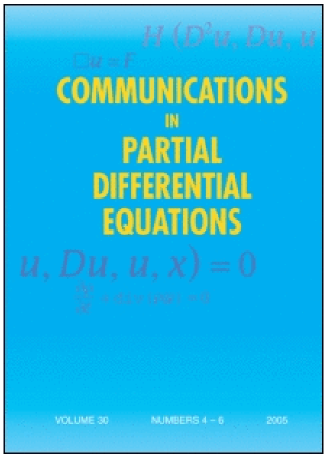

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Vladimír Šverák [a]; Tai-Peng Tsai [b]
[a] School of Mathematics, Universtiy of Minnesota, Minneapolis, MN, U.S.A
[b] Courant Institute of Mathematical Sciences, New York University, New York, NY, U.S.A

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ON THE SPATIAL DECAY OF 3-D STEADY-STATE NAVIER-STOKES FLOWS

Vladimír Šverák

School of Mathematics
University of Minnesota
Minneapolis, MN 55455, U.S.A.
sverak@math.umn.edu

and

Tai-Peng Tsai

Courant Institute of Mathematical Sciences
New York University
New York, NY 10012, U.S.A.
ttsai@cims.nyu.edu

Abstract

This note shows that any solution $\mathbf{u}$ of the Navier-Stokes equations in a 3-D exterior domain which decays as $O(|x|^{-1})$ at ∞ has the optimal decay $\nabla^k \mathbf{u}(x) = O(|x|^{-1-k})$ for $k \geq 1$, if the body force $\mathbf{f}$ satisfies $\nabla^m \mathbf{f}(x) = O(|x|^{-3-m})$ for all $m < k$. The main tool is an interior estimate for the Stokes system.

1 Introduction

In this short note we consider the question of the spatial decay of the solutions of the Navier-Stokes equations in three-dimensional exterior domains with zero velocity at infinity, which was raised by R. Finn [10] in 1960's. Our main tool is an interior estimate for the Stokes system, which seems to be of independent interest. This estimate is probably known to experts, but we were unable to locate it in the literature.

We consider the Navier-Stokes equations (with unit viscosity) in a domain $\Omega \subset \mathbf{R}^n$,

$$-\Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}, \quad \operatorname{div} \mathbf{u} = 0 \qquad \text{in } \Omega, \tag{1.1}$$

where $\mathbf{u} = (u_1, ..., u_n)$ stands for the velocity of the fluid, p the pressure, and $\mathbf{f} = (f_1, ..., f_n)$ the body force. When Ω is an exterior domain (a domain whose complement is compact), one often considers the boundary conditions

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_*, \qquad \mathbf{u}(x) \to \mathbf{u}_\infty \quad \text{as } |x| \to \infty. \tag{1.2}$$

In 1933 Leray [22] studied the existence of solutions to (1.1), (1.2) with finite Dirichlet integrals (so called *D-solutions*). His results were later extended in [21], [13], [9]–[11], [24], [19], [3]–[5], and other papers. However, many important questions still remain open. (See [12] for a survey and [16] for recent results.) In this note we will only consider the case $n = 3$ and $\mathbf{u}_\infty = 0$.

In 1965 Finn [10] showed (for small data) the existence of solutions with the following spatial decay at infinity

$$\mathbf{u}(x) = O\left(|x|^{-1}\right) \quad \text{as } |x| \to \infty, \tag{1.3}$$

(so called *physically reasonable solutions*). Finn, Babenko and Vasil'ev, Clark, etc. were able to show that solutions satisfying (1.3) possess many properties which are in agreement with conjectures based on a suitable linearization of the Navier-Stokes equations near infinity. (See e.g. [9]–[11], [5], [8].) Finn also showed ([10])

$$\nabla \mathbf{u}(x) = O\left(|x|^{-2} \ln |x|\right) \quad \text{as } |x| \to \infty. \tag{1.4}$$

If one compares (1.3) and (1.4) with the decay of the fundamental solution of the Stokes system, one finds that (1.3) is optimal, but (1.4) is not. The optimal decay for $\nabla \mathbf{u}(x)$ would be

$$\nabla \mathbf{u}(x) = O\left(|x|^{-2}\right) \quad \text{as } |x| \to \infty. \tag{1.5}$$

The question whether (1.5) holds has received a lot of attention recently because of its relevance to the stability of steady solutions. Novotny and Padula [23] showed, for small data, the existence of solutions satisfying both (1.3) and (1.5). Important results were also obtained by Borchers and Miyakawa [6], Kozono and Yamazaki [20], and Galdi and Simader [17]. (For the case $\mathbf{u}_\infty \neq 0$ see [9]–[11], [4], [5]. For the 2-D case, which is very different, see e.g. [18] and the references therein.)

The approach in these papers is to prove, under some smallness assumptions, the existence of solutions which have the desired decay properties.

In this note we prove that *any* solution with the decay (1.3) must satisfy (1.5), under natural assumptions on the body force $\mathbf{f}(x)$. In fact, under the assumption (1.3), we have

$$\nabla^k \mathbf{u}(x) = O\left(|x|^{-1-k}\right) \quad \text{as } |x| \to \infty \tag{1.6}$$

if $\mathbf{f}$ satisfies $|\nabla^m \mathbf{f}(x)| \le C_m |x|^{-3-m}$ at infinity for all $m < k$, where C_m are not necessarily small. Moreover, this assertion still holds if one assumes Ω is any unbounded domain. (One replaces $|x|$ in (1.3) and (1.6) by $\operatorname{dist}(x, \partial\Omega)$.) Our result does not say anything about the open question of existence of physically reasonable solutions ($\mathbf{u}_\infty = 0$) when the data are large. (Of course, for large data the steady-state solutions are likely to be unstable, so the term "physically reasonable" should probably not be taken too seriously in that case.)

Our proof involves an interior estimate for the Stokes system and a scaling argument. The usual interior estimate for the Stokes system is of the form

$$\|\mathbf{u}\|_{1,q,B_1} + \inf_{c\in\mathbf{R}} \|p-c\|_{q,B_1} \le C\left(\|p\|_{-1,q,B_2-B_1} + \|\mathbf{u}\|_{q,B_2-B_1} + \|\mathbf{f}\|_{-1,q,B_2}\right) \tag{1.7}$$

for $B_1 \subset B_2 \subset \Omega$, ([14] p.210). Our interior estimate is

$$\|\mathbf{u}\|_{1,q,B_1} + \inf_{c\in\mathbf{R}} \|p-c\|_{q,B_1} \le C\left(\|\mathbf{u}\|_{q,B_2-B_1} + \|\mathbf{f}\|_{-1,q,B_2}\right), \tag{1.8}$$

which differs from (1.7) by dropping the pressure term $\|p\|_{-1,q,B_2-B_1}$ from the right hand side. The usual proof for (1.7) is obtained by applying the hydrodynamical potential theory in the whole space, together with standard localization techniques. When one follows this procedure, it does not seem immediately obvious that one can drop the pressure term. However, (1.8) is not surprising since the Stokes system is elliptic in the sense of [2].

Our interest in such estimate was motivated by our previous investigations regarding Leray's self-similar solutions of the Navier-Stokes equations [27]. We will discuss this connection in another paper [26].

This paper is organized as follows. In Section 2 we prove the interior estimate (1.8), after giving some preliminary definitions and results. In Section 3 we use this interior estimate to prove the decay (1.6).

Notation. The letter n always denotes the space dimension, $(n > 1)$, and C denotes a generic constant. Ω denotes an open domain in $\mathbf{R}^n$, and B_1, B_2 denote two concentric balls inside Ω with radii R and $2R$ for some $R > 0$. Summation convention is used. We denote $\phi_{i,j} = \partial\phi_i/\partial x_j$. L^q, $W^{1,q}$, and $W^{-1,q}$ denote the usual Lebesque spaces, Sobolev spaces, and negative Sobolev spaces, with norms $\|\cdot\|_q$, $\|\cdot\|_{1,q}$, $\|\cdot\|_{-1,q}$ respectively. Also, $\|\cdot\|_{1-\frac{1}{q},q,\partial\Omega}$ denotes the boundary-trace norm. See Adams [1].

2 Interior estimates

We recall that, a *q-weak solution* (a terminology used in [14], [15]) of the Navier-Stokes equations (1.1) is a function $\mathbf{u} = (u_1, ..., u_n) \in W^{1,q}_{\text{loc}}$, $\operatorname{div} \mathbf{u} = 0$, which satisfies

$$\int_\Omega \nabla \mathbf{u} \cdot \nabla \boldsymbol{\phi} + \int_\Omega u_i\, u_{j,i}\, \phi_j = \langle \mathbf{f}, \boldsymbol{\phi} \rangle \tag{2.1}$$

for all $\boldsymbol{\phi} = (\phi_1, ..., \phi_n) \in C_c^\infty$, $\operatorname{div} \boldsymbol{\phi} = 0$; where $\mathbf{f}$ lies in a suitable distribution class. (Usually we call $\mathbf{u}$ a *weak solution* if $q = 2$.) Similarly we define *q-weak solutions* of the Stokes system by dropping the middle term in (2.1). Given a q-weak solution, one can find a corresponding pressure p (unique up to an adding constant) such that the Navier-Stokes equations (1.1) (resp. Stokes system) are satisfied in distribution sense, see [14] p.180, [15] p.8.

We will also use the following existence result, which can be found in [14] p.225.

Lemma 2.1 *Let Ω be a bounded C^2 domain in $\mathbf{R}^n$, $n \geq 2$, and $1 < q < \infty$. Let $\mathbf{u}_* \in W^{1-1/q,q}(\partial\Omega)$ and $\mathbf{f} \in W^{-1,q}(\Omega)$. Then there exists a q-weak solution $(\mathbf{u}, p) \in W^{1,q}(\Omega) \times L^q(\Omega)$ of the Stokes system, satisfying $\mathbf{u} = \mathbf{u}_*$ on $\partial\Omega$. It is unique up to an adding constant of p. Moreover, we have*

$$\|\mathbf{u}\|_{1,q,\Omega} + \inf_{c \in \mathbf{R}} \|p - c\|_{q,\Omega} \leq C_1 \left(\|\mathbf{u}_*\|_{1-\frac{1}{q},q,\partial\Omega} + \|\mathbf{f}\|_{-1,q,\Omega} \right), \tag{2.2}$$

where $C_1 = C_1(n, q, \Omega)$.

Now we can prove the interior estimate. We remark that the first half of the proof is similar to Struwe [25], p.444.

Theorem 2.2 *Let Ω be a domain in $\mathbf{R}^n$, $n \geq 2$, and $B_1 \subset B_2$ be concentric balls of radii R and $2R$, strictly contained in Ω. Let $1 < q < \infty$ and $\mathbf{f} \in W^{-1,q}_{loc}(\Omega)$. If $(\mathbf{u}, p) \in W^{1,q}_{loc}(\Omega) \times L^q_{loc}(\Omega)$ is a pair of q-weak solution of the Stokes system in Ω, then*

$$\|\mathbf{u}\|_{1,q,B_1} + \inf_{c \in \mathbf{R}} \|p - c\|_{q,B_1} \leq C_2 \left(\|\mathbf{u}\|_{1,B_2 - B_1} + \|\mathbf{f}\|_{-1,q,B_2} \right), \tag{2.3}$$

where $C_2 = C_2(n, q, R)$.

Proof. By Lemma 2.1, there exists a pair of auxiliary functions $(\mathbf{w}, \pi)$, $\mathbf{w} \in W^{1,q}_0(B_2)$ and $\pi \in L^q(B_2)$, which satisfies

$$-\Delta \mathbf{w} + \nabla \pi = \mathbf{f}, \quad \operatorname{div} \mathbf{w} = 0 \qquad \text{in } B_2$$

in weak sense. Moreover, this pair $(\mathbf{w}, \pi)$ satisfies

$$\|\mathbf{w}\|_{1,q,B_2} + \inf_{c \in \mathbf{R}} \|\pi - c\|_{q,B_2} \leq C_1 \|\mathbf{f}\|_{-1,q,B_2}. \tag{2.4}$$

Let $(\mathbf{v}, \chi) = (\mathbf{u} - \mathbf{w}, p - \pi)$. It satisfies (in weak sense)

$$-\Delta \mathbf{v} + \nabla \chi = 0, \quad \operatorname{div} \mathbf{v} = 0 \qquad \text{in } B_2.$$

It follows, by a well-known argument, that $\mathbf{v}$ is smooth and satisfies $\Delta^2 \mathbf{v} = 0$. Moreover, by the classical estimates of elliptic equations, (see for instance Browder [7]), there is an absolute constant $C_3 = C_3(n, R)$ such that

$$\|\mathbf{v}\|_{W^{2,\infty}(B_1)} \leq C_3 \|\mathbf{v}\|_{L^1(B_2 - B_1)}. \tag{2.5}$$

Since $\mathbf{u} = \mathbf{v} + \mathbf{w}$, (2.4) and (2.5) together give

$$\|\mathbf{u}\|_{1,q,B_1} \leq C \left(\|\mathbf{u}\|_{1,B_2 - B_1} + \|\mathbf{f}\|_{-1,q,B_2} \right).$$

The estimate for p is given by [14] p.181 Remark 1.3,

$$\inf_{c \in \mathbf{R}} \|p - c\|_{q,B_1} \leq C \|\mathbf{f}\|_{-1,q,B_1} + C \|\nabla \mathbf{u}\|_{q,B_1}.$$

The proof is complete. **Q.E.D.**

3 Spatial decay in unbounded domains

In this section we prove the

Theorem 3.1 *Let Ω be an unbounded domain ($\partial\Omega$ may not be compact) in $\mathbf{R}^n$, $n \geq 2$, and we denote $\delta(x) = dist(x, \partial\Omega)$. Let $k \geq 1$ be an integer. If $\mathbf{u}$ is a (weak) solution of the Navier-Stokes equations (1.1) with the decay*

$$\mathbf{u}(x) = O(\delta(x)^{-1}), \quad \nabla^m \mathbf{f}(x) = O(\delta(x)^{-3-m}) \qquad as\ \delta(x) \to \infty \tag{3.1}$$

for all $m < k$, then we have

$$\nabla^k \mathbf{u}(x) = O(\delta(x)^{-1-k}) \qquad as\ \delta(x) \to \infty. \tag{3.2}$$

Proof. For $x \in \Omega$ and $\lambda > 0$, let

$$\mathbf{u}_\lambda(y) = \lambda \mathbf{u}(\lambda y + x), \quad \mathbf{f}_\lambda(y) = \lambda^3 \mathbf{f}(\lambda y + x).$$

Then $\mathbf{u}_\lambda(y)$ and $\mathbf{f}_\lambda(y)$ are defined for $\{y : |y| < \delta(x)/\lambda\}$ and $\mathbf{u}_\lambda$ solves the Navier-Stokes equations with the body force $\mathbf{f}_\lambda$. In other words, $\mathbf{u}_\lambda$ solves the Stokes system with the body force $\tilde{\mathbf{f}}_\lambda = \mathbf{f}_\lambda - (\mathbf{u}_\lambda \cdot \nabla)\mathbf{u}_\lambda$. If we choose $\lambda = \delta(x)/2$ and consider $\mathbf{u}_\lambda$ in the unit ball $\{|y| < 1\}$, we find that $\{\mathbf{u}_\lambda\}$ and $\{\nabla^m \mathbf{f}_\lambda\}$, $m < k$, are uniformly bounded in $L^\infty(B_1)$, by virtue of (3.1). Hence $\{\tilde{\mathbf{f}}_\lambda\}$ are uniformly bounded in $W^{-1,q}(B_1)$ for any $q < \infty$. By Theorem 2.2 and by following the bootstrap argument for regularity, we know $\nabla^k \mathbf{u}_\lambda$ are also bounded uniformly in λ, (their dependence on the bounds in (3.1) are polynomials). Scaling back, we get (3.2). **Q.E.D.**

Remarks. (a) We did not specify $\mathbf{u}_*$ in Theorem 3.1 because for our purpose it is not important. (b) Although the theorem is true in any dimension $n \geq 3$, it is only interesting in the $n = 3$ case since 3 is the only dimension that the decay (3.1) holds for the fundamental solutions. (c) If Ω is an exterior domain, then $\delta(x) \sim |x|$ for $|x|$ large enough. This is the usual setting for the spatial decay problem. (d) In Theorem 3.1 we required that $|\nabla^m \mathbf{f}(x)| \cdot |x|^{3+m}$ be bounded, but not necessarily small.

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