alpha\Phi + e^{-t\omega_*}\beta\sigma_3\Phi\}$.

(5) *We have the orthogonality relations in Eqs.* (2.10) *and* (2.11). *Moreover,* $\sigma_1\bar{\Phi} \perp \{\bar{\Phi}, \sigma_3\Phi, \sigma_3\bar{\Phi}\}$, $\sigma_1\Phi \perp \{\Phi, \sigma_3\Phi, \sigma_3\bar{\Phi}\}$, *and* $\int \bar{u}v\,dx = 0$, *etc. For any function* $\psi \in \mathbb{C}L^2$, *if we decompose*

$$\psi = a\begin{bmatrix} R_1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ Q_1 \end{bmatrix} + \alpha_1\Phi + \alpha_2\bar{\Phi} + \beta_1\sigma_3\Phi + \beta_2\sigma_3\bar{\Phi} + \eta, \tag{2.21}$$

where $a, b, \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$ *and* $\eta \in \mathbf{H}_c(\mathcal{L}_1)$, *then we have*

$$\begin{aligned} &a = c_1\left(\sigma_1\begin{bmatrix} 0 \\ Q_1 \end{bmatrix}, \psi\right), \qquad b = c_1\left(\sigma_1\begin{bmatrix} R_1 \\ 0 \end{bmatrix}, \psi\right), \\ &\alpha_1 = c_2(\sigma_1\bar{\Phi}, \psi), \qquad \alpha_2 = \bar{c}_2(\sigma_1\Phi, \psi), \\ &\beta_1 = -c_2(\sigma_1\sigma_3\bar{\Phi}, \psi), \qquad \beta_2 = -\bar{c}_2(\sigma_1\sigma_3\Phi, \psi), \end{aligned} \tag{2.22}$$

where $c_1^{-1} = (Q_1, R_1)$ *and* $c_2^{-1} = (\sigma_1\bar{\Phi}, \Phi) = \int 2uv\,dx$. (*Note* $c_1\lambda > 0$.) *The statement that* $\psi \in L^2$ *is equivalent to that* $a, b \in \mathbb{R}$, $\alpha_1 = \alpha_2 = \alpha/2$, $\beta_1 = \beta_2 = \beta/2$ *and* $\mathbf{RE}\,\eta = \eta$. *In this case,*

$$\psi = a\begin{bmatrix} R_1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ Q_1 \end{bmatrix} + \mathbf{RE}\{\alpha\Phi + \beta\sigma_3\Phi\} + \eta, \tag{2.23}$$

with $a, b \in \mathbb{R}$, $\eta \in \mathbf{H}_c(\mathcal{L}_1)$ *with* $\mathbf{RE}\,\eta = \eta$, $\alpha, \beta \in \mathbb{C}$, *and*

$$\alpha = P_\alpha(\psi) \equiv 2c_2(\sigma_1\bar{\Phi}, \psi), \qquad \beta = P_\beta(\psi) \equiv -2c_2(\sigma_1\sigma_3\bar{\Phi}, \psi). \tag{2.24}$$

P_α *and* P_β *are maps from* L^2 *to* $\mathbb{C}$.

(6) *There is a constant* $C > 1$ *such that, for all* $\eta \in \mathbf{H}_c(\mathcal{L}_1)$ *and all* $t \in \mathbb{R}$, *we have*

$$C^{-1}\|\eta\|_{H^k} \le \|e^{t\mathcal{L}_1}\eta\|_{H^k} \le C\|\eta\|_{H^k}, \quad (k = 1, 2).$$

(7) *Decay estimates: For all* $\eta \in \mathbf{H}_c(\mathcal{L}_1)$, *for all* $p \in [2, \infty]$, *one has*

$$\|e^{t\mathcal{L}_1}\eta\|_{L^p} \le C|t|^{-3(1/2-1/p)}\|\eta\|_{L^{p'}},$$

where $C = C(n, p)$ *depends on* n.

Remark. (i). In (6), we restrict ourselves to $\mathbf{H}_c(\mathcal{L}_1)$, not M_1 as in Theorem 2.1. (ii). In (3), Φ is not a perturbation of $\left[\begin{smallmatrix} \phi_0 \\ -i\phi_0 \end{smallmatrix}\right]$. Also, the L^2 functions u_1 and u_2 are independent of each other. So are v_1 and v_2. (iii) In (7) the constant depends on n since there are eigenvalues which are very close to the continuous spectrum.

Since the proof of Theorem 2.1 is easier, we postpone it to the last subsection, §2.8. We will focus on proving Theorem 2.2 in the following subsections.

2.1. Perturbation of Embedded Eigenvalues and Their Eigenvectors

In this subsection we study the eigenvalues of $\mathcal{L}_1$ near ie_{01}. By symmetry we also get the information near $-ie_{01}$. For our fixed nonlinear excited state $Q_1 = Q_{1,E_1}$, let $H = -\Delta + V - E_1 + \lambda Q_1^2$. ($H$ is L_- in Eq. (2.6).) Let ϕ_0 denote a positive normalized ground state of H, with ground state energy $-\rho$ which is very close to $-e_{01}$. Hence the bottom of the continuous spectrum of H, which is close to $|e_1|$, is less than ρ.

We have

$$\begin{aligned} &HQ_1 = 0, \qquad H\widetilde{\phi}_0 = -\rho\widetilde{\phi}_0. \\ &Q_1 = n\phi_1 + O(n^3), \qquad \widetilde{\phi}_0 = \phi_0 + O(n^2). \end{aligned} \tag{2.25}$$

We want to solve the eigenvalue problem $\mathcal{L}_1\Phi = \omega_*\Phi$ with ω_* near ie_{01}. Write $\Phi = \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$. The problem has the form

$$\begin{bmatrix} 0 & H \\ -(H + 2\lambda Q_1^2) & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \omega_*\begin{bmatrix} u \\ v \end{bmatrix},$$

for some ω_* near ie_{01} and for some complex L^2-functions u, v. We have

$$Hv = \omega_* u, \qquad (H + 2\lambda Q_1^2)u = -\omega_* v.$$

Thus $H(H + 2\lambda Q_1^2)u = -\omega_*^2 u$. Suppose $\omega_* = i\kappa + \gamma$ with $\kappa \sim e_{01}$ and $\gamma \geq 0$. Since $\operatorname{Im}(-\omega_*^2) \leq 0$ and H is real, it is more convenient to solve

$$(H^2 + A)\bar{u} = z\bar{u}, \tag{2.26}$$

where

$$A \equiv H2\lambda Q_1^2, \qquad z \equiv -\bar{\omega}_*^2. \tag{2.27}$$

Note $z \sim e_{01}^2$ with $\operatorname{Im} z$ small. We may and will assume $\operatorname{Im} z \geq 0$. Note that $\gamma > 0$ corresponds to $\operatorname{Im} z > 0$. We will assume $\operatorname{Im} z \neq 0$ in this subsection. The non-existence of eigenvalues with $\operatorname{Im} z = 0$ will be proved in §2.4.

If we decompose $\bar{u} = a\widetilde{\phi}_0 + bQ_1 + h$ with $h \in \mathbf{H}_c(H)$, we find $b = 0$ since $\bar{u} \in \operatorname{Image} H$. If $a = 0$, we have $(H^2 + \mathbf{P}_c A\mathbf{P}_c - z)h = 0$. Here $\mathbf{P}_c = \mathbf{P}_c(H)$. We will show later that the resolvent

$$(H^2 + \mathbf{P}_c A\mathbf{P}_c - z)^{-1}\,\mathbf{P}_c \tag{2.28}$$

is well-defined if $\operatorname{Im} z \neq 0$. It can be proven by expanding

$$\begin{aligned} &(H^2 + \mathbf{P}_c A\mathbf{P}_c - z)^{-1}\mathbf{P}_c \\ &\quad = \frac{H}{H^2 - z}\mathbf{P}_c - \frac{H}{H^2 - z}\mathbf{P}_c 2\lambda Q_1 \sum_{j=0}^{\infty}\left[Q_1 \frac{-2\lambda H}{H^2 - z}\mathbf{P}_c Q_1\right]^j Q_1 \frac{1}{H^2 - z}\mathbf{P}_c, \end{aligned} \tag{2.29}$$

and summing the estimate for each term provided by Lemma 2.3. Hence $h = 0$ and there is no such solution.

Suppose now $a \neq 0$. We may assume $a = 1$ and $\bar{u} = \widetilde{\phi}_0 + h$. We have

$$(H^2 + A)(\widetilde{\phi}_0 + h) = z(\widetilde{\phi}_0 + h),$$

i.e.,

$$z\widetilde{\phi}_0 + zh = \rho^2\widetilde{\phi}_0 + A\widetilde{\phi}_0 + (H^2 + A)h. \tag{2.30}$$

Taking projection $\mathbf{P}_c = \mathbf{P}_c(H)$, we get

$$zh = \mathbf{P}_c A\widetilde{\phi}_0 + (H^2 + \mathbf{P}_c A\ \mathbf{P}_c)h.$$

Hence

$$h = -(H^2 + \mathbf{P}_c A\mathbf{P}_c - z)^{-1}\mathbf{P}_c A\widetilde{\phi}_0. \tag{2.31}$$

Note, if $\operatorname{Im} z = 0$, the function H defined above is generically not in L^2. Taking inner product of Eq. (2.30) with $\widetilde{\phi}_0$, we get

$$z = \rho^2 + (\widetilde{\phi}_0, A\widetilde{\phi}_0) + (\widetilde{\phi}_0, Ah).$$

Substituting Eq. (2.31), we get

$$z = \rho^2 + (\widetilde{\phi}_0, A\widetilde{\phi}_0) - (\widetilde{\phi}_0, A(H^2 + \mathbf{P}_c A\mathbf{P}_c - z)^{-1}\mathbf{P}_c A\widetilde{\phi}_0). \tag{2.32}$$

Remark. If A is self-adjoint, then the signs of the imaginary parts of the two sides of the above equation are different. This can be seen by expanding the right side into series and taking the leading term of the imaginary part. Thus z is real and generically h is not in L^2. In our case, $A = H2\lambda Q_1^2$ is not self-adjoint and hence a solution is not excluded.

Using $A = H2\lambda Q_1^2$ and $H\widetilde{\phi}_0 = -\rho\widetilde{\phi}_0$, Eq. (2.32) becomes the following fixed point problem,

$$z = f(z), \tag{2.33}$$

where

$$\begin{aligned} f(z) &= \rho^2 - \rho(\widetilde{\phi}_0 2\lambda Q_1^2\widetilde{\phi}_0) \\ &\quad + \rho(\widetilde{\phi}_0 2\lambda Q_1^2, (H^2 + H\mathbf{P}_c 2\lambda Q_1^2\mathbf{P}_c - z)^{-1} H\mathbf{P}_c 2\lambda Q_1^2\widetilde{\phi}_0). \end{aligned} \tag{2.34}$$

Let

$$R(z) = (H^2 - z)^{-1}H = \frac{1}{2(H - \sqrt{z})} + \frac{1}{2(H + \sqrt{z})}, \tag{2.35}$$

where $\sqrt{z}$ takes the branch $\operatorname{Im}\sqrt{z} > 0$ if $\operatorname{Im} z > 0$. We can expand $f(z)$ as

$$f(z) = \rho^2 - \rho(\widetilde{\phi}_0 2\lambda Q_1^2 \widetilde{\phi}_0) - \sum_{k=1}^{\infty} \rho 2\lambda(\widetilde{\phi}_0 Q_1, [-2\lambda Q_1 \mathbf{P}_c R(z) \mathbf{P}_c Q_1]^k Q_1 \widetilde{\phi}_0). \tag{2.36}$$

Let

$$z_0 = \rho^2 - \rho(\widetilde{\phi}_0 2\lambda Q_1^2 \widetilde{\phi}_0),$$
$$z_1 = z_0 + 4\rho\lambda^2(\widetilde{\phi}_0 Q_1^2, R(z_0 + 0i)\mathbf{P}_c Q_1^2 \widetilde{\phi}_0).$$

We have $|z_1 - z_0| \le Cn^4$ from its explicit form, (cf. Eq. (2.39) of Lemma 2.3 below). We also have, by Eqs. (2.25) and (1.5),

$$\operatorname{Im} z_1 = \operatorname{Im} 4\rho\lambda^2\left(\widetilde{\phi}_0 Q_1^2, \frac{1}{2(H - \sqrt{z_0} - 0i)} \mathbf{P}_c Q_1^2 \widetilde{\phi}_0\right) \ge \frac{7}{4} e_{01}\lambda^2 \gamma_0 n^4 + O(n^6) > 0.$$

Let $r_0 = \frac{1}{4}((e_{01})^2 - |e_1|^2)$ be a length of order 1. Denote the regions

$$G = \{x + iy : |x - \rho^2| < r_0, 0 < y < r_0\}, \tag{2.37}$$

$$D = B(z_1, n^5) = \{z : |z - z_1| \le n^5\}. \tag{2.38}$$

Clearly $z_0 \in \overline{G}$ and $z_1 \in D \subset G$. Also observe that the real part of all points in G are greater than $|E_1|^2$. We will solve the fixed point problem (2.33) in D. We need the following two lemmas.

Lemma 2.3. Fix $r > 3$. *There is a constant* $C_1 > 0$ *such that, for all* $z \in G$,

$$\left\| \langle x \rangle^{-r} \mathbf{P}_c R(z) \mathbf{P}_c \langle x \rangle^{-r} \right\|_{(L^2, L^2)} \le C_1, \tag{2.39}$$

$$\left\| \langle x \rangle^{-r} \mathbf{P}_c \frac{d}{dz} R(z) \mathbf{P}_c \langle x \rangle^{-r} \right\|_{(L^2, L^2)} \le C_1 (\operatorname{Im} z)^{-1/2}. \tag{2.40}$$

Here $\mathbf{P}_c = \mathbf{P}_c(H)$. *Moreover, for* $w_1, w_2 \in G$,

$$\left\| \langle x \rangle^{-r} \mathbf{P}_c [R(w_1) - R(w_2)] \mathbf{P}_c \langle x \rangle^{-r} \right\|_{(L^2, L^2)} \le C_1 (\max(\operatorname{Im} w_1, \operatorname{Im} w_2))^{-1/2} |w_1 - w_2|. \tag{2.41}$$

Proof. We have

$$R(z) = (H^2 - z)^{-1} H = \frac{1}{2(H - \sqrt{z})} + \frac{1}{2(H + \sqrt{z})}. \tag{2.42}$$

Since $1/(2(H + \sqrt{z}))$ is regular in a neighborhood of $\overline{G}$, it is sufficient to prove the lemma with $R(z)$ replaced by $R_1(z) := (H - \sqrt{z})^{-1}$.

That $\|\langle x\rangle^{-r}\mathbf{P}_c R_1(z)\mathbf{P}_c\langle x\rangle^{-r}\|_{(L^2, L^2)} \le C_1$ is well-known, see e.g. Refs. [1,9]. The estimate (2.40) will follow from Eq. (2.41) by taking limit. We now show Eq. (2.41) for $R_1(z)$. For any $w_1, w_2 \in G$, we have $|\sqrt{w_1} - \sqrt{w_2}| \le |w_1 - w_2|$. Write $\sqrt{w_1} = a_1 + ib_1$ and $\sqrt{w_2} = a_2 + ib_2$. We may assume $0 < b_1 \le b_2$. Let $w_3 \in G$ be the unique number such that $\sqrt{w_3} = a_1 + ib_2$.

For any $u, v \in L^2$ with $\|u\|_2 = \|v\|_2 = 1$, let $u_1 = \mathbf{P}_c\langle x\rangle^{-r}u$ and $v_1 = \mathbf{P}_c\langle x\rangle^{-r}v$. We have $u_1, v_1 \in L^1 \cap L^2(\mathbb{R}^3)$ and

$$\begin{aligned}
&\left|(u, \langle x\rangle^{-r}\mathbf{P}_c[R_1(w_1) - R_1(w_3)]\mathbf{P}_c\langle x\rangle^{-r}v)\right| \\
&\quad = \left|\int_0^\infty (u_1, e^{-it(H-a_1)}v_1)(e^{-b_1 t} - e^{-b_2 t})\,dt\right| \\
&\quad \le \int_0^\infty C(1+t)^{-3/2}(e^{-b_1 t} - e^{-b_2 t})\,dt \le C b_2^{-1/2}(b_2 - b_1).
\end{aligned}$$

Here we have used the decay estimate for e^{-itH} with $H = -\Delta + V - E_1 - \lambda Q_1^2$, namely,

$$\left\|e^{-itH}\mathbf{P}_c\phi\right\|_{L^\infty} \le C|t|^{-3/2}\|\phi\|_{L^1} \tag{2.43}$$

under our Assumption A1. See Refs. [9,10,13,27]. The bound $b_2^{-1/2}(b_2 - b_1)$ can be proved by considering two cases: If $b_1 \le b_2/2$, the integral is bounded by

$$\lesssim \int_0^{1/b_2} (1+t)^{-3/2}(b_2 - b_1)t\,dt + \int_{1/b_2}^\infty (1+t)^{-3/2}e^{-b_1 t}\,dt \lesssim b_2^{-1/2}(b_2 - b_1).$$

If $b_2/2 \le b_1 \le b_2$, the integral is bounded by

$$\lesssim \int_0^\infty (1+t)^{-3/2}(b_2 - b_1)te^{-b_1 t}\,dt \lesssim (b_2 - b_1)(1/b_1)^{1/2},$$

which is similar to $b_2^{-1/2}(b_2 - b_1)$. Hence we have the bound $b_2^{-1/2}(b_2 - b_1)$.

We also have

$$\begin{aligned}
&\left|(u, \langle x\rangle^{-r}\mathbf{P}_c[R_1(w_3) - R_1(w_2)]\mathbf{P}_c\langle x\rangle^{-r}v)\right| \\
&\quad = \left|\int_0^\infty (u_1, e^{-it(H-a_2-ib_2)}v_1)(e^{i(a_1-a_2)t} - 1)\,dt\right| \\
&\quad \le \int_0^\infty C(1+t)^{-3/2}e^{-b_2 t}|e^{i(a_1-a_2)t} - 1|\,dt \le Cb_2^{-1/2}|a_1 - a_2|.
\end{aligned}$$

Since $|a_1 - a_2| + |b_1 - b_2| \sim |\sqrt{w_1} - \sqrt{w_2}| \le |w_1 - w_2|$, we conclude

$$\left|(u, \langle x\rangle^{-r}\mathbf{P}_c[R_1(w_1) - R_1(w_2)]\mathbf{P}_c\langle x\rangle^{-r}v)\right| \le Cb_2^{-1/2}|w_1 - w_2|.$$

Hence we have Eq. (2.41). Q.E.D.

Lemma 2.4. *Recall the regions G and D are defined in Eqs.* (2.37)–(2.38).

(1) *$f(z)$ defined by Eq.* (2.34) *is well-defined and analytic in G.*
(2) $|f'(z)| \le Cn^4(\operatorname{Im} z)^{-1/2}$ *in G and* $|f'(z)| \le 1/2$ *in D.*
(3) *for* $w_1, w_2 \in G$,

$$|f(w_1) - f(w_2)| \le Cn^4(\max(\operatorname{Im} w_1, \operatorname{Im} w_2))^{-1/2}|w_1 - w_2|.$$

(4) *$f(z)$ maps D into D.*

Proof. By Eq. (2.39), the expansion (2.36) can be bounded by

$$|f(z)| \le C + CC_1n^4 + CC_1^2n^6 + \cdots$$

and thus converges. Since every term in Eq. (2.36) is analytic, $f(z)$ is well-defined and analytic. We also get the estimates in (2) using Eqs. (2.36) and (2.40). To prove (3), let $b = \max(\operatorname{Im} w_1, \operatorname{Im} w_2)$. From Eqs. (2.36), (2.39), (2.41),

$$|f(w_1) - f(w_2)| \le \sum_{k=1}^\infty CkC_1^k n^{2k+2}b^{-1/2}|w_1 - w_2| \le Cn^4b^{-1/2}|w_1 - w_2|.$$

It remains to show (4). We first estimate $|f(z_1) - z_1|$. Write $z_1 = z_0 + a + bi$. Recall that $|a| < Cn^4$ and $e_{01}\lambda^2\gamma_0 n^4 < |b| < Cn^4$. Using

Eqs. (2.39) and (2.41) we have

$$\begin{aligned}|f(z_1)-z_1| &= \Big|(\widetilde{\phi}_0 Q_1^2, [R(z_1)-R(z_0+0i)]\mathbf{P}_c Q_1^2\widetilde{\phi}_0) \\ &\quad + \sum_{k=2}^{\infty}(\widetilde{\phi}_0 Q_1, [Q_1\mathbf{P}_c R(z_1)\mathbf{P}_c Q_1]^k Q_1\widetilde{\phi}_0)\Big| \\ &\le Cn^4 b^{-1/2}(|a|+|b|) + CC_1^2 n^6 + CC_1^3 n^8 + \cdots \le Cn^6.\end{aligned}$$

Hence $|f(z_1)-z_1| \le Cn^6$. For any $z \in D$, we have

$$|f(z)-z_1| \le |f(z)-f(z_1)| + |f(z_1)-z_1| \le \frac{1}{2}|z-z_1| + Cn^6 \le n^5.$$

Hence $f(z) \in D$. This proves (4). Q.E.D.

We are ready to solve Eq. (2.33) in G. By Lemma 2.4 (1), (2) and (4), the map $f \to f(z)$ is a contraction mapping in D and hence has a unique fixed point z_* in D. By (3), for any $z \in G$ we have $|f(z)-f(z_*)| \le Cn^4(\operatorname{Im} z_*)^{-1/2}|z-z_*| \le 1/2|z-z_*|$. Hence there is no other fixed point of $f(z)$ in G.

By symmetry, there is another unique fixed point with negative imaginary part. Moreover, they have the size indicated in Theorem 2.2. We will prove in §2.3 and §2.4 that ω_* does not admit generalized eigenvectors and that there is no purely imaginary eigenvalue near ie_{01}, i.e., there is no embedded eigenvalue. Hence ω_*, and $-\bar{\omega}_*$ are simple and are the only eigenvalues near ie_{01}.

We now look more carefully on z_* and u_*, where u_* denotes the unique solution of $H(H+2\lambda Q_1^2)u_* = -\omega_*^2 u_*$ with the form $u_* = \widetilde{\phi}_0 + \bar{h}_*$. Recall $|z_1 - z_*| \le n^5$ and

$$z_1 = \rho^2 - \rho(\widetilde{\phi}_0 2\lambda Q_1^2\widetilde{\phi}_0) + 4\rho\lambda^2\Big(\widetilde{\phi}_0 Q_1^2, R(z_0+0i)\,\mathbf{P}_c Q_1^2\widetilde{\phi}_0\Big),$$

where $z_0 = \rho^2 - \rho(\widetilde{\phi}_0 2\lambda Q_1^2\widetilde{\phi}_0)$. Hence

$$\begin{aligned}\sqrt{z_*} &= \sqrt{z_1} + O(n^5) \\ &= \rho - (\widetilde{\phi}_0\lambda Q_1^2\widetilde{\phi}_0) + 2\lambda^2(\widetilde{\phi}_0 Q_1^2, R(z_0+0i)\mathbf{P}_c Q_1^2\widetilde{\phi}_0) \\ &\quad - \frac{1}{2\rho}(\widetilde{\phi}_0\lambda Q_1^2\widetilde{\phi}_0)^2 + O(n^5).\end{aligned}$$

Since $z_* = -\bar{\omega}_*^2$, we have $\bar{\omega}_* = i\sqrt{z_*}$. Thus if we write $\omega_* = i\kappa + \gamma$, then

$$\begin{aligned} \kappa &= \rho - (\widetilde{\phi}_0 \lambda Q_1^2 \widetilde{\phi}_0) - \frac{1}{2\rho}(\widetilde{\phi}_0 \lambda Q_1^2 \widetilde{\phi}_0)^2 \\ &\quad + \operatorname{Re} 2\lambda^2 (\widetilde{\phi}_0 Q_1^2, R(z_0 + 0i)\mathbf{P}_c Q_1^2 \widetilde{\phi}_0) + O(n^5), \\ \gamma &= -\operatorname{Im} 2\lambda^2 (\widetilde{\phi}_0 Q_1^2, R(z_0 + 0i)\mathbf{P}_c Q_1^2 \widetilde{\phi}_0) + O(n^5). \end{aligned} \tag{2.44}$$

By Eqs. (2.35), (2.25) and expansion into series,

$$\begin{aligned} \gamma &= \operatorname{Im} \lambda^2 (\widetilde{\phi}_0 Q_1^2, (H - \sqrt{z_0} - 0i)\mathbf{P}_c Q_1^2 \widetilde{\phi}_0) + O(n^5) \\ &= \operatorname{Im} \lambda^2 n^4 \left(\phi_0 \phi_1^2, \frac{1}{-\Delta + V - E_1 - \sqrt{z_0} - 0i} P_c \phi_1^2 \phi_0 \right) + O(n^5). \end{aligned} \tag{2.45}$$

By Eq. (1.5), $\gamma \geq \lambda^2 n^4 \gamma_0 + O(n^5)$.

We now consider the eigenvector. Since $\operatorname{Im} z_* \neq 0$, the resolvent Eq. (2.28) is invertible and hence there is a unique eigenvector h_* given by (2.31) with $z = z_*$. Since $A = H 2\lambda Q_1^2$, we have

$$h_* = -(H^2 + \mathbf{P}_c H 2\lambda Q_1^2 \mathbf{P}_c - z_*)^{-1} H \mathbf{P}_c 2\lambda Q_1^2 \widetilde{\phi}_0, \tag{2.46}$$

where $\mathbf{P}_c = \mathbf{P}_c(H)$. We now expand the resolvent on the right side using Eq. (2.29). By Lemma 2.3, we obtain $|(\phi, h)| \leq Cn^2 \|\langle x \rangle^r \phi\|_2$, for any $r > 3$.

We now show that h_* is bounded in L^2 with a bound uniform in n. Recall $\sqrt{z_*} = \kappa + i\gamma$ with $\kappa \sim e_{01}$, $\gamma > \frac{1}{2}\lambda^2 \gamma_0 n^4$. Since $Q_1 = n\phi_1 + O(n^3)$, by expansion and Eq. (2.25) we have

$$\begin{aligned} h_* &= -(H^2 - z_*)^{-1} H \mathbf{P}_c(H) 2\lambda \phi_0 Q_1^2 + O(n^2) \\ &= -(H - \sqrt{z_*})^{-1} \mathbf{P}_c(H) \lambda \phi_0 Q_1^2 + O(n^2) \\ &= -\frac{1}{-\Delta + V - s - \gamma i} \mathbf{P}_c(H_0) \lambda \phi_0 {Q_1}^2 + O(n^2), \end{aligned} \tag{2.47}$$

where $s = E_1 + \kappa = 2e_1 - e_0 + O(n^2) > 0$. Here we have used the fact that

$$\mathbf{P}_c(H)\phi = \mathbf{P}_c(H_0)\phi + n^2 \sum_{k=1}^{N} (\psi_k^*, \phi)\psi_k,$$

for some local functions ψ_k, ψ_k^* of order one. We will show that the leading term on the right of Eq. (2.47) is of order one in L^2. It follows from the same proof that $O(n^2)$ on the right is also in L^2 sense.

We first consider the case $V = 0$. For $f(p) \in L^2 \cap L^\infty$ of order 1,

$$\int \frac{1}{-\Delta - s + \gamma i} \hat{f}(x) \frac{1}{-\Delta - s - \gamma i} \bar{\hat{f}}(x)\, dx = \int |f(p)|^2 \frac{1}{(p^2 - s)^2 + \gamma^2}\, dp.$$

We can divide the integral into two parts: $|p| \notin I$ and $|p| \in I$, where $I = (\sqrt{s}/2, 3\sqrt{s}/2)$. Note s is of order 1. For $|p| \notin I$, we have $1/((p^2 - s)^2 + \gamma^2) \leq C$. Hence the integral is bounded by $\|f\|_{L^2}^2$. For $|p| \in I$, we first bound $|f(p)|^2$ by $\|f\|_{L^\infty}^2$ and then integrate out the angular directions. Hence the whole integral is bounded by

$$C + C\int_{\sqrt{s}/2}^{3\sqrt{s}/2} \frac{r^2}{(|r-\sqrt{s}|+\gamma)^2}\,dr \leq C + C\int_0^{\sqrt{s}/2} \frac{1}{(\tau+\gamma)^2}\,d\tau \leq C + C/\gamma.$$

Here $r \geq 0$ denotes the radial direction and $\tau = r - \sqrt{s}$.

Using wave operator for $-\Delta + V$, we have similar estimates if $-\Delta$ is replaced by $-\Delta + V$. Since $\gamma \sim n^4$ and $\lambda\phi_0 Q_1^2 = O(n^2)$ is smooth and localized (similarly for $O(n^2)$ on the right side of Eq. (2.47)), we get

$$(h_*, h_*) \leq Cn^2\gamma^{-1}n^2 \leq C,$$

where C is independent of n. Since $u_* = \widetilde{\phi}_0 + \bar{h}_* = \phi_0 + \bar{h}_* + O(n^2)$, we have obtained the u part of the estimates $\|\Phi\|_{L^2} \leq C$ and Eq. (2.17). The corresponding estimate for v can be proved using $v = (-L_+)u/\omega_*$.

2.2. Resolvent Estimates

In this subsection we study the resolvent $R(w) = (w - \mathcal{L}_1)^{-1}$. Note that $R(w)$ had a different meaning in the previous subsection.

Let L_r^2 denote the weighted L^2 spaces for $r \in \mathbb{R}$:

$$L_r^2 = \left\{f : (1+x^2)^{r/2} f(x) \in L^2(\mathbb{R}^3)\right\}.$$

We will prove the following lemma on resolvent estimates along the continuous spectrum Σ_c. As a corollary of the proof, we also show that $\{0, \pm\omega_*, \pm\overline{\omega_*}\}$ consists of all eigenvalues outside of Σ_c.

Lemma 2.5. *Let $R(w) = (w - \mathcal{L}_1)^{-1}$ be the resolvent of $\mathcal{L}_1$. Let $\mathbf{B} = B(L_r^2, L_{-r}^2)$, the space of bounded operators from L_r^2 to L_{-r}^2 with $r > 3$. Recall $\omega_* = i\kappa + \gamma$. For $\tau \geq |E_1|$ we have*

$$\left\|R(i\tau \pm 0)\right\|_{\mathbf{B}} + \left\|R(-i\tau \pm 0)\right\|_{\mathbf{B}} \leq C(1+\tau)^{-1/2} + C(|\tau - \kappa| + n^4)^{-1}. \tag{2.48}$$

The constant C is independent of n. We also have

$$\left\|R^{(k)}(i\tau \pm 0)\right\|_{\mathbf{B}} + \left\|R^{(k)}(-i\tau \pm 0)\right\|_{\mathbf{B}} \le C(1+\tau)^{-(1+k)/2} + C(|\tau-\kappa| + n^4)^{-1} \tag{2.49}$$

for derivatives, where $k = 1, 2$.

We first consider $R_0(w) = (w - JH_1)^{-1}$. Recall $H_1 = -\Delta + V - E_1$. Since

$$\begin{aligned}(w - JH_1)^{-1} &= \begin{bmatrix} w & -H_1 \\ H_1 & w \end{bmatrix}^{-1} = \frac{1}{H_1^2 + w^2}\begin{bmatrix} w & H_1 \\ -H_1 & w \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}(H_1 - iw)^{-1} + \frac{1}{2}\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}(H_1 + iw)^{-1},\end{aligned} \tag{2.50}$$

the estimates of $R_0(w)$ can be derived from those of $(H_1 - iw)^{-1}$ and $(H_1 + iw)^{-1}$. By assumption, the bottom of the continuous spectrum of H_1, $-E_1$, is not an eigenvalue nor a resonance of H_1. Hence $(H_1 - z)^{-1}$ is uniformly bounded in **B** for z away from $e_0 - E_1$ and $e_1 - E_1$, see Ref. [9]. By Eqs. (2.4) and (2.50), $R_0(w)$ is uniformly bounded in **B** for w with $\mathrm{dist}(w, \Sigma_p) \ge n$, where $\Sigma_p = \{0, ie_{01}, -ie_{01}\}$.

Write

$$\mathcal{L}_1 = JH_1 + W, \quad W = \begin{bmatrix} 0 & \lambda Q_1^2 \\ -3\lambda Q_1^2 & 0 \end{bmatrix}.$$

For $R(w) = (w - \mathcal{L}_1)^{-1}$ we have

$$R(w) = (1 - R_0(w)W)^{-1}R_0(w) = \sum_{k=0}^{\infty}[R_0(w)W]^k R_0(w). \tag{2.51}$$

Since $R_0(w)$ is uniformly bounded in **B** for w with $\mathrm{dist}(w, \Sigma_p) > n$, and W is localized and small, Eq. (2.51) converges and $(w - \mathcal{L}_1)^{-1}$ is uniformly bounded in **B** for w with $\mathrm{dist}(w, \Sigma_p) > n$ and we have

$$\left\|R(w)\right\|_B \le C\,\mathrm{dist}(w, \Sigma_p)^{-1}, \quad (n \le \mathrm{dist}(w, \Sigma_p) \le 1). \tag{2.52}$$

Recall $\Sigma_c = \{is : |s| \ge |E_1|\}$ is the continuous spectrum of JH_1 and $\mathcal{L}_1$. For w in the region

$$\left\{w : \mathrm{dist}(w, \Sigma_p) \ge n, w \notin \Sigma_c\right\}, \tag{2.53}$$

we have

$$\|R_0(w)\|_{(L^2,L^2)} \le C\,\mathrm{dist}(w,\Sigma_c)^{-1}.$$

By Eq. (2.51), and because W is localized and small,

$$\begin{aligned}\|R(w)\|_{(L^2,L^2)} \le{}& \|R_0(w)\|_{(L^2,L^2)} \\ &+ \sum_{k=1}^{\infty} C\|R_0(w)\|_{(L^2,L^2)}\left\{Cn^2\|R_0(w)\|_{\mathbf{B}}\right\}^{k-1}\|R_0(w)\|_{(L^2,L^2)} \\ \le{}& C\,\mathrm{dist}(w,\Sigma_c)^{-1} + C\,\mathrm{dist}(w,\Sigma_c)^{-2}.\end{aligned}$$

Hence $R(w)$ is uniformly bounded in (L^2, L^2) in a neighborhood of w. In particular, there is no eigenvalue of $\mathcal{L}_1$ in the region (2.53) above. Note that this region includes a neighborhood of the bottom of the continuous spectrum Σ_c, $\pm iE_1$, except those in Σ_c. Hence the eigenvalues can occur only in $\{w : \mathrm{dist}(w,\Sigma_p) < n\}$ or Σ_c.

The circle $\{w : |w| = \sqrt{n}\}$ is in the resolvent set of $\mathcal{L}_1$. By Ref. [15] Theorem XII.6, the Cauchy integral

$$P = \frac{1}{2\pi i}\oint_{|w|=\sqrt{n}} (w - \mathcal{L}_1)^{-1}\,dw$$

gives the L^2-projection onto the generalized eigenspaces with eigenvalues inside the disk $\{w : |w| < \sqrt{n}\}$. Moreover, the dimension of P is an upper bound for the sum of the dimensions of those eigenspaces. However, since the projection $P_0 = (2\pi i)^{-1}\oint_{|w|=\sqrt{n}} R_0(w)\,dw$ has dimension 2 (see Eqs. (2.3)–(2.4)), and

$$P - P_0 = \frac{1}{2\pi i}\oint_{|w|=\sqrt{n}} \sum_{k=1}^{\infty} [R_0(w)W]^k R_0(w)\,dw$$

is convergent and bounded in (L^2, L^2) by

$$\begin{aligned}&\le C\|R_0(w)\|_{(L^2,L^2)} n^2 \sum_{k=0}^{\infty}\left(Cn^2\|R_0(w)\|_{\mathbf{B}}\right)^k \|R_0(w)\|_{(L^2,L^2)} \\ &\le Cn^{-1/2}Cn^2n^{-1/2} = Cn,\end{aligned}$$

(here we have used Eq. (2.52)), the dimension of P is also 2. Since we already have two generalized eigenvectors $\begin{bmatrix}0\\ Q_1\end{bmatrix}$ and $\begin{bmatrix}R_1\\ 0\end{bmatrix}$ with eigenvalue 0, we have obtained all generalized eigenvectors with eigenvalues in the disk $|w| < \sqrt{n}$. Together with the results in §2.1, we have obtained all eigenvalues outside of Σ_c: 0, $\pm\omega_*$, and $\pm\overline{\omega_*}$.

We next study $R(w) = (w - \mathcal{L}_1)^{-1}$ for w near $\pm ie_{01}$: $|w - ie_{01}| < n$ or $|w + ie_{01}| < n$. Let us assume $w = i\tau - \varepsilon$ with $\tau, \varepsilon > 0$, thus $-w^2$ lies in G (defined in Eq. (2.37)). The other cases are similar. Let $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in \mathbb{C}L^2$. We want to solve the equation

$$(w - \mathcal{L}_1)\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}. \tag{2.54}$$

We have

$$wu - Hv = f, \qquad wv + (H + 2\lambda Q_1^2)u = g.$$

Cancelling v, we get (recall $A = H2\lambda Q_1^2$)

$$w^2 u + (H^2 + A)u = F, \qquad F = wf + Hg.$$

Write $u = \alpha\widetilde{\phi}_0 + \beta\widehat{Q}_1 + \eta$ with $\eta \in \mathbf{H}_c(H)$ and $\widehat{Q}_1 = Q_1/\|Q_1\|_2$. Also denote $\zeta = \alpha\widetilde{\phi}_0 + \beta\widehat{Q}_1 = u - \eta$. We have

$$\begin{aligned} \left(w^2 + H^2 + \mathbf{P}_c A\right)\eta &= \mathbf{P}_c F - \mathbf{P}_c A\zeta, \\ \left(w^2 + H^2 + \mathbf{P}^{\perp} A\right)\zeta &= \mathbf{P}^{\perp} F - \mathbf{P}^{\perp} A\eta. \end{aligned}$$

Here $\mathbf{P}_c = \mathbf{P}_c(H)$ and $\mathbf{P}^{\perp} = 1 - \mathbf{P}_c$. Solving η in terms of ζ, we get

$$\eta = \Omega(\mathbf{P}_c F - \mathbf{P}_c A\zeta), \qquad \Omega \equiv \left(w^2 + H^2 + \mathbf{P}_c A \mathbf{P}_c\right)^{-1}. \tag{2.55}$$

Note that Ω is the resolvent in Eq. (2.28) with $z = -w^2$. Substituting the above into the ζ equation we get

$$\left(w^2 + H^2 + \mathbf{P}^{\perp} A - \mathbf{P}^{\perp} A\Omega\, \mathbf{P}_c A\right)\zeta = \widetilde{F}, \tag{2.56}$$

$$\widetilde{F} = \mathbf{P}^{\perp} F - \mathbf{P}^{\perp} A\Omega\, \mathbf{P}_c F.$$

Using $\widetilde{\phi}_0$ and $\widehat{Q}_1$ as basis, we can put Eq. (2.56) into matrix form

$$\begin{bmatrix} a & b \\ 0 & w^2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} (\widetilde{\phi}_0, \widetilde{F}) \\ (\widehat{Q}_1, \widetilde{F}) \end{bmatrix}, \tag{2.57}$$

where (recall $H\widetilde{\phi}_0 = -\rho\widetilde{\phi}_0$, $H\widehat{Q}_1 = 0$ and $A = H2\lambda Q_1^2$)

$$\begin{aligned} a &= w^2 + \rho^2 - \rho(\widetilde{\phi}_0 2\lambda Q_1^2 \widetilde{\phi}_0) + \rho(\widetilde{\phi}_0 2\lambda Q_1^2, \Omega H \mathbf{P}_c 2\lambda Q_1^2 \widetilde{\phi}_0), \\ b &= -\rho(\widetilde{\phi}_0 2\lambda Q_1^2 \widehat{Q}_1) + \rho(\widetilde{\phi}_0 2\lambda Q_1^2, \Omega H \mathbf{P}_c 2\lambda Q_1^2 \widehat{Q}_1). \end{aligned}$$

Thus

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1/a & -b/(aw^2) \\ 0 & w^{-2} \end{bmatrix} \begin{bmatrix} (\widetilde{\phi}_0, \widetilde{F}) \\ (\widehat{Q}_1, \widetilde{F}) \end{bmatrix}. \tag{2.58}$$

Note that we have $(\widehat{Q}_1, \widetilde{F}) = (\widehat{Q}_1, F) = (\widehat{Q}_1, wf)$ and

$$\begin{aligned}(\widetilde{\phi}_0, \widetilde{F}) &= (\widetilde{\phi}_0, F) - (-\rho\widetilde{\phi}_0 2\lambda Q_1^2, \Omega \mathbf{P}_c F) \\ &= (\widetilde{\phi}_0, wf) - (\rho\widetilde{\phi}_0, g) + (\rho\widetilde{\phi}_0 2\lambda Q_1^2, \Omega \mathbf{P}_c wf + \Omega H \mathbf{P}_c g).\end{aligned}$$

By Eq. (2.55), $F = wf + Hg$ and $A = H2\lambda Q_1^2$,

$$\eta = \Omega w \mathbf{P}_c f + \Omega H \mathbf{P}_c g - \Omega H \mathbf{P}_c 2\lambda Q_1^2 \zeta. \tag{2.59}$$

The above computation from Eqs. (2.54)–(2.59) is valid as long as Ω is invertible, in particular, if $z = -w^2 \in G$. We now consider the case $w = i\tau - \varepsilon$ with $|\tau - e_{01}| < 2n$ and $0 < \varepsilon \ll n^4$. It follows that $z = -w^2 \in G$ and $\operatorname{Re} z > 0$ is small. Recall $f(z)$ defined in Eq. (2.34), and the fixed point $z_* = -\bar{\omega}_*^2$ found in §2.1. We have $a = f(z) - z = (z_* - z) + (f(z) - f(z_*))$. Using Lemma 2.4 (3) with $w_1 = z$ and $w_2 = z_*$, we have

$$|a| \geq |z - z_*| - |f(z) - f(z_*)| \geq \frac{1}{2}|z - z_*| = \frac{1}{2}|w^2 - \bar{\omega}_*^2| \geq C|w + \bar{\omega}_*|.$$

Since $\omega_* = i\kappa + \gamma$ with $\gamma \sim n^4$ and $w = i\tau - \varepsilon$ with $0 < \varepsilon \ll n^4$, we have $|a| \geq C(|\tau - \kappa| + n^4)$.

We will bound α, β, and η using Eqs. (2.58) and (2.59). Note that the operators $\Omega = (w^2 + H^2 + \mathbf{P}_c A \mathbf{P}_c)^{-1}$ and ΩH do not have a uniform bound in (L^2, L^2) as ε goes to zero. They are, however, uniformly bounded in $\mathbf{B}$. It can be proven by first expanding Ω into a series as in Eq. (2.29), and then by using formulas like Eq. (2.42) and the usual weighted estimates near the continuous spectrum. Therefore, if $f, g \in L_r^2$, using Eqs. (2.58), (2.59), and the explicit forms of $(\widetilde{\phi}_0, \widetilde{F})$ and $(\widehat{Q}_1, \widetilde{F})$,

$$\begin{aligned}&|\alpha| + |\beta| \leq C(1 + |a|^{-1})\|f, g\|_{L_r^2} \leq C(|\tau - \kappa| + n^4)^{-1}\|f, g\|_{L_r^2}, \\ &\|\eta\|_{L_{-r}^2} \leq C\|f, g\|_{L_r^2} + Cn^2(|\alpha| + |\beta|).\end{aligned}$$

We conclude, for $u = \alpha\widetilde{\phi}_0 + \beta\widehat{Q}_1 + \eta$,

$$\|u\|_{L_{-r}^2} \leq (C + C(|\tau - \kappa| + n^4)^{-1})(\|f\|_{L_r^2} + \|g\|_{L_r^2}).$$

We can estimate v similarly. Thus, for $\tau \in (e_{01} - n, e_{01} + n)$,

$$\|R(i\tau - 0)\|_{\mathbf{B}} \leq C + C(|\tau - \kappa| + n^4)^{-1}, \quad (|\tau - e_{01}| < n).$$

The estimate for $\|R(i\tau + 0)\|_{\mathbf{B}}$ is similar.

For $\tau > e_{01} + n$ and $w = i\tau + 0$, using $R(w) = (1 + R_0(w)W)^{-1} R_0(w)$ and the fact that $\|R_0(w)\|_{\mathbf{B}} \le C(1+\tau)^{-1/2}$, (see Ref. [9] Theorem 9.2), we have $\|R(i\tau + 0)\|_{\mathbf{B}} \le C\tau^{-1/2}$. For $\tau \in [|E_1|, e_{01} - n]$, the same argument gives $\|R(i\tau + 0)\|_{\mathbf{B}} \le C$. The derivative estimates for the resolvent are obtained by induction argument, by differentiating the relation $R(1 + WR_0) = R_0$ and by using the relations $(1 + WR_0)^{-1} = 1 - WR$ and $(1 + R_0 W)^{-1} = 1 - RW$. See the proof of Ref. [9] Theorem 9.2. We have proved Lemma 2.5.

2.3. Nonexistence of Generalized ω_*-Eigenvector

Since the resolvent in Eq. (2.28) with $z = z_*$ is invertible, h_* given by Eq. (2.31) is unique and hence Φ is the only ω_*-eigenvector satisfying $(\mathcal{L}_1 - \omega_*)\Phi = 0$. We now show that there is no other *generalized* ω_*-eigenvector, i.e., there is no vector ϕ with $(\mathcal{L}_1 - \omega_*)\phi \neq 0$ but $(\mathcal{L}_1 - \omega_*)^k \phi = 0$ for some $k \ge 2$. Suppose the contrary, then we may find a vector $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ with $(\omega_* - \mathcal{L}_1)\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] = \left[\begin{smallmatrix} u_* \\ v_* \end{smallmatrix}\right]$. That is, $w = \omega_*$ and $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] = \left[\begin{smallmatrix} u_* \\ v_* \end{smallmatrix}\right]$ in the system (2.54). We have $F = wu_* + Hv_* = 2\omega_* u_*$. Since $u_* = \widetilde{\phi}_0 + \bar{h}_*$ with $\bar{h}_* \in \mathbf{H}_c(H)$, we have $(\widehat{Q}_1, \widetilde{F}) = (\widehat{Q}_1, F) = (\widehat{Q}_1, 2\omega_* u_*) = 0$. Hence $\beta = 0$. Also

$$
\begin{aligned}
(\widetilde{\phi}_0, \widetilde{F}) &= (\widetilde{\phi}_0, F) - (\widetilde{\phi}_0 H 2\lambda Q_1^2 (w^2 + H^2 + \mathbf{P}_c A \mathbf{P}_c)^{-1} \mathbf{P}_c F) \\
&= 2\omega_* + \rho(\widetilde{\phi}_0 2\lambda Q_1^2 (w^2 + H^2 + \mathbf{P}_c A \mathbf{P}_c)^{-1} 2\omega_* \bar{h}_*) \\
&= 2\omega_* \left[1 + \rho(\Psi, \Omega\overline{\Omega} H \Psi)\right],
\end{aligned}
$$

where $\Omega = (w^2 + H^2 + \mathbf{P}_c A \mathbf{P}_c)^{-1}$ and $\Psi = \mathbf{P}_c \widetilde{\phi}_0 2\lambda Q_1^2$. Since the main term in $(\Psi, \Omega\overline{\Omega} H \Psi)$,

$$
(\Psi, (w^2 + H^2)^{-1} (\bar{w}^2 + H^2)^{-1} H \Psi),
$$

is positive, $(\widetilde{\phi}_0, \widetilde{F})$ is not zero. On the other hand, $a = \omega_*^2 + f(-\omega_*^2) = -\bar{z}_* + f(\bar{z}_*) = 0$. Hence there is no solution for α. This shows ω_* is simple (and so are $-\omega_*$, $\pm\bar{\omega}_*$).

2.4. Nonexistence of Embedded Eigenvalues

In this subsection we prove that there is no embedded eigenvalue $i\tau$ with $|\tau| > |E_1|$. Suppose the contrary, we may assume $\tau > -E_1 > 0$ and $\mathcal{L}_1 \psi = i\tau\psi$ for some $\psi \in \mathbb{C}L^2$. We will derive a contradiction.

Let $H_* = -\Delta - E_1$. We can decompose

$$\mathcal{L}_1 = JH_* + A, \qquad A = \begin{bmatrix} 0 & V + \lambda Q_1^2 \\ -V - 3\lambda Q_1^2 & 0 \end{bmatrix}. \tag{2.60}$$

Hence $(i\tau - JH_*)\psi = A\psi$. By the same computation of Eq. (2.50) we have

$$(w - JH_*)^{-1} = (H_* - iw)^{-1} M_+ + (H_* + iw)^{-1} M_-,$$

where

$$M_+ = \frac{1}{2}\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}, \qquad M_- = \frac{1}{2}\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}.$$

Thus, with $w = i\tau$, we have

$$\psi = (i\tau - JH_*)^{-1} A\psi = (H_* + \tau)^{-1}\phi_+ + (H_* - \tau)^{-1}\phi_-, \tag{2.61}$$

where $\phi_+ = M_+ A\psi$ and $\phi_- = M_- A\psi$. By Assumption A1 on the decay of V and that $\psi \in L^2$, both $\phi_+, \phi_- \in L^2_{5+\sigma}$ with $\sigma > 0$. Since $-\tau$ is outside the spectrum of H_*, we have $(H_* + \tau)^{-1}\phi_+ \in L^2_{5+\sigma}$. Let $s = E_1 + \tau > 0$. We have $H_* - \tau = -\Delta - s$. By assumption $\psi \in \mathbb{C}L^2$, hence so is $(H_* - \tau)^{-1}\phi_-$. Therefore $(p^2 - s)^{-1}\widehat{\phi_-}(p) \in L^2$. Since $\phi_- \in L^2_{5+\sigma}$, $\widehat{\phi_-}$ is continuous and we can conclude

$$\widehat{\phi_-}(p)\big|_{|p|=\sqrt{s}} = 0. \tag{2.62}$$

We now recall Ref. [14] page 82, Theorem IX.41: Suppose $f \in L^2_r$ with $r > 1/2$ and let $B_s f = ((p^2 - s)^{-1}\hat{f})^\vee$. Suppose $\hat{f}(p)\big|_{|p|=\sqrt{s}} = 0$. Then for any $\varepsilon > 0$, one has $B_s f \in L^2_{r-1-2\varepsilon}$ and $\|B_s f\|_{L^2_{r-1-2\varepsilon}} \le C_{r,\varepsilon,s}\|f\|_{L^2_r}$ for some constant $C_{r,\varepsilon,s}$.

In our case, we have $f = \phi_-$, $\varepsilon = \sigma/2$ and $r = 5 + \sigma$. We conclude $(H_* - \tau)^{-1}\phi_- = B_s f \in L^2_4$. Thus $\psi \in L^2_4$.

However, since $(z - \mathcal{L}_1)\psi = (z - i\tau)\psi$, we have $R(z)\psi = (z - i\tau)^{-1}\psi$. Thus we have

$$\big\|(z - i\tau)^{-1}\psi\big\|_{L^2_{-r}} \le C\big\|\psi\big\|_{L^2_4},$$

where the constant C remains bounded as $z \to i\tau$ by Lemma 2.5. This is clearly a contradiction. Thus ψ does not exist.

2.5. Absence of Eigenvector and Resonance at Bottom of Continuous Spectrum

We want to show that $\pm iE_1$, the bottom of the continuous spectrum, are not eigenvalue nor resonance. That is, the null space of $\mathcal{L}_1 \mp iE_1$ in $X = L^2_{-r}$, $r > 1/2$, is zero. In fact, since the resolvent are bounded near $\pm iE$ by Lemma 2.5, the same argument in Ref. [9] for the expansion formula of the resolvent near the bottom of the continuous spectrum, trivially extended for non-self adjoint perturbations, shows the claim. Here we provide another proof for completeness.

Let us consider the case at $i|E_1|$. Suppose otherwise, we have a sequence $Q_{1,E_1(k)} \to 0$ and $\psi_k \in X = L^2_{-r}$ so that

$$\left(\mathcal{L}_{1,E_1(k)} + iE_1(k)\right)\psi_k = 0, \quad \|\psi_k\|_X = 1.$$

As in Eq. (2.60) we write $\mathcal{L}_{1,E_1(k)} = JH_* + A_k$, where $H_* = -\Delta - E_1(k)$ and $A_k = JV + \left[\begin{smallmatrix} 0 & 1 \\ -3 & 0 \end{smallmatrix}\right]\lambda Q^2_{1,E_1(k)}$. By Eq. (2.61) with $\tau = |E_1(k)|$ we have

$$\psi_k = (i\tau - JH_*)^{-1}A_k\psi_k = (-\Delta + 2\tau)^{-1}M_+A_k\psi_k + (-\Delta)^{-1}M_-A_k\psi_k$$

in X. Note that $(-\Delta + 2\tau)^{-1}M_+A_k$ and $(-\Delta)^{-1}M_-A_k$ are compact operators in X, with a bound uniform in k. Since X is a reflexive Banach space, we can find a subsequence, which we still denote by ψ_k, converging weakly to some $\psi_* \in X$. Thus $\tau \to |e_1|$, $(-\Delta + 2\tau)^{-1}M_+A_k\psi_k \to (-\Delta - 2e_1)^{-1} \times M_+JV\psi_*$ and $(-\Delta)^{-1}M_-A_k\psi_k \to (-\Delta)^{-1}M_+JV\psi_*$ strongly in X. Thus

$$\psi_* = (-\Delta - 2e_1)^{-1}M_+JV\psi_* + (-\Delta)^{-1}M_+JV\psi_*$$

and $\psi_k \to \psi_*$ strongly. Hence $\|\psi_*\|_X = \lim\|\psi_k\|_X = 1$ and $(JH_1 + ie_1)\psi_* = 0$ by Eq. (2.61) again. One can show that $(-\Delta + V)\psi_* = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$, which contradicts Assumption A1 and thus shows the claim.

2.6. Proof of Theorem 2.2 (4)–(6)

Once we have an eigenvector Φ with $\mathcal{L}_1\Phi = \omega_*\Phi$ and ω_* complex, we have three other eigenvalues and eigenvectors as given in Eq. (2.19). Hence we have found all eigenvalues and eigenvectors of $\mathcal{L}_1$. $\mathbb{C}\mathbf{E}_1$ is the combined eigenspace of $\pm\omega_*$ and $\pm\bar{\omega}_*$. It is easy to check that $\mathbf{RE}\ \mathbb{C}\mathbf{E}_1 = \mathbf{E}_1$. We have proved parts (1)–(3) of Theorem 2.2.

We now show the orthogonality conditions. Recall $\sigma_1 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. It is self-adjoint in $\mathbb{C}L^2$. Let $\mathcal{L}_1^*$ be the adjoint of $\mathcal{L}_1$ in $\mathbb{C}L^2$. We have $\mathcal{L}_1^* = \left[\begin{smallmatrix} 0 & -L_+ \\ L_- & 0 \end{smallmatrix}\right]$ and $\mathcal{L}_1^* = \sigma_1\mathcal{L}_1\sigma_1$. Suppose $\mathcal{L}_1 f = \omega_1 f$ and $\mathcal{L}_1 g = \omega_2 g$ with

$\bar{\omega}_1 \neq \omega_2$. We have $\mathcal{L}_1^* \sigma_1 f = \sigma_1 \mathcal{L}_1 f = \omega_1 \sigma_1 f$. Thus

$$\omega_2(\sigma_1 f, g) = (\sigma_1 f, \omega_2 g) = (\sigma_1 f, \mathcal{L}_1 g)$$
$$= (\mathcal{L}_1^* \sigma_1 f, g) = (\omega_1 \sigma_1 f, g) = \bar{\omega}_1(\sigma_1 f, g).$$

Hence $(\sigma_1 f, g) = 0$. Therefore we have $\sigma_1 \bar{\Phi} \perp \bar{\Phi}, \sigma_3 \Phi, \sigma_3 \bar{\Phi}$, $\sigma_1 \Phi \perp \Phi, \sigma_3 \Phi$, $\sigma_3 \bar{\Phi}$, etc. If we write $u = u_1 + iu_2$, $v = v_1 + iv_2$ and $\Phi = \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$, then we have

$$\int \bar{u} v \, dx = 0. \tag{2.63}$$

In other words, $(u_1, v_1) + (u_2, v_2) = 0$ and $(u_1, v_2) = (u_2, v_1)$.

If $f \in S(\mathcal{L}_1)$ and $\mathcal{L}_1 g = \omega_2 g$ with $\omega_2 \neq 0$. We have $(\mathcal{L}_1^*)^2 \sigma_1 f = 0$, hence

$$(\sigma_1 f, \omega_2^2 g) = (\sigma_1 f, \mathcal{L}_1^2 g) = ((\mathcal{L}_1^*)^2 \sigma_1 f, g) = (0, g).$$

Hence $(\sigma_1 f, g) = 0$. In terms of components, we get $(Q_1, u_1) = (Q_1, u_2) = 0$, $(R_1, v_1) = (R_1, v_2) = 0$. The above shows Eq. (2.22). The rest of (4) and (5) follows directly.

To prove (6), we first prove the following spectral gap

$$L_+ \Big|_{\{Q_1, v_1, v_2\}^\perp} > \frac{1}{2}|e_1|, \qquad L_- \Big|_{\{R_1, u_1, u_2\}^\perp} > \frac{1}{2}|e_1|. \tag{2.64}$$

We will show the first assertion. Note that by Eq. (2.17) we have

$$v_1 = \mathbf{P}_c(L_-) v_1 + O(n^2), \qquad v_2 = -\phi_0 + \mathbf{P}_c(H_1) v_2 + O(n^2)$$

in L^2. In particular $\|v_2\|_{L^2} \geq 1/2$, and $(v_1, L_- v_1) \geq (v_1, L_- \mathbf{P}_c(L_-) v_1) - Cn^2 \geq -Cn^2$. By Eq. (2.63)

$$(v_1, L_- v_1) + (v_2, L_- v_2) = (v, L_- v) = (v, \omega u) = 0.$$

Hence $(v_2, L_+ v_2) = (v_2, L_- v_2) + O(n^2) \leq Cn^2$. We also have $(Q_1, L_+ Q_1) = (Q_1, L_- Q_1) + O(n^4) = 0 + O(n^4)$. Let $Q_1' = Q_1 - (Q_1, v_2) v_2 / \|v_2\|_2^2$. We have $Q_1' \perp v_j$ and $Q_1' = Q_1 + O(n^3)$ by Eq. (2.17) again. Hence $(Q_1', L_+ Q_1') = (Q_1, L_+ Q_1) + O(n^4) = O(n^4) \leq Cn^2 (Q_1', Q_1')$. We conclude that $L_+|_{\mathrm{span}\{Q_1, v_2\}} \leq Cn^2$. Since L_+ is a perturbation of H_1, it has exactly two eigenvalues below $\frac{1}{2}|e_1|$. By minimax principle we have $L_+|_{\{Q_1, v_2\}^\perp} > (1/2)|e_1|$. This shows the first assertion of Eq. (2.64). The second assertion is proved similarly.

Let $\mathbf{Q}(\psi)$ denote the quadratic form: (see e.g. Refs. [25,26])

$$\mathbf{Q}(\psi) = (f, L_+ f) + (g, L_- g), \quad \text{if } \psi = f + ig. \tag{2.65}$$

One can show for any $\psi \in L^2$

$$\mathbf{Q}(e^{t\mathcal{L}_1} \psi) = \mathbf{Q}(\psi), \quad \text{for all } t, \tag{2.66}$$

by direct differentiation in t. By Eq. (2.64) one has

$$\mathbf{Q}(\eta) \sim \|\eta\|_{H^1}^2, \quad \text{for any} \quad \eta \in \mathbf{H}_c(\mathcal{L}_1).$$

Thus

$$\|e^{t\mathcal{L}_1}\eta\|_{H^1}^2 \sim \mathbf{Q}(e^{t\mathcal{L}_1}\eta) = \mathbf{Q}(\eta) \sim \|\eta\|_{H^1}^2.$$

Similarly, we have by Eq. (2.64) and the above relation

$$\|\eta\|_{H^3}^2 \sim \|\mathcal{L}_1\eta\|_{H^1}^2 \sim \mathbf{Q}(\mathcal{L}_1\eta).$$

Since $\mathbf{Q}(\mathcal{L}_1\eta) = \mathbf{Q}(e^{t\mathcal{L}_1}\mathcal{L}_1\eta)$, we have $\|\eta\|_{H^3} \sim \|e^{t\mathcal{L}_1}\eta\|_{H^3}$. By interpolation we have $\|\eta\|_{H^2} \sim \|e^{t\mathcal{L}_1}\eta\|_{H^2}$. We have proven (6).

2.7. Wave Operator and Decay Estimate

It remains to prove the decay estimate (7). We will use the wave operator. We will compare $\mathcal{L}_1$ with JH_*, where $H_* = -\Delta - E_1$. Recall we write $\mathcal{L}_1 = JH_* + A$ in §2.4, Eq. (2.60). Keep in mind that H_* has no bound states and A is local. Define $W_+ = \lim_{t\to+\infty} e^{-t\mathcal{L}_1}e^{tJH_*}$. Let $R(z) = (z - \mathcal{L}_1)^{-1}$ and $R_*(z) = (z - JH_*)^{-1}$. We have

$$\begin{aligned} &W_+ f - f \\ &\quad = \lim_{\varepsilon\to 0+} \int_{|E_1|}^{+\infty} R(i\tau + \varepsilon) A[R_*(i\tau - \varepsilon) - R_*(i\tau + \varepsilon)] f \, d\tau \\ &\quad\quad - \lim_{\varepsilon\to 0+} \int_{|E_1|}^{+\infty} R(-i\tau + \varepsilon) A[R_*(-i\tau - \varepsilon) - R_*(-i\tau + \varepsilon)] f \, d\tau. \end{aligned}$$

Yajima[27,28] was the first to give a general method for proving the $(W^{k,p}, W^{k,p})$ estimates for the wave operators of self-adjoint operators. This method was extended by Cuccagna[5] to non-selfadjoint operators in the form we are considering. (He also used idea from Kato[11]). One key ingredient in this approach is the resolvent estimates near the continuous spectrum, which in many cases can be obtained by the Jensen–Kato[9] method. (See Ref. [27], Lemmas 3.1–3.2 and Ref. [5], Lemmas 3.9–3.10). In our current setting, this estimate is provided by the Lemma 2.5. We can thus follow the proof of Ref. [5] to obtain that W_+ is an operator from $\mathbb{C}L^2$ onto $\mathbf{H}_c(\mathcal{L}_1)$. Furthermore, W_+ and its inverse (restricted to $\mathbf{H}_c(\mathcal{L}_1)$) are bounded in (L^p, L^p)-norm for any $p \in [1, \infty]$. (Note this bound depends on n since our bound on $R(w)$ depends on n.) By the intertwining property of the wave operator we have

$$e^{t\mathcal{L}_1}\,\mathbf{P}_c = W_+ e^{tJH_*}(W_+)^*\,\mathbf{P}_c.$$

The decay estimate in (7) follows from the decay estimate of e^{tJH_*}.

The proof of Theorem 2.2 is complete.

2.8. Proof of Theorem 2.1

By the same Cauchy integral argument as in subsection 2.2, the only eigenvalues of $\mathcal{L}_1$ are inside the disks $\{w : |w| < \sqrt{n}\}$, $\{w : |w - ie_{01}| < \sqrt{n}\}$ and $\{w : |w + ie_{01}| < \sqrt{n}\}$. Moreover, their dimensions are 2, 1, and 1, respectively, the same as that of JH_1. It counts the dimension of (generalized) eigenspaces of $\mathcal{L}_1$ in $\mathbb{C}L^2$. It also counts the dimensions of the restriction of these spaces in $L^2 = L^2(\mathbb{R}^3, \mathbb{R}^2)$ as a real-valued vector space.

By Eq. (2.9), we already have two generalized eigenvectors near 0. Hence we have everything near 0. Since the dimension is 1 near ie_{01}, there is only a simple eigenvalue ω_* near ie_{01}. We have $\omega_* = ie_{01} + O(n^2)$ since the difference between $\mathcal{L}_1$ and JH_1 is of order $O(n^2)$. ω_* has to be purely imaginary, otherwise $-\bar{\omega}_*$ is another eigenvalue near ie_{01}, cf. Eq. (2.19), and the dimension cannot be 1. (This also follows from the Theorem of Grillakis.)

By the same arguments in §2.2–2.4 we can prove resolvent estimates and the non-existence of embedded eigenvalues. Also, the bottoms of the continuous spectrum are not eigenvalue nor resonance.

Let Φ be an eigenvector corresponding to ω_*. Since $\mathcal{L}_1\Phi = \omega_*\Phi$ and $\bar{\omega}_* = -\omega_*$, we have $\mathcal{L}_1\bar{\Phi} = -\omega_*\bar{\Phi}$. Hence the (unique) eigenvalue near $-ie_{01}$ is $-\omega_*$ with eigenvector $\bar{\Phi}$. Write $\Phi = \left[\begin{smallmatrix} u \\ -iv \end{smallmatrix}\right]$. We may assume u is real. Writing out $\mathcal{L}_1\Phi = i\kappa\Phi$ we get $L_- v = -\kappa u$ and $L_+ u = -\kappa v$. Hence v is also real. We can normalize u so that $(u, v) = 1$ or -1. Since Φ is a perturbation of $\left[\begin{smallmatrix} \phi_0 \\ -i\phi_0 \end{smallmatrix}\right]$, we have $(u, v) = 1$.

With this choice of u, v, let $\mathbb{C}\mathbf{E}_1$ and $\mathbf{E}_1$ be defined as in Eq. (2.12). $\mathbb{C}\mathbf{E}_1$ is the combined eigenspace corresponding to $\pm\omega_*$. Clearly $\mathbf{RE}\;\mathbb{C}\mathbf{E}_1 \subset \mathbf{E}_1$. Since

$$a\begin{bmatrix} u \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ v \end{bmatrix} = \mathbf{RE}\,\alpha\Phi, \quad \alpha = a + bi,$$

we have $\mathbf{RE}\,\mathbb{C}\mathbf{E}_1 = \mathbf{E}_1$. That the choice of α is unique can be checked directly. The statement that if $\zeta = \mathbf{RE}\;\alpha\Phi$ then $\mathcal{L}_1\zeta = \mathbf{RE}\;\omega_*\alpha\Phi$ and $e^{t\mathcal{L}_1}\zeta = \mathbf{RE}\;e^{t\omega_*}\alpha\Phi$ is clear. We have proved (3) and (4).

Clearly, $S(\mathcal{L}_1)$, $\mathbf{E}_1(\mathcal{L}_1)$, and $\mathbf{H}_c(\mathcal{L}_1)$ defined as in Eqs. (2.9), (2.10), and (2.12) are invariant subspaces of L^2 under $\mathcal{L}_1$, and we have the decomposition Eq. (2.7). This is (2).

For (5), note that Eq. (2.10) is by definition. For Eq. (2.11), we have

$$(Q_1, u) = (Q_1, (-\kappa)^{-1} L_- v) = (L_- Q_1, (-\kappa)^{-1} v) = 0,$$
$$(R_1, v) = (R_1, (-\kappa)^{-1} L_+ u) = (-\kappa)^{-1}(L_+ R_1, u) = (-\kappa)^{-1}(Q_1, u) = 0.$$

Equation (2.14) comes from the orthogonal relations directly.

The first statement of (6) is because of (5). For the rest of (6), we first prove the following spectral gap

$$L_+|_{\{Q_1, v\}^\perp} > \frac{1}{2}|e_1|, \qquad L_-|_{\{R_1, u\}^\perp} > \frac{1}{2}|e_1|. \tag{2.67}$$

Since L_+ is a perturbation of H_1, it has exactly two eigenvalues below $(1/2)|e_1|$. Notice that $(Q_1, L_+ Q_1) = (Q_1 L_- Q_1) + O(n^4) = O(n^4)$ and $(v, L_+ v) = (v, -\kappa u) = -\kappa$. Since $Q_1 = n\phi_1 + O(n^3)$ and $v = \phi_0 + O(n^2)$, one has $(Q_1, v) = O(n^3)$. Thus one can show $L_+|_{\mathrm{span}\{Q_1, v\}} \le Cn^2$. If there is a $\phi \perp Q_1, v$ with $(\phi, L_+\phi) \le \left(\frac{1}{2}\right)|e_1|(\phi, \phi)$, then we have $L_+|_{\mathrm{span}\{Q_1, v, \phi\}} \le (1/2)|e_1|$, which contradicts with the fact that L_+ has exactly two eigenvalues below $(1/2)|e_1|$ by minimax principle. This shows the first part of Eq. (2.67). The second part is proved similarly.

Recall the quadratic form $\mathbf{Q}(\psi)$ defined in Eq. (2.65) in §2.6. Also recall Eq. (2.66) that $\mathbf{Q}(e^{t\mathcal{L}_1}\psi) = \mathbf{Q}(\psi)$ for all t and all $\psi \in L^2$. By the spectral gap Eq. (2.67) one has

$$\mathbf{Q}(\eta) \sim \|\eta\|_{H^1}^2, \quad \mathbf{Q}(\mathcal{L}_1\eta) \sim \|\eta\|_{H^3}^2, \quad \text{for any} \quad \eta \in \mathbf{H}_c(\mathcal{L}_1). \tag{2.68}$$

For $\psi \in M_1$, we can write $\psi = \zeta + \eta$, where $\zeta = \mathbf{RE}\ \alpha\Phi$, $\alpha \in \mathbb{C}$ and $\eta \in \mathbf{H}_c(\mathcal{L}_1)$. Notice that, by orthogonality in Eq. (2.10),

$$\mathbf{Q}(\psi) = -|\alpha|^2\kappa(u, v) + \mathbf{Q}(\eta),$$

which is not positive definite, (recall $(u, v) = 1$). However,

$$\|\psi\|_{H^1}^2 \sim |\alpha|^2 + \|\eta\|_{H^1}^2. \tag{2.69}$$

To see it, one first notes that $\|\psi\|_{H^1}^2$ is clearly bounded by the right side. Because of Eq. (2.14), one has $|\alpha|^2 \le C\|\psi\|_{H^1}^2$. One also has $\|\eta\|_{H^1}^2 \le C\|\phi\|_{H^1}^2 + C|\alpha|^2$. Hence Eq. (2.69) is true.

Therefore for $\psi = (\mathbf{RE}\ \alpha\Phi) + \eta$ we have

$$\begin{aligned}
\left\| e^{t\mathcal{L}_1}\psi \right\|_{H^1}^2 &\sim \left\| e^{t\mathcal{L}_1}\mathbf{RE}\,\alpha\Phi \right\|_{H^1}^2 + \left\| e^{t\mathcal{L}_1}\eta \right\|_{H^1}^2 && \text{(by Eq.(2.69))}\\
&\sim |e^{-it\omega_*}\alpha|^2 + \mathbf{Q}(e^{t\mathcal{L}_1}\eta) && \text{(by (4), Eq. (2.68))}\\
&\sim |\alpha|^2 + \mathbf{Q}(\eta) && \text{(by Eq. (2.66)).}
\end{aligned}$$

Hence we have $\|e^{t\mathcal{L}_1}\psi\|_{H^1}^2 \sim \|\psi\|_{H^1}^2$ for all t. By an argument similar to that in §2.6, we have $\|e^{t\mathcal{L}_1}\psi\|_{H^k} \sim \|\psi\|_{H^k}$ for $k = 3, 2$. We have shown (6). The decay estimate in (7) is obtained as in Theorem 2.2 (7). The constant C, however, is independent of n in the non-resonant case. The proof of Theorem 2.1 is complete.

3. SOLUTIONS CONVERGING TO EXCITED STATES

In this section we prove Theorem 1.1 using Theorems 2.1 and 2.2. Since the proof for the non-resonant case is easier, we will first prove the resonant case and then sketch the non-resonant case. Note that we could follow the approach of Theorem 1.5 of Ref. [23] if we had the transform $\mathcal{L}_1\,\mathbf{P}_c^{\mathcal{L}_1} = -U^{-1}iAU\,\mathbf{P}_c^{\mathcal{L}_1}$ as in Ref. [23]. However, it is not easy to define A and U for $\mathcal{L}_1$ and hence we choose another approach. This new approach also gives another proof for Theorem 1.5 of Ref. [23].

Note that, if we reverse the time direction, the same proof below gives the "*unstable manifold*," i.e., solutions $\psi(t)$ which converge to excited states as $t \to -\infty$.

Fix E_1 and $Q_1 = Q_{1,E_1}$. Let $\mathcal{L}_1$ be the corresponding linearized operator, and $\mathbf{P}_{M_1}$, $\mathbf{P}_{\mathbf{E}_1}$ and $\mathbf{P}_c^{\mathcal{L}_1}$ the corresponding projections with respect to $\mathcal{L}_1$. For any $\xi_\infty \in \mathbf{H}_c(\mathcal{L}_1)$ with small $H^2 \cap W^{2,1}$ norm, we want to construct a solution $\psi(t)$ of the nonlinear Schrödinger Eq. (1.1) with the form

$$\psi(t) = [Q_1 + a(t)R_1 + h(t)]e^{-iE_1t + i\theta(t)},$$

where $a(t), \theta(t) \in \mathbb{R}$ and $h(t) \in M_1 = \mathbf{E}_1 \oplus \mathbf{H}_c(\mathcal{L}_1)$. Substituting the above ansatz into Eq. (1.1) and using $\mathcal{L}_1 iQ_1 = 0$ and $\mathcal{L}_1 R_1 = -iQ_1$, we get

$$\partial_t h = \mathcal{L}_1 h + i^{-1}F(aR_1 + h) - i\dot{\theta}(Q_1 + aR_1 + h) - aiQ_1 - \dot{a}R_1,$$

where

$$F(k) = \lambda Q_1(2|k|^2 + k^2) + \lambda|k|^2 k, \qquad k = aR_1 + h. \tag{3.1}$$

The condition $h(t) \in M_1$ can be satisfied by requiring that $h(0) \in M_1$ and

$$\dot{a} = (c_1 Q_1, \mathrm{Im}(F + \dot{\theta} h)), \tag{3.2}$$

$$\dot{\theta} = -[a + (c_1 R_1,\ \mathrm{Re} F)][1 + (c_1 R_1, R_1)a + (c_1 R_1,\ \mathrm{Re}\, h)]^{-1}, \tag{3.3}$$

where $c_1 = (Q_1, R_1)^{-1}$ and $F = F(aR_1 + h)$. The equation for h becomes

$$\partial_t h = \mathcal{L}_1 h + P_M F_{\mathrm{all}}, \qquad F_{\mathrm{all}} = i^{-1}(F + \dot{\theta}(aR_1 + h)).$$

The proofs of the two cases diverge here. For the resonant case we decompose, using the decomposition of M_1 and Eq. (2.20) of Theorem 2.2,

$$h(t) = \zeta(t) + \eta(t), \qquad \zeta(t) = \ \mathbf{RE}\{\alpha(t)\Phi + \beta(t)\sigma_3\Phi\},$$

where $\alpha(t), \beta(t) \in \mathbb{C}$ and $\eta(t) \in \mathbf{H}_c(\mathcal{L}_1)$. Note

$$\mathcal{L}_1 \zeta = \mathbf{RE}\big\{\omega_* \alpha \Phi - \omega_* \beta \sigma_3 \Phi\big\}.$$

Recall $\omega_* = i\kappa + \gamma$ with $\kappa, \gamma > 0$. Taking the projections P_α and P_β defined in Eq. (2.24) of Theorem 2.2 of the h-equation, we have

$$\dot{\alpha} = \omega_* \alpha + P_\alpha F_{\mathrm{all}}, \tag{3.4}$$

$$\dot{\beta} = -\omega_* \beta + P_\beta F_{\mathrm{all}}. \tag{3.5}$$

Taking projection $\mathbf{P}_c^{\mathcal{L}_1}$ we get the equation for η,

$$\partial_t \eta = \mathcal{L}_1 \eta + \ \mathbf{P}_c^{\mathcal{L}_1} i^{-1} \dot{\theta} \eta + \ \mathbf{P}_c^{\mathcal{L}_1} \widetilde{F}, \qquad \widetilde{F} = i^{-1}(F + \dot{\theta}(aR_1 + \zeta)).$$

We single out $\mathbf{P}_c^{\mathcal{L}_1} i^{-1} \dot{\theta} \eta$ since it is a global linear term in η and cannot be treated as error. Let

$$\widetilde{\eta} = \ \mathbf{P}_c^{\mathcal{L}_1} e^{i\theta} \eta.$$

Note $\eta = \widetilde{\eta} + \mathbf{P}_c^{\mathcal{L}_1}(1 - e^{i\theta})\eta$ and $\mathbf{P}_c^{\mathcal{L}_1}(1 - e^{i\theta})$ is a bounded map from $\mathbf{H}_c(\mathcal{L}_1) \cap H^2$ into itself with its norm bounded by $C|\theta|$. Hence if θ is sufficiently small, we can solve η in terms of $\widetilde{\eta}$ by expansion:

$$\eta = U_\theta \widetilde{\eta}, \qquad U_\theta \equiv \sum_{j=0}^{\infty} [\,\mathbf{P}_c^{\mathcal{L}_1}(1 - e^{i\theta})]^j. \tag{3.6}$$

The equation for $\widetilde{\eta}$ is

$$\begin{aligned}\partial_t\widetilde{\eta} &= \mathbf{P}_c^{\mathcal{L}_1}e^{i\theta}(i\dot{\theta}\eta+\partial_t\eta)\\ &= \mathcal{L}_1\widetilde{\eta}+\Big\{\mathbf{P}_c^{\mathcal{L}_1}e^{i\theta}\mathcal{L}_1-\mathcal{L}_1\mathbf{P}_c^{\mathcal{L}_1}e^{i\theta}\Big\}\eta\\ &\quad+\mathbf{P}_c^{\mathcal{L}_1}e^{i\theta}\Big\{i\dot{\theta}\eta-\mathbf{P}_c^{\mathcal{L}_1}i\dot{\theta}\eta+\mathbf{P}_c^{\mathcal{L}_1}\widetilde{F}\Big\}.\end{aligned}$$

Note that

$$\begin{aligned}\Big\{\mathbf{P}_c^{\mathcal{L}_1}e^{i\theta}\mathcal{L}_1-\mathcal{L}_1\mathbf{P}_c^{\mathcal{L}_1}e^{i\theta}\Big\}\eta &= \mathbf{P}_c^{\mathcal{L}_1}[e^{i\theta},\mathcal{L}_1]\eta\\ &= \mathbf{P}_c^{\mathcal{L}_1}\sin\theta[i,\mathcal{L}_1]\eta\\ &= \mathbf{P}_c^{\mathcal{L}_1}\sin\theta 2\lambda Q_1^2\bar{\eta}.\end{aligned}$$

Hence we have

$$\partial_t\widetilde{\eta}=\mathcal{L}_1\widetilde{\eta}+\mathbf{P}_c^{\mathcal{L}_1}\Big\{\sin\theta 2\lambda Q_1^2\bar{\eta}+e^{i\theta}(1-\mathbf{P}_c^{\mathcal{L}_1})i\dot{\theta}\eta+e^{i\theta}\,\mathbf{P}_c^{\mathcal{L}_1}\widetilde{F}\Big\}.$$

For a given profile ξ_∞, let

$$\widetilde{\eta}(t)=e^{t\mathcal{L}_1}\xi_\infty+g(t). \tag{3.7}$$

We have the equation

$$\partial_t g=\mathcal{L}_1 g+\mathbf{P}_c^{\mathcal{L}_1}\Big\{\sin\theta 2\lambda Q_1^2\bar{\eta}+e^{i\theta}(1-\mathbf{P}_c^{\mathcal{L}_1})i\dot{\theta}\eta+e^{i\theta}\,\mathbf{P}_c^{\mathcal{L}_1}\widetilde{F}\Big\}. \tag{3.8}$$

We want $g(t)\to 0$ as $t\to\infty$ in some sense.

Summarizing, we write the solution $\psi(t)$ in the form

$$\begin{aligned}\psi(t)=\Big\{&Q_1+a(t)R_1+\mathbf{RE}\big\{\alpha(t)\Phi+\beta(t)\sigma_3\Phi\big\}\\ &+U_{\theta(t)}(e^{t\mathcal{L}_1}\xi_\infty+g(t))\Big\}\,e^{-iE_1t+i\theta(t)},\end{aligned} \tag{3.9}$$

with $a(t)$, $\theta(t)$, $\alpha(t)$, $\beta(t)$, and $g(t)$ satisfying Eqs. (3.2)–(3.5) and (3.8), respectively.

The main term of F is

$$F_0=\lambda Q_1\big(2|\xi|^2+\xi^2\big)+\lambda|\xi|^2\xi, \qquad \xi(t)=U_{\theta(t)}e^{t\mathcal{L}_1}\xi_\infty.$$

Notice that, if $\|\xi_\infty\|_{H^2\cap W^{2,1}}\le\varepsilon\ll 1$, then $\xi(t)$ satisfies

$$\begin{aligned}&\|\xi(t)\|_{H^2}\le C(n)\varepsilon, \quad \|\xi(t)\|_{W^{2,\infty}}\le C(n)\varepsilon|t|^{-3/2},\\ &\||\xi|^2\xi(t)\|_{H^2}\le C(n)\varepsilon^3\langle t\rangle^{-3}.\end{aligned}$$

Here we have used the boundedness and decay estimates for $e^{t\mathcal{L}_1}\mathbf{P}_c^{\mathcal{L}_1}$ in Theorem 2.2 (6)–(7). Since Q_1 is fixed, it does not matter that the constant depends on n. The main term of F_0 is quadratic in ξ. Hence

$$\|F_0(t)\|_{H^2} \le C\varepsilon^2\langle t\rangle^{-3}.$$

As it will become clear, we have the freedom to choose ξ_∞ and $\beta_0 = \beta(0)$. We require that $\xi_\infty \in \mathbf{H}_c(\mathcal{L}_1)$ and

$$\|\xi_\infty\|_{H^2\cap W^{2,1}} \le \varepsilon, \qquad |\beta_0| \le \varepsilon^2/4, \tag{3.10}$$

with $\varepsilon \le \varepsilon_0(n)$ sufficiently small. With given ξ_∞ and β_0, we will define a contraction mapping Ω in the following space

$$\begin{aligned}\mathcal{A} = \Big\{&(a,\theta,\alpha,\beta,g) : [0,\infty) \to \mathbb{R}\times\mathbb{R}\times\mathbb{C}\times\mathbb{C}\times(\mathbf{H}_c(\mathcal{L}_1)\cap H^2),\\ &|a(t)|, |\alpha(t)|, |\beta(t)|, \le \varepsilon^{7/4}(1+t)^{-2},\\ &\|g(t)\|_{H^2} \le \varepsilon^{7/4}(1+t)^{-7/4}, |\theta(t)| \le 2\varepsilon^{7/4}(1+t)^{-1}\Big\}.\end{aligned}$$

For convenience, we introduce a variable $b = \dot\theta$. Our map Ω is defined by

$$\Omega : (a,\theta,\alpha,\beta,\eta) \longrightarrow (a^\triangle, \theta^\triangle, \alpha^\triangle, \beta^\triangle, \eta^\triangle),$$

$$\begin{aligned}a^\triangle(t) &= \int_\infty^t (c_1Q_1, \operatorname{Im}(F+bh))\,ds,\\ \theta^\triangle(t) &= \int_\infty^t b(s)\,ds,\\ \alpha^\triangle(t) &= \int_\infty^t e^{\omega_*(t-s)}P_\alpha i^{-1}(F+b(aR+h))\,ds,\\ \beta^\triangle(t) &= e^{-\omega_* t}\beta_0 + \int_0^t e^{-\omega_*(t-s)}P_\beta i^{-1}(F+b(aR+h))\,ds,\\ g^\triangle(t) &= \int_\infty^t e^{\mathcal{L}_1(t-s)}\mathbf{P}_c^{\mathcal{L}_1}\Big\{\sin\theta 2\lambda Q_1^2\bar\eta + e^{i\theta}(1-\mathbf{P}_c^{\mathcal{L}_1})ib\eta\\ &\qquad + e^{i\theta}\mathbf{P}_c^{\mathcal{L}_1} i^{-1}(F+b(aR+\zeta))\Big\}\,ds,\end{aligned}$$

where $c_1 = (Q_1, R_1)^{-1}$, $F = F(aR+h)$ is defined in Eq. (3.1), and

$$\begin{aligned}h(t) &= \zeta(t)+\eta(t),\\ \zeta(t) &= \mathbf{RE}\big\{\alpha(t)\Phi + \beta(t)\sigma_3\Phi\big\}, \quad \eta(t) = U_{\theta(t)}(e^{t\mathcal{L}_1}\xi_\infty + g(t)),\\ b(t) &= -[a + (c_1R_1, \operatorname{Re}F)][1 + (c_1R_1, R_1)a + (c_1R_1, \operatorname{Re}h)]^{-1}.\end{aligned}$$

We will use Strichartz estimate for the term $\sin\theta 2\lambda Q_1^2\bar\eta$ in the g-integral:

$$\left\| \int_\infty^t e^{\mathcal{L}_1(t-s)}\,\mathbf{P}_c^{\mathcal{L}_1} f(s,\cdot)\,ds \right\|_{L_x^2} \le C(n)\left\{ \int_\infty^t \|\,f(s,\cdot)\|_{L_x^{r'}}^{q'}\,ds \right\}^{1/q'} \tag{3.11}$$

for $3/r + 2/q = 3/2$, $2 < q \le \infty$. Here $'$ means the usual conjugate exponent. Equation (3.11) can be proved by either using wave operator to map $e^{t\mathcal{L}_1}$ to $e^{-it(-\Delta-E_1)}$, or by using the decay estimate Theorem 2.2 (7) and repeating the usual proof for Strichartz estimate. We will also use

$$\|\phi\|_{H^2} \sim \|\mathcal{L}_1\phi\|_{L^2} \quad \text{for} \quad \phi \in \mathbf{H}_c(\mathcal{L}_1),$$

which follows from the spectral gap Eq. (2.64). Since $\sin\theta 2\lambda Q_1^2\bar\eta$ is local and bounded by $C(n)\varepsilon^{7/4}\langle t\rangle^{-1}\varepsilon\langle t\rangle^{-3/2}$, by choosing q large we have

$$\begin{aligned}
&\left\| \int_\infty^t e^{\mathcal{L}_1(t-s)}\mathbf{P}_c^{\mathcal{L}_1} \sin\theta 2\lambda Q_1^2\bar\eta\,ds \right\|_{H^2} \\
&\quad \le C\left\| \int_\infty^t e^{\mathcal{L}_1(t-s)}\mathbf{P}_c^{\mathcal{L}_1}\mathcal{L}_1 \sin\theta 2\lambda Q_1^2\bar\eta\,ds \right\|_{L_x^2} \\
&\quad \le C\left\{ \int_\infty^t [\varepsilon^{11/4}(1+s)^{-(5/2)}]^{q'}\,ds \right\}^{1/q'} = C\varepsilon^{11/4}(1+t)^{-5/2+1/q'}.
\end{aligned}$$

Here $C = C(n)$. In particular, we get $C(n)\varepsilon^{11/4}(1+t)^{-7/4}$ by choosing $q = 4$. Note that we would only get $t^{-3/2}$ if we estimate this term directly without using Eq. (3.11).

Note $|b(t)| \le 2|a(t)|$. Since $t - s < 0$ in the integrand of α, $\operatorname{Re}\omega_*(t-s) < 0$ and the α-integral converges. Similarly $\operatorname{Re}\omega_*(t-s) > 0$ in the integrand of β and hence the β-integration converges. Observe that we have the freedom of choosing β_0 and ξ_∞. Since $e^{-\omega_* t}\beta_0$ decays exponentially, the main term of $\beta(t)$ when t large is given by F_0, not $e^{-\omega_* t}\beta_0$. Direct estimates show that

$$\begin{aligned}
&|\alpha(t)| \le C(n)\varepsilon^2(1+t)^{-3}, \quad |\beta(t)| \le \varepsilon^2 e^{-\gamma t}/4 + C(n)\varepsilon^2(1+t)^{-3}, \\
&|a(t)|, |b(t)| \le C(n)\varepsilon^2(1+t)^{-2}, \quad |\theta(t)| \le C(n)\varepsilon^2(1+t)^{-1}, \\
&\|g(t)\|_{H^2} \le C(n)\varepsilon^2(1+t)^{-7/4}.
\end{aligned}$$

It is easy to check that the map Ω is a contraction if ε is sufficiently small. Thus we have a fixed point in $\mathcal{A}$, which gives a solution to the system (3.2)–(3.5), and (3.8). Since it lies in $\mathcal{A}$, we also have the desired estimates. We obtain $\alpha(0)$, $a(0)$, and $\theta(0)$ as functions of ξ_∞ and β_0.

Recall $\psi_{as}(t) = Q_1 e^{-iE_1 t + i\theta(t)} + e^{-iE_1 t} e^{t\mathcal{L}_1}\xi_\infty$ and we have

$$\psi(t) = [Q_1 + U_{\theta(t)} e^{t\mathcal{L}_1}\xi_\infty]\, e^{-iE_1 t + i\theta(t)} + O(t^{-7/4}) \quad \text{in} \quad H^2.$$

Since $\mathbf{P}_c^{\mathcal{L}_1}(1 - e^{i\theta}) = O(\theta(t)) = O(t^{-1})$, by the definition (3.6) of U_θ,

$$\begin{aligned} U_{\theta(t)} e^{t\mathcal{L}_1}\xi_\infty &= [1 + \mathbf{P}_c^{\mathcal{L}_1}(1 - e^{i\theta})] e^{t\mathcal{L}_1}\xi_\infty + O(t^{-2}) \\ &= (2 - e^{i\theta}) e^{t\mathcal{L}_1}\xi_\infty + (1 - \mathbf{P}_c^{\mathcal{L}_1})(1 - e^{i\theta}) e^{t\mathcal{L}_1}\xi_\infty + O(t^{-2}) \end{aligned}$$

in H^2. Since $(1 - \mathbf{P}_c^{\mathcal{L}_1})$ is a local operator, $(1 - \mathbf{P}_c^{\mathcal{L}_1})(1 - e^{i\theta}) e^{t\mathcal{L}_1}\xi_\infty = O(t^{-1} \cdot t^{-3/2})$. Also, $e^{i\theta}(2 - e^{i\theta}) = 1 + O(\theta^2) = 1 + O(t^{-2})$. Hence we have $\psi(t) - \psi_{as}(t) = O(t^{-7/4})$ in H^2. We have proven Theorem 1.1 under assumption (R).

We now sketch the proof for the non-resonant case. The only difference is that we define $\zeta(t)$ as $\mathbf{RE}\ \alpha(t)\Phi$ and write $\psi(t)$ in the form

$$\psi(t) = \Big\{ Q_1 + a(t) R_1 + \mathbf{RE}(\alpha(t)\Phi) + U_{\theta(t)}(e^{t\mathcal{L}_1}\xi_\infty + g(t)) \Big\} e^{-iE_1 t + i\theta(t)}.$$

The function $\alpha(t)$ still satisfies Eq. (3.4) but with a purely imaginary eigenvalue ω_*. The previous proof will go through if we remove all terms related to β.

4. APPENDIX

In this appendix we prove Proposition 1.2 on the existence of vanishing solutions. Recall $H_0 = -\Delta + V$. The propagator $e^{-iH_0 t}$ is bounded in H^s, $s \ge 0$, and satisfies the decay estimate,

$$\left\| e^{-itH_0}\, \mathbf{P}_c^{H_0}\phi \right\|_{L^\infty} \le C|t|^{-3/2} \|\phi\|_{L^1} \tag{4.1}$$

under assumption A1. See Refs. [9,10,13,27].

For any $\xi_\infty \ne 0 \in \mathbf{H}_c(H_0)$ with $\|\xi_\infty\|_{H^2 \cap W^{2,1}} = \varepsilon$ small, we want to construct a solution $\psi(t)$ of Eq. (1.1) with the form

$$\psi(t) = e^{-iH_0 t}\xi_\infty + g(t), \quad g(t) = \text{error}. \tag{4.2}$$

Let $\xi(t) = e^{-iH_0 t}\xi_\infty$. By Eq. (4.1) we have,

$$\|\xi(t)\|_{H^2} \le C_1 \varepsilon, \qquad \|\xi(t)\|_{W^{2,\infty}} \le C_1 \varepsilon |t|^{-3/2}, \qquad \left\|\xi^2 \bar{\xi}(t)\right\|_{H^2} \le C_1 \varepsilon^3 \langle t \rangle^{-3},$$

for some constant C_1. The error term $g(t)$ satisfies

$$\partial_t g = -iH_0 g + F,$$

with $g(t) \to 0$ as $t \to \infty$ in certain sense, and

$$F(t) = -i\lambda|\psi|^2\psi, \qquad \psi = \xi(t) + g(t), \qquad \xi(t) = e^{-iH_0 t}\xi_\infty. \tag{4.3}$$

We define a solution by Eq. (4.3) and

$$g(t) = \int_\infty^t e^{-iH_0(t-s)}F(s)\,ds. \tag{4.4}$$

Note that $g(t)$ belongs to L^2 and is not restricted to the continuous spectrum component of H_0. Also note that the main term in F is $|\xi|^2\xi(t)$, which is of order t^{-3} in H^2. Hence $g(t) \lesssim t^{-2}$.

We define a contraction mapping in the following class

$$\mathcal{A} = \left\{g(t) : [0, \infty) \to H^2(\mathbb{R}^3),\ \|h(t)\|_{H^2} \le C_1\varepsilon^3(1+t)^{-2}\right\}.$$

This class is not empty since it contains the zero function. We also define the norm

$$\|g\|_{\mathcal{A}} := \sup_{t>0}(1+t)^2\|g(t)\|_{H^2}.$$

For $g(t) \in \mathcal{A}$ we define

$$\Omega : g(t) \longrightarrow g^\triangle(t) = -i\lambda \int_\infty^t e^{-iH_0(t-s)}\big(|\xi+g|^2(\xi+g)\big)(s)\,ds.$$

It is easy to check that

$$\begin{aligned}\left\|g^\triangle(t)\right\|_{H^2} &\le \int_t^\infty \|F(t)\|_{H^2}\,ds \\ &\le \int_t^\infty C_1\varepsilon^3\langle s\rangle^{-3} + C\varepsilon^5\langle s\rangle^{-7/2}\,ds \le C_1\varepsilon^3\langle t\rangle^{-2},\end{aligned}$$

if ε_0 is sufficiently small. This shows that the map Ω maps $\mathcal{A}$ into itself. Similarly one can show $\|\Omega g_1 - \Omega g_2\|_{\mathcal{A}} \le \frac{1}{2}\|g_1 - g_2\|_{\mathcal{A}}$, if $g_1, g_2 \in \mathcal{A}$. Therefore our map is a contraction mapping and we have a fixed point. Hence we have a solution $\psi(t)$ of the form (4.2) with $e^{-itH_0}\xi_\infty$ as the main profile.

Remark. The above existence result holds no matter how many bound states H_0 has. The situation is different if we linearize around a nonlinear excited state. In that case, the propagator $e^{t\mathcal{L}_1}$, ($\mathcal{L}_1$ is the linearized operator), may not be bounded in whole L^2.

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