Nontriviality of Rankin-Selberg L-functions and CM points.

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1 Introduction

1.1 Rankin-Selberg L-functions

Let π be an irreducible cuspidal automorphic representation of GL_2 over a totally real number field F. Let K be a totally imaginary quadratic extension of F. Given a quasi-character χ of $\mathbf{A}_K^\times/K^\times$, we denote by $L(\pi, \chi, s)$

the Rankin-Selberg L-function associated to π and $\pi(\chi)$, where $\pi(\chi)$ is the automorphic representation of GL_2 attached to χ – see [16] and [15] for the definitions. This L-function, which is first defined as a product of Euler factors over all places of F, is known to have a meromorphic extension to $\mathbf{C}$ with functional equation

$$L(\pi, \chi, s) = \epsilon(\pi, \chi, s)L(\tilde{\pi}, \chi^{-1}, 1-s)$$

where $\tilde{\pi}$ is the contragredient of π and $\epsilon(\pi, \chi, s)$ is a certain ϵ-factor.

Let $\omega : \mathbf{A}_F^\times/F^\times \to \mathbf{C}^\times$ be the central quasi-character of π. The condition

$$\chi \cdot \omega = 1 \quad \text{on} \quad \mathbf{A}_F^\times \subset \mathbf{A}_K^\times \tag{1}$$

implies that $L(\pi, \chi, s)$ is entire and equal to $L(\tilde{\pi}, \chi^{-1}, s)$. The functional equation thus becomes

$$L(\pi, \chi, s) = \epsilon(\pi, \chi, s)L(\pi, \chi, 1-s)$$

and the parity of the order of vanishing of $L(\pi, \chi, s)$ at $s = 1/2$ is determined by the value of

$$\epsilon(\pi, \chi) \stackrel{\text{def}}{=} \epsilon(\pi, \chi, 1/2) \in \{\pm 1\}.$$

We say that the pair (π, χ) is *even* or *odd*, depending upon whether $\epsilon(\pi, \chi)$ is $+1$ or -1. It is expected that the order of vanishing of $L(\pi, \chi, s)$ at $s = 1/2$ should 'usually' be minimal, meaning that either $L(\pi, \chi, 1/2)$ or $L'(\pi, \chi, 1/2)$ should be nonzero, depending upon whether (π, χ) is even or odd.

Calculation of sign

For the computation of $\epsilon(\pi, \chi)$, one first writes it as the product over all places v of F of the local signs $\epsilon(\pi_v, \chi_v)$ which are attached to the local components of π and χ, normalized as in [9, Section 9]. Let η be the quadratic Hecke character of F attached to K/F, and denote by η_v and ω_v the local components of η and ω. Then

$$\epsilon(\pi, \chi) = (-1)^{S(\chi)} \tag{2}$$

where

$$S(\chi) \stackrel{\text{def}}{=} \{v;\, \epsilon(\pi_v, \chi_v) \neq \eta_v \cdot \omega_v(-1)\}. \tag{3}$$

Indeed, $S(\chi)$ is finite because $\epsilon(\pi_v, \chi_v) = 1 = \eta_v \cdot \omega_v(-1)$ for all but finitely many v's, and (2) then follows from the product formula: $\eta \cdot \omega(-1) = 1 = \prod_v \eta_v \cdot \omega_v(-1)$.

The various formulae for the local ϵ-factors that are spread throughout [16] and [15] allow us to decide whether a given place v of F belongs to $S(\chi)$, provided that the local components π_v and $\pi(\chi_v)$ of π and $\pi(\chi)$ are not simultaneously supercuspidal. At the remaining places, one knows that χ ramifies and we may use a combination of [16, Proposition 3.8] and [15, Theorem 20.6] to conclude that, when χ is *sufficiently ramified* at v, v does not belong to $S(\chi)$. For our purposes, we just record the following facts.

For any finite place v of F in $S(\chi)$, K_v is a field and π_v is either special or supercuspidal. Conversely, if v is inert in K, χ is unramified at v and π_v is either special or supercuspidal, then v belongs to $S(\chi)$ if and only if the v-adic valuation of the conductor of π_v is odd. Finally an archimedean (real) place v of F belongs to $S(\chi)$ if $\chi_v = 1$ and π_v is the holomorphic discrete series of weight $k_v \geq 2$.

Ring class characters

In this paper, we regard the automorphic representation π as being fixed and let χ vary through the collection of ring class characters of P-power conductor, where P is a fixed maximal ideal in the ring of integers $\mathcal{O}_F \subset F$.

Here, we say that χ is a *ring class character* if there exists some $\mathcal{O}_F$-ideal $\mathcal{C}$ such that χ factors through the finite group

$$\mathbf{A}_K^\times / K^\times K_\infty^\times \widehat{\mathcal{O}}_{\mathcal{C}}^\times \simeq \mathrm{Pic}(\mathcal{O}_{\mathcal{C}})$$

where $K_\infty = K \otimes \mathbf{R}$ and $\mathcal{O}_{\mathcal{C}} \stackrel{\text{def}}{=} \mathcal{O}_F + \mathcal{C}\mathcal{O}_K$ is the $\mathcal{O}_F$-order of conductor $\mathcal{C}$ in K. The conductor $c(\chi)$ of χ is the largest such $\mathcal{C}$. Note that this definition differs from the classical one — the latter yields an ideal $c'(\chi)$ of $\mathcal{O}_K$ such that $c'(\chi) \mid c(\chi)\mathcal{O}_K$.

Equivalently, a ring class character is a finite order character whose restriction to $\mathbf{A}_F^\times$ is *everywhere unramified*. In view of (1), it thus make sense to require that ω is a finite order, everywhere unramified character of $\mathbf{A}_F^\times / F^\times$. Then, there are ring class characters of conductor P^n satisfying (1) for any sufficiently large n. Concerning our fixed representation π, we also require that

π is cuspidal of parallel weight $(2, \cdots, 2)$ and level $\mathcal{N}$, and the prime to P part $\mathcal{N}'$ of $\mathcal{N}$ is relatively prime to the discriminant $\mathcal{D}$ of K/F.

In this situation, we can give a fairly complete description of $S(\chi)$.

Lemma 1.1 *For a ring class character χ of conductor P^n, $S(\chi) = S$ or $S \cup \{P\}$, where S is the union of all archimedean places of F, together with those finite places of F which do not divide P, are inert in K, and divide $\mathcal{N}$ to an odd power. Moreover, $S(\chi) = S$ if either P does not divide $\mathcal{N}$, or P splits in K, or n is sufficiently large.*

Remark 1.2 Note that π_v is indeed special or supercuspidal for any finite place v of F which divides $\mathcal{N}$ to an odd power, as the conductor of a principal series representation with unramified central character is necessarily a square.

It follows that the sign of the functional equation essentially does not depend upon χ, in the sense that for all but finitely many ring class characters of P-power conductor,

$$\epsilon(\pi, \chi) = (-1)^{|S|} = (-1)^{[F:\mathbf{Q}]}\eta(\mathcal{N}').$$

If $P \nmid \mathcal{N}$ or splits in K, this formula even holds for all χ's. We say that the triple (π, K, P) is *definite* or *indefinite* depending upon whether this generic sign $(-1)^{[F:\mathbf{Q}]}\eta(\mathcal{N}')$ equals $+1$ or -1.

Exceptional cases

In the definite case, it might be that the L-function $L(\pi, \chi, s)$ actually factors as the product of two odd L-functions, and therefore vanishes to order at least 2. This leads us to what Mazur calls the *exceptional case.*

Definition 1.3 We say that (π, K) is exceptional if $\pi \simeq \pi \otimes \eta$.

This occurs precisely when $\pi \simeq \pi(\alpha)$ for some quasi-character α of $\mathbf{A}_K^\times/K^\times$, in which case $L(\pi, \chi, s) = L(\alpha\chi, s) \cdot L(\alpha\chi', s)$ where χ' is the outer twist of χ by $\mathrm{Gal}(K/F)$. Moreover, both factors have a functional equations with sign ± 1 and it can and does happen that both signs are -1, in which case $L(\pi, \chi, s)$ has at least a double zero. It is then more natural to study the individual factors than the product; this is the point of view taken in [20]. In this paper, we will always assume that (π, K) is not exceptional when (π, K, P) is definite.

Mazur's conjectures

With this convention, we now do expect, in the spirit of Mazur's conjectures in [17], that the order of vanishing of $L(\pi, \chi, 1/2)$ should generically be 0 in the definite case and 1 in the indefinite case. Let us say that χ is *generic* if it follows this pattern. We will show that there are many generic χ's of conductor P^n for all sufficiently large n.

More precisely, let $K[P^n]/K$ be the abelian extension of K associated by class field theory to the subgroup $K^\times K_\infty^\times \widehat{\mathcal{O}}_{P^n}^\times$ of $\mathbf{A}_K^\times$, so that

$$G(n) \stackrel{\text{def}}{=} \mathrm{Gal}(K[P^n]/K) \simeq \mathbf{A}_K^\times / K^\times K_\infty^\times \widehat{\mathcal{O}}_{P^n}^\times \simeq \mathrm{Pic}(\mathcal{O}_{P^n}).$$

Put $K[P^\infty] = \cup K[P^n]$, $G(\infty) = \mathrm{Gal}(K[P^\infty]/K) = \varprojlim G(n)$ and let G_0 be the torsion subgroup of $G(\infty)$. It is shown in section 2 below that G_0 is a finite group and $G(\infty)/G_0$ is a free $\mathbf{Z}_p$-module of rank $[F_P : \mathbf{Q}_p]$, where p is the residue characteristic of P. Moreover, the reciprocity map of K maps $\mathbf{A}_F^\times \subset \mathbf{A}_K^\times$ onto a subgroup $G_2 \simeq \mathrm{Pic}(\mathcal{O}_F)$ of G_0 (the missing group G_1 will make an appearance latter). Using this reciprocity map to identify ring class characters of P-power conductor with finite order characters of $G(\infty)$, and ω with a character of G_2, we see that the condition (1) on χ is equivalent to the requirement that $\chi \cdot \omega = 1$ on G_2.

Conversely, a character χ_0 of G_0 induces a character on $\mathbf{A}_F^\times$, and it make sense therefore to require that $\chi_0 \cdot \omega = 1$ on $\mathbf{A}_F^\times$. Given such a character, we denote by $P(n, \chi_0)$ the set of characters of $G(n)$ which induce χ_0 on G_0 and do not factor through $G(n-1)$ – these are just the ring class characters of conductor P^n which, beyond (1), satisfy the stronger requirement that $\chi = \chi_0$ on G_0.

Theorem 1.4 *Let the data of (π, K, P) be given and* definite. *Let χ_0 be any character of G_0 with $\chi_0 \cdot \omega = 1$ on $\mathbf{A}_F^\times$. Then for all n sufficiently large, there exists a character $\chi \in P(n, \chi_0)$ for which $L(\pi, \chi, 1/2) \neq 0$.*

For the indefinite case, we obtain a slightly more restrictive result.

Theorem 1.5 *Let the data of (π, K, P) be given and* indefinite*. Suppose also that $\omega = 1$, and that $\mathcal{N}$, $\mathcal{D}$ and P are pairwise coprime. Let χ_0 be any character of G_0 with $\chi_0 = 1$ on $\mathbf{A}_F^\times$. Then for all n sufficiently large, there exists a character $\chi \in P(n, \chi_0)$ for which $L'(\pi, \chi, 1/2) \neq 0$.*

We prove these theorems using Gross-Zagier formulae to reduce the nonvanishing of L-functions and their derivatives to the nontriviality of certain CM points. The extra assumptions in the indefinite case are due to the fact that these formulae are not yet known in full generality, although great progress has been made by Zhang [29, 28, 30] in extending the original work of Gross and Zagier. We prove the relevant statements about CM points without these restrictions.

1.2 Gross-Zagier Formulae

Roughly speaking, the general framework of a Gross-Zagier formula yields a discrete set of *CM points* on which the Galois group of the maximal abelian extension of K acts continuously, together with a function ψ on this set with values in a complex vector space such that the following property holds: a character χ as above is generic if and only if

$$\mathbf{a}(x, \chi) \stackrel{\text{def}}{=} \int_{\mathrm{Gal}_K^{\mathrm{ab}}} \chi(\sigma)\psi(\sigma \cdot x) d\sigma \neq 0 \tag{4}$$

where x is any CM point whose conductor equals that of χ – we will see that CM points have conductors. Note that the above integral is just a finite sum.

In the indefinite case, the relevant set of CM points consists of those special points with complex multiplication by K in a certain Shimura curve M defined over F, and ψ takes its values in (the complexification of) the Mordell-Weil groups of a suitable quotient A of $J = \mathrm{Pic}^0_{M/F}$. In the definite case, a finite set M plays the role of the Shimura curve. The CM points project onto this M and the function ψ is the composite of this projection with a suitable complex valued function on M.

Quaternion algebras

In both cases, these objects are associated to a quaternion algebra B over F whose isomorphism class is uniquely determined by π, K and P. To describe this isomorphism class, we just need to specify the set $\mathrm{Ram}(B)$ of places of F

where B ramifies. In the definite case, the set S of Lemma 1.1 has even order and we take $\mathrm{Ram}(B) = S$, so that B is totally definite. In the indefinite case, S is odd but it still contains all the archimedean (real) places of F. We fix arbitrarily a real place τ of F and take $\mathrm{Ram}(B) = S - \{\tau\}$.

We remark here that in both cases, B splits at P. Moreover, the Jacquet-Langlands correspondence implies that there is a unique cuspidal automorphic representation π' on B associated to $\pi = \mathrm{JL}(\pi')$, and π' occurs with multiplicity one in the space of automorphic cuspforms on B – this is the space denoted by $\mathcal{A}_0(G)$ in [27]. Finally, since K_v is a field for all v's in S, we may embed K into B as a maximal commutative F-subalgebra. We fix such an embedding.

Let $G \stackrel{\text{def}}{=} \mathrm{Res}_{F/\mathbf{Q}}(B^\times)$ be the algebraic group over $\mathbf{Q}$ whose set of points on a commutative $\mathbf{Q}$-algebra A is given by $G(A) = (B \otimes A)^\times$. Thus, G is a reductive group with center $Z \stackrel{\text{def}}{=} \mathrm{Res}_{F/\mathbf{Q}}(F^\times)$ and the reduced norm $\mathrm{nr} : B \to F$ induces a morphism $\mathrm{nr} : G \to Z$ which also identifies Z with the cocenter $G/[G,G]$ of G. Our chosen embedding $K \hookrightarrow B$ allows us to view $T \stackrel{\text{def}}{=} \mathrm{Res}_{F/\mathbf{Q}}(K^\times)$ as a maximal subtorus of G which is defined over $\mathbf{Q}$.

CM points.

For any compact open subgroup H of $G(\mathbf{A}_f)$, we define a set of CM points by

$$\mathrm{CM}_H \stackrel{\text{def}}{=} T(\mathbf{Q})\backslash G(\mathbf{A}_f)/H.$$

There is an action of $T(\mathbf{A}_f)$ on CM_H, given by left multiplication in $G(\mathbf{A}_f)$. This action factors through the reciprocity map

$$\mathrm{rec}_K : T(\mathbf{A}_f) \twoheadrightarrow \mathrm{Gal}_K^{\mathrm{ab}}$$

and thus defines a Galois action on CM_H. For $x = [g]$ in CM_H (with g in $G(\mathbf{A}_f)$), the stabilizer of x in $T(\mathbf{A}_f)$ equals

$$U(x) \stackrel{\text{def}}{=} T(\mathbf{Q}) \cdot (T(\mathbf{A}_f) \cap gHg^{-1})$$

and we say that x is *defined* over the abelian extension of K which is fixed by $\mathrm{rec}_K(U(x))$. When $H = \widehat{R}^\times$ for some $\mathcal{O}_F$-order $R \subset B$, $T(\mathbf{A}_f) \cap gHg^{-1} = \widehat{\mathcal{O}(x)}^\times$ for some $\mathcal{O}_F$-order $\mathcal{O}(x) \subset K$, and we define the conductor of x to be that of $\mathcal{O}(x)$. In particular, a CM point of conductor P^n is defined over $K[P^n]$.

We shall also need a some what more technical notion, namely that of a *good* CM point.

Definition 1.6 Assume therefore that $H = \widehat{R}^\times$ as above, and that the P-component of R is an Eichler order of level P^δ in $B_P \simeq M_2(F_P)$. Then R is uniquely expressed as the (unordered) intersection of two $\mathcal{O}_F$-orders R_1 and R_2 in B, which are both maximal at P but agree with R outside P. We say that a CM point $x = [g] \in \mathrm{CM}_H$ is *good* if either $\delta = 0$ or $K_P \cap g_P R_1 g_P^{-1} \neq K_P \cap g_P R_2 g_P^{-1}$, and we say that x is *bad* otherwise.

It is relatively easy to check that if CM_H contains any CM point of P-power conductor, then it contains good CM points of conductor P^n for all sufficiently large n.

Automorphic forms

Let $\mathcal{S}$ denote the space of automorphic forms on B in the definite case, and the space of automorphic cuspforms on B in the indefinite case. As a first step towards the construction of the function ψ of (4), we shall now specify a certain line $\mathbf{C} \cdot \Phi$ in the realization $\mathcal{S}(\pi')$ of π' in $\mathcal{S}$. We will first define an admissible $G(\mathbf{A}_f)$-submodule $\mathcal{S}_2$ of $\mathcal{S}$, using the local behavior of π' at infinity. The line we seek then consists of those vectors in $\mathcal{S}_2(\pi') = \mathcal{S}_2 \cap \mathcal{S}(\pi')$ which are fixed by a suitable compact open subgroup H of $G(\mathbf{A}_f)$. We refer to [8] for a more comprehensive discussion of these issues.

Recall from [16] that $\mathcal{S}$ and π' are representations of $G(\mathbf{A}_f) \times \mathcal{H}_\infty$ where $\mathcal{H}_\infty$ is a certain sort of group algebra associated to $G(\mathbf{R})$. As a representation of $\mathcal{H}_\infty$, π' is the direct sum of copies of the irreducible representation $\pi'_\infty = \otimes_{v|\infty} \pi'_v$ of $\mathcal{H}_\infty$. Let V_∞ be the representation space of π'_∞. We claim that V_∞ is one dimensional in the definite case, while V_∞ has a "weight decomposition"

$$V_\infty = \oplus_{k \in 2\mathbf{Z} - \{0\}} V_{\infty,k} \tag{5}$$

into one dimensional subspaces in the indefinite case. Indeed, the compatibility of the global and local Jacquet-Langlands correspondence, together with our assumptions on $\pi = \mathrm{JL}(\pi')$, implies that for a real place v of F, π'_v is the trivial one dimensional representation of $B_v^\times$ if v ramifies in B, while for $v = \tau$ in the indefinite case, $\pi'_v \simeq \pi_v$ is the holomorphic discrete series of weight 2 which is denoted by σ_2 in [6, section 11.3], and the representation space of σ_2 is known to have a weight decomposition similar to (5).

Remark 1.7 In the indefinite case, the above decomposition is relative to the choice of an isomorphism between B_τ and $M_2(\mathbf{R})$. Given such an isomorphism, the subspace $V_{\infty,k}$ consists of those vectors in V_∞ on which $SO_2(\mathbf{R})$ acts by the character

$$\left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right) \mapsto e^{2ki\theta}.$$

Definition 1.8 We denote by $\mathcal{S}_2$ the admissible $G(\mathbf{A}_f)$-submodule of $\mathcal{S}$ which is the image of the $G(\mathbf{A}_f)$-equivariant morphism

$$\mathrm{Hom}_{\mathcal{H}_\infty}(V_\infty, \mathcal{S}) \hookrightarrow \mathcal{S} : \quad \varphi \mapsto \varphi(v_\infty)$$

where v_∞ is any nonzero element of V_∞ in the definite case, and any nonzero element of $V_{\infty,2}$ (a *lowest weight* vector) in the indefinite case. By construction, the $G(\mathbf{A}_f)$-submodule $\mathcal{S}_2(\pi') = \mathcal{S}_2 \cap \mathcal{S}(\pi')$ of $\mathcal{S}_2$ is isomorphic to $\mathrm{Hom}_{\mathcal{H}_\infty}(V_\infty, \mathcal{S}(\pi'))$. It is therefore irreducible.

Level subgroups

Turning now to the construction of H, let δ be the exponent of P in $\mathcal{N}$, so that $\mathcal{N} = P^\delta \mathcal{N}'$. Let $R_0 \subset B$ be an Eichler order of level P^δ such that the conductor of the $\mathcal{O}_F$-order $\mathcal{O} = \mathcal{O}_K \cap R_0$ is a power of P. The existence of R_0 is given by [26, II.3], and we may even require that $\mathcal{O} = \mathcal{O}_K$ if P does not divide $\mathcal{N}$ or splits in K. On the other hand, recall that the reduced discriminant of B/F is the squarefree product of those primes of F which are inert in K and divide $\mathcal{N}'$ to an odd power. We may thus find an ideal $\mathcal{M}$ in $\mathcal{O}_K$ such that

$$\mathrm{Norm}_{K/F}(\mathcal{M}) \cdot \mathrm{Disc}_{B/F} = \mathcal{N}'.$$

We then take

$$H \overset{\text{def}}{=} \widehat{R}^\times \quad \text{where} \quad R \overset{\text{def}}{=} \mathcal{O} + \mathcal{M} \cap \mathcal{O} \cdot R_0. \tag{6}$$

Note that R is an $\mathcal{O}_F$-order of reduced discriminant $\mathcal{N}$ in B. Since $R_P = R_{0,P}$ is an Eichler order (of level P^δ), we have the notion of good and bad CM points on CM_H. Since $x = [1]$ is a CM point of P-power conductor (with $\mathcal{O}(x) = \mathcal{O}$), there are good CM points of conductor P^n for all sufficiently large n.

We claim that $\mathcal{S}_2(\pi')^H$ is 1-dimensional. Indeed, for every finite place v of F, $(\pi'_v)^{R_v^\times}$ is 1-dimensional: this follows from [3, Theorem 1] when v does not divide $\mathcal{N}'$ (including $v = P$) and from [8, Proposition 6.4], or a mild generalization of [29, Theorem 3.2.2] in the remaining cases.

1.3 The indefinite case

Suppose first that (π, K, P) is indefinite, so that

$$B \otimes \mathbf{R} = \prod_{v|\infty} B_v \simeq M_2(\mathbf{R}) \times \mathbf{H}^{[F:\mathbf{Q}]-1}$$

where $\mathbf{H}$ is Hamilton's quaternion algebra and the $M_2(\mathbf{R})$ factor corresponds to $v = \tau$. We fix such an isomorphism, thus obtaining an action of

$$G(\mathbf{R}) \simeq GL_2(\mathbf{R}) \times (\mathbf{H}^\times)^{[F:\mathbf{Q}]-1}$$

on $X \overset{\text{def}}{=} \mathbf{C} - \mathbf{R}$ by combining the first projection with the usual action of $GL_2(\mathbf{R})$ on X.

For any compact open subgroup H of $G(\mathbf{A}_f)$, we then have a Shimura curve $\mathrm{Sh}_H(G, X)$ whose complex points are given by

$$\mathrm{Sh}_H(G, X)(\mathbf{C}) = G(\mathbf{Q}) \backslash (G(\mathbf{A}_f)/H \times X) .$$

The reflex field of this curve is the subfield $\tau(F)$ of $\mathbf{C}$, and its pull-back to F is a smooth curve M_H over F whose isomorphism class does not depend upon our choice of τ. When $S = \emptyset$, $F = \mathbf{Q}$, $G = GL_2$ and the M_H's are the classical (affine) modular curves over $\mathbf{Q}$. These curves can be compactified by adding finitely many cusps, and we denote by M_H^* the resulting proper curves. In all other cases, M_H is already proper over F and we put $M_H^* = M_H$. We denote by J_H the connected component of the relative Picard scheme of M_H^*/F.

Let x be the unique fixed point of $T(\mathbf{R})$ in the upper half plane $X^+ \subset X$. The map $g \mapsto (g, x)$ then defines a bijection between CM_H and the set of *special points with complex multiplication by* K in M_H. It follows from Shimura's theory that these points are defined over the maximal abelian extension K^{ab} of K, and that the above bijection is equivariant with respect to the Galois actions on both sides.

On the other hand, there is a natural $G(\mathbf{A}_f)$-equivariant isomorphism between the subspace $\mathcal{S}_2$ of $\mathcal{S}$ and the inductive limit (over H) of the spaces of holomorphic differentials on M_H^*. This is well-known in the classical case where $S = \emptyset$ – see for instance [6], section 11 and 12. For the general case, we sketch a proof in section 3.6 of this paper.

In particular, specializing now to the level structure H defined by (6), we obtain a line $\mathcal{S}_2(\pi')^H = \mathbf{C} \cdot \Phi$ in the space $\mathcal{S}_2^H$ of holomorphic differentials on

M_H^*, a space isomorphic to the cotangent space of $J_H/\mathbf{C}$ at 0. By construction, this line is an eigenspace for the action of the universal Hecke algebra $\mathbf{T}_H$, with coefficients in $\mathbf{Z}$, which is associated to our H. Since the action of $\mathbf{T}_H$ on the cotangent space factors through $\mathrm{End}_F J_H$, the annihilator of $\mathbf{C}\cdot\Phi$ in $\mathbf{T}_H$ cuts out a quotient A of J_H:

$$A \stackrel{\text{def}}{=} J_H/\mathrm{Ann}_{\mathbf{T}_H}(\mathbf{C}\cdot\Phi)\cdot J_H.$$

The Zeta function of A is essentially the product of the L-function of π together with certain conjugates – see [29, Theorem B] for a special case.

The function ψ of (4) is now the composite of

- the natural inclusions $\mathrm{CM}_H \hookrightarrow M_H \hookrightarrow M_H^*$,
- a certain morphism $\iota_H \in \mathrm{Mor}(M_H^*, J_H)\otimes\mathbf{Q}$, and
- the quotient map $J_H \twoheadrightarrow A$.

In the classical case where $S=\emptyset$, ι_H is a genuine morphism $M_H^* \to J_H$ which is defined using the cusp at ∞ on M_H^*. In the general case, one has to use the so-called *Hodge class*. For a discussion of the Hodge class, we refer to [30, section 6], or [9, section 23]. A variant of this construction, adapted to out purposes, is given in section 3.5 below.

Statement of results

Now, let χ be a ring class character of conductor P^n such that $\chi\cdot\omega = 1$ on $\mathbf{A}_F^\times$. Suppose also that $\epsilon(\pi,\chi)=-1$: this holds true for any $n\geq 0$ if $P\nmid\mathcal{N}$ or P splits in K, but only for $n\gg 0$ in the general case. Then $L(\pi,\chi,1/2)=0$ and the Birch and Swinnerton-Dyer conjecture predicts that the χ^{-1}-component of $A(K[P^n])\otimes\mathbf{C}$ should be non-trivial. If moreover $L'(\pi,\chi,1/2)\neq 0$, the Gross-Zagier philosophy tells us more, namely that this non-triviality should be accounted for by the CM points of conductor P^n: if x is such a point, there should exists a formula relating $L'(\pi,\chi,1/2)$ to the canonical height of

$$\mathbf{a}(x,\chi) \stackrel{\text{def}}{=} \frac{1}{|G(n)|}\sum_{\sigma\in G(n)} \chi(\sigma)\psi(\sigma x) \in A(K[P^n])\otimes\mathbf{C},$$

thereby showing that $L'(\pi,\chi,1/2)$ is nonzero precisely when $\mathbf{a}(x,\chi)$ is a nonzero element in the χ^{-1}-component of $A(K[P^n])\otimes\mathbf{C}$.

Unfortunately, such a formula has not yet been proven in this degree of generality. For our purposes, the most general case of which we are aware is Theorem 6.1 of Zhang's paper [30], which gives a precise formula of this type under the hypotheses that the central character of π is trivial and that $\mathcal{N}$, $\mathcal{D}$ and P are pairwise prime to each other.

Remark 1.9 We point out that Zhang works with the Shimura curves attached to G/Z instead of G, and uses $\mathbf{a}(x, \chi^{-1})$ instead of $\mathbf{a}(x, \chi)$. The first distinction is not a real issue, and the second is irrelevant, as long as we are restricting our attention to the *anticyclotomic* situation where $\chi = \omega = 1$ on $\mathbf{A}_F^\times$. Indeed, χ^{-1} is then equal to the outer twist of χ by $\mathrm{Gal}(K/F)$, so that $L(\pi, \chi, s) = L(\pi, \chi^{-1}, s)$ and any lift of the non-trivial element of $\mathrm{Gal}(K/F)$ to $\mathrm{Gal}(K[P^n]/F)$ interchanges the eigenspaces for χ and χ^{-1} in $A(K[P^n]) \otimes \mathbf{C}$.

One has to be more careful when χ is non-trivial on $\mathbf{A}_F^\times$. To be consistent with the BSD conjecture, a Gross-Zagier formula should relate $L'(\pi, \chi, 1/2)$ to a point in the χ^{-1}-component of $A(K[P^n]) \otimes \mathbf{C}$.

In any case, Zhang's Gross-Zagier formula implies that Theorem 1.5 is now a consequence of the following result, which itself is a special case of Theorem 4.1 in the text.

Theorem 1.10 *Let χ_0 be any character of G_0 such that $\chi_0 \cdot \omega = 1$ on $\mathbf{A}_F^\times$. Then, for any good CM point x of conductor P^n with n sufficiently large, there exists a character $\chi \in P(n, \chi_0)$ such that $\mathbf{a}(x, \chi) \neq 0$.*

1.4 The definite case

Suppose now that the triple (π, K, P) is *definite*, so that π'_∞ is the trivial 1-dimensional representation of

$$G(\mathbf{R}) = \prod_{v|\infty} B_v^\times \simeq (\mathbf{H}^\times)^{[F:\mathbf{Q}]}.$$

Then $\mathcal{S}(\pi')$ is contained in $\mathcal{S}_2$, and the latter is simply the subspace of $\mathcal{S}$ on which $G(\mathbf{R})$ acts trivially; this is the space of all smooth functions

$$\phi : G(\mathbf{Q})\backslash G(\mathbf{A})/G(\mathbf{R}) = G(\mathbf{Q})\backslash G(\mathbf{A}_f) \longrightarrow \mathbf{C},$$

with $G(\mathbf{A}_f)$ acting by right translation. Note that the $G(\mathbf{A}_f)$-module underlying $S(\pi') = \mathcal{S}_2(\pi')$ is admissible, infinite dimensional and irreducible; it contains no nonzero function which factors through the reduced norm, because any such function spans a finite dimensional $G(\mathbf{A}_f)$-invariant subspace.

For any compact open subgroup H of $G(\mathbf{A}_f)$, we may identify $\mathcal{S}_2^H$ with the set of complex valued functions on the finite set

$$M_H \stackrel{\text{def}}{=} G(\mathbf{Q})\backslash G(\mathbf{A}_f)/H,$$

and any such function may be evaluated on $\mathrm{CM}_H = T(\mathbf{Q})\backslash G(\mathbf{A}_f)/H$.

Specializing now to the H which is defined by (6), let ψ be the function induced on CM_H by some nonzero element Φ in the 1-dimensional space $\mathcal{S}_2(\pi')^H = \mathcal{S}(\pi')^H = \mathbf{C} \cdot \Phi$:

$$\psi : \mathrm{CM}_H \to M_H \xrightarrow{\Phi} \mathbf{C}.$$

For a ring class character χ of conductor P^n such that $\chi \cdot \omega = 1$ on $\mathbf{A}_F^\times$, the Gross-Zagier philosophy predicts that there should exist a formula relating $L(\pi, \chi, 1/2)$ to $|\mathbf{a}(x, \chi)|^2$, for some CM point $x \in \mathrm{CM}_H$ of conductor P^n, with

$$\mathbf{a}(x, \chi) \stackrel{\text{def}}{=} \frac{1}{|G(n)|} \sum_{\sigma \in G(n)} \chi(\sigma)\psi(\sigma \cdot x) \in \mathbf{C}.$$

Such a formula has indeed been proven by Zhang [30, Theorem 7.1], under the assumption that $\omega = 1$, and that $\mathcal{N}$, $\mathcal{D}$ and P are pairwise coprime. On the other hand, there is a more general theorem of Waldspurger which, although it does not give a precise *formula* for the central value of $L(\pi, \chi, s)$, still gives a criterion for its non-vanishing.

Statement of results

Thus, let χ be any character of $T(\mathbf{Q})\backslash T(\mathbf{A})$ such that $\chi \cdot \omega = 1$ on $\mathbf{A}_F^\times$. Such a character yields a linear form ℓ_χ on $\mathcal{S}(\pi')$, defined by

$$\ell_\chi(\phi) \stackrel{\text{def}}{=} \int_{Z(\mathbf{A})T(\mathbf{Q})\backslash T(\mathbf{A})} \chi(t)\phi(t)dt$$

where dt is any choice of Haar measure on $T(\mathbf{A})$. By a fundamental theorem of Waldspurger [27, Théorème 2], this linear form is nonzero on $\mathcal{S}(\pi')$ if and only if $L(\pi, \chi, 1/2) \neq 0$ and certain local conditions are satisfied. The results of Tunell and Saito which are summarized in [9, Section 10] show that these local conditions are satisfied if and only if the set $S(\chi)$ of (3) is equal to the set S of places where B ramifies. For a ring class character χ of P-power conductor, Lemma 1.1 shows that $S(\chi) = S$ if and only if (π, χ) is even. We thus obtain the following simple criterion.

Theorem 1.11 *[Waldspurger] For a ring class character χ of P-power conductor such that $\chi \cdot \omega = 1$ on $\mathbf{A}_F^\times$,*

$$L(\pi, \chi, 1/2) \neq 0 \quad \Leftrightarrow \quad \exists\, \phi \in \mathcal{S}(\pi') : \ \ell_\chi(\phi) \neq 0.$$

Remark 1.12 Waldspurger's theorem does not give a precise formula for the *value* of $L(\pi, \chi, 1/2)$, and it does not specify a canonical choice of ϕ (a *test vector* in the language of [10]) on which to evaluate the linear functional ℓ_χ. The problem of finding such a test vector ϕ and a Gross-Zagier formula relating $\ell_\chi(\phi)$ to $L(\pi, \chi, 1/2)$ is described in great generality in [9], and explicit formulae are proven in [7] (for $F = \mathbf{Q}$) and for a general F in [28, 30], under various assumptions. A leisurely survey of this circle of ideas may be found in [25].

Recall that ψ is the function which is induced on CM_H by some nonzero Φ in $\mathcal{S}(\pi')^H$. For $\phi = g \cdot \Phi \in \mathcal{S}(\pi')$, with $g \in G(\mathbf{A}_f)$ corresponding to a CM point $x = [g] \in \mathrm{CM}_H$ whose conductor P^n equals that of χ, we find that, up to a nonzero constant,

$$\ell_\chi(\phi) \sim \mathbf{a}(x, \chi).$$

Theorem 1.4 therefore is a consequence of the following result, which itself is a special case of Theorem 5.10 in the text.

Theorem 1.13 *Let χ_0 be any character of G_0 such that $\chi_0 \cdot \omega = 1$ on $\mathbf{A}_F^\times$. Then, for any good CM point x of conductor P^n with n sufficiently large, there exists a character $\chi \in P(n, \chi_0)$ such that $\mathbf{a}(x, \chi) \neq 0$.*

1.5 Applications

As we have already explained, the present work was firstly motivated by a desire to prove non-vanishing of L-functions and their derivatives. However, the results we prove on general CM points have independent applications to Iwasawa theory, even when they are not yet known to be related to L-functions.

For instance, Theorem 1.10 implies directly that certain Euler systems of Bertolini-Darmon (when $F = \mathbf{Q}$) and Howard (for general F) are actually non-trivial. The non-triviality of these Euler systems is used by B. Howard in [14] to establish half of the relevant Main Conjecture, for the anti-cyclotomic Iwasawa theory of abelian varieties of GL(2)-type. It is also used by J. Nekovář in [18] to prove new cases of parity in the Bloch-Kato conjecture, for Galois representations attached to Hilbert modular newforms over F with trivial central character and parallel weight $(2k, \cdots, 2k)$, $k \geq 1$.

1.6 Sketch of proof

We want to briefly sketch the proof of our nontriviality theorems for CM points. The basic ideas are drawn from our previous papers [4, 23, 24] with a few simplifications and generalizations.

Thus, let χ_0 be a character of G_0 such that $\chi_0 \cdot \omega = 1$ on $\mathbf{A}_F^\times$, let $H = \widehat{R}^\times$ be the compact open subgroup of $G(\mathbf{A}_f)$ defined by (6), and let x be a CM point of conductor P^n in CM_H. We have defined a function ψ on CM_H with values in a complex vector space, and we want to show that

$$\mathbf{a}(x, \chi) \stackrel{\text{def}}{=} \frac{1}{|G(n)|} \sum_{\sigma \in G(n)} \chi(\sigma) \psi(\sigma \cdot x)$$

is nonzero for at least some $\chi \in P(n, \chi_0)$, provided that n is sufficiently large. The analysis of such sums proceeds in a series of reductions.

From $G(n)$ to G_0

To prove that $\mathbf{a}(x, \chi) \neq 0$ for some $\chi \in P(n, \chi_0)$, it suffices to show that the *sum* of these values is nonzero. A formal computation in the group algebra of $G(n)$ shows that this sum,

$$\mathbf{b}(x, \chi_0) \stackrel{\text{def}}{=} \sum_{\chi \in P(n, \chi_0)} \mathbf{a}(x, \chi)$$

is given by

$$\mathbf{b}(x, \chi_0) = \frac{1}{q\,|G_0|} \sum_{\sigma \in G_0} \chi_0(\sigma) \psi_* (\sigma \cdot \tilde{x})$$

where ψ_* is the extension of ψ to $\mathbf{Z}[\mathrm{CM}_H]$ and

$$\tilde{x} = q \cdot x - \mathrm{Tr}_{Z(n)}(x) = \sum_{\sigma \in Z(n)} x - \sigma \cdot x.$$

Here, $Z(n) = \mathrm{Gal}(K[P^n]/K[P^{n-1}])$ and $q = |Z(n)| = |\mathcal{O}_F/P|$.

Distribution relations

To deal with $\tilde{x}$, we have to use distribution relations and Hecke correspondences, much as in the case of $F = \mathbf{Q}$ treated in our previous works. However, there are numerous technicalities to overcome, owing to the fact that we are now working over a more general field, with automorphic forms that may have a nontrivial central character, and with a prime P that may divide the level. Although the necessary arguments are ultimately quite simple, the details are somewhat tedious, and we request forgiveness for what might seem to be a rather opaque digression. To avoid obscuring the main lines of the argument, we have banished the discussion of distribution relations to the appendices.

Basically, these distribution relations will produce for us a level structure $H^+ \subset H$, a function ψ^+ on CM_{H^+} and a CM point $x^+ \in \mathrm{CM}_{H^+}$ of conductor P^n such that

$$\forall \sigma \in \mathrm{Gal}_K^{\mathrm{ab}}: \quad \psi^+(\sigma \cdot x^+) = \psi_*(\sigma \cdot \tilde{x}).$$

In fact, $H^+ = \widehat{R^+}^\times$ for some $\mathcal{O}_F$-order $R^+ \subset B$ which agrees with R outside P, and is an Eichler order of level $P^{\max(\delta,2)}$ at P.

Note that this part of the proof is responsible for the goodness assumptions in our theorems. Indeed, the *bad* CM points simply do not seem to satisfy any distribution relations, and the above construction may therefore only be applied to a *good* CM point x. We also mention that this computation would not work with a more general function ψ: one needs ψ to be *new* at P, in some suitable sense.

From G_0 to G_0/G_2

We now have

$$\mathbf{b}(x, \chi_0) = \frac{1}{|G_0|} \sum_{\sigma \in G_0} \chi_0(\sigma) \psi^+(\sigma \cdot x^+).$$

Using the fact that $\chi_0 \cdot \omega = 1$ on G_2, we prove that

$$\mathbf{b}(x, \chi_0) = \frac{1}{|G_0/G_2|} \sum_{\sigma \in G_0/G_2} \chi_0(\sigma)\psi^+(\sigma \cdot x^+).$$

Indeed, the map $\sigma \mapsto \chi_0(\sigma)\psi^+(\sigma \cdot x^+)$ factors through G_0/G_2.

From G_0/G_2 to G_0/G_1

We can reduce the above sum to something even simpler. Indeed, it turns out that there is a subgroup $G_1 \subset G_0$, containing G_2, such that the *Galois* action of the elements in G_1 can be realized by *geometric* means. In fact, G_1 is the maximal such subgroup, and G_1/G_2 is generated by the classes of the frobeniuses in $G(\infty)$ of those primes of K which are ramified over F but do not divide P.

More precisely, we construct yet another level structure $H_1^+ \subset H^+$, a function ψ_1^+ on $\mathrm{CM}_{H_1^+}$, and a CM point $x_1^+ \in \mathrm{CM}_{H_1^+}$ of conductor P^n, such that

$$\forall \gamma \in \mathrm{Gal}_K^{\mathrm{ab}} : \quad \psi_1^+(\gamma \cdot x_1^+) = \sum_{\sigma \in G_1/G_2} \chi_0(\sigma)\psi^+(\sigma\gamma \cdot x^+).$$

This H_1^+ corresponds to an $\mathcal{O}_F$-order $R_1^+ \subset B$ which only differs from R^+ at those finite places $v \neq P$ of F which ramify in K.

This part of the proof is responsible for our general assumption that $\mathcal{N}'$ and $\mathcal{D}$ are relatively prime. Indeed, to establish the above formula, we need to know that for all $v \neq P$ that ramify in K, the local component $H_v^+ = H_v = R_v^\times$ of H^+ is a *maximal* order in a *split* quaternion algebra. It seems likely that the case where R_v is an Eichler order of level v in a split algebra could still be handled by similar methods.

Dealing with G_0/G_1

We finally obtain

$$\mathbf{b}(x, \chi_0) = \frac{1}{|G_0/G_2|}\mathbf{c}(x_1^+) \quad \text{with} \quad \mathbf{c}(y) = \sum_{\sigma \in G_0/G_1} \chi_0(\sigma)\psi_1^+(\sigma \cdot y).$$

We prove that $\mathbf{c}(y) \neq 0$ for sufficiently many y's in $\mathrm{CM}_{H_1^+}(P^n)$, $n \gg 0$, using a theorem of M. Ratner on uniform distribution of unipotent orbits on p-adic Lie groups. Just as in our previous work, we show that the elements of

G_0/G_1 act *irrationally* on the relevant CM points. Slightly more precisely, we prove that for y as above, the images $\bar{\mathbf{r}}(z)$ of vectors of the form

$$\mathbf{r}(z) = (\sigma \cdot z)_{\sigma \in G_0/G_1} \in \mathrm{CM}_{H_1^+}(P^n)^{G_0/G_1}$$

are uniformly distributed in some appropriate space, as z runs through the Galois orbit of y, and n goes to infinity. In the indefinite case, $\bar{\mathbf{r}}(z)$ is the vector of supersingular points in characteristic ℓ which is obtained by reducing the coordinates of $\mathbf{r}(z)$ at some suitable place of $K[P^\infty]$. In the definite case, M_H itself plays the role of the supersingular locus. The uniform distribution theorem implies that the image of the Galois orbit of y tends to be large, and it easily follows that $\mathbf{c}(y)$ is nonzero.

We have chosen to present a more general variant of the uniform distribution property alluded to above in a separate paper [5], which is quoted here in propositions 4.17 and 5.6 (for respectively the indefinite and the definite case). Although it really is the kernel of our proof, one may consider [5] as a black box while reading this paper.

1.7 Notations.

For any place v of F, F_v is the completion of F at v and $\mathcal{O}_{F,v}$ is its ring of integers (if v is finite). If E is a vector field over F, such as K or B, we put $E_v = E \otimes_F F_v$. More generally, if R is a module over the ring of integers $\mathcal{O}_F$ of F, we put $R_v = R \otimes_{\mathcal{O}_F} \mathcal{O}_{F,v}$. We denote by $\mathbf{A}$ (resp. $\mathbf{A}_f$) the ring of adeles (resp. finite adeles) of $\mathbf{Q}$, so that $\mathbf{A} = \mathbf{A}_f \times \mathbf{R}$, and $\mathbf{A}_f$ is the restricted product of the F_v's with respect to the $\mathcal{O}_{F,v}$'s. We put $\mathbf{A}_F = \mathbf{A} \otimes_{\mathbf{Q}} F$ and $\mathbf{A}_K = \mathbf{A} \otimes_{\mathbf{Q}} K$. For the finite adeles, we write $\widehat{F} = \mathbf{A}_f \otimes_{\mathbf{Q}} F$ and $\widehat{K} = \mathbf{A}_f \otimes_{\mathbf{Q}} K$. Thus $\widehat{F} = \widehat{\mathcal{O}}_F \otimes \mathbf{Q}$ where $\widehat{M} = M \otimes \widehat{\mathbf{Z}}$ denotes the profinite completion of a finite generated $\mathbf{Z}$-module M. For any affine algebraic group $G/\mathbf{Q}$, we topologize $G(\mathbf{A}) = G(\mathbf{A}_f) \times G(\mathbf{R})$ in the usual way. When G is the Weil restriction $G = \mathrm{Res}_{F/\mathbf{Q}} G'$ of an algebraic group G'/F, we denote by $g_v \in G'(F_v)$ the v-component of $g \in G(\mathbf{A}) = G'(\mathbf{A}_F)$ (or $G(\mathbf{A}_f) = G'(\widehat{F}^\times)$), and we identify $G'(F_v)$ with the subgroup $\{g \in G(\mathbf{A}); \forall w \neq v, g_w = 1\}$ of $G(\mathbf{A})$ (or $G(\mathbf{A}_f)$).

We put $\mathrm{Gal}_F = \mathrm{Gal}(\overline{F}/F)$ and $\mathrm{Gal}_K = \mathrm{Gal}(\overline{F}/K)$ where $\overline{F}$ is a fixed algebraic closure of F containing K. We denote by F^{ab} and K^{ab} the maximal abelian extensions of F and K inside $\overline{F}$, with Galois groups $\mathrm{Gal}_F^{\mathrm{ab}}$ and $\mathrm{Gal}_K^{\mathrm{ab}}$. We denote by Frob_v the *geometric* Frobenius at v (the inverse of $x \mapsto x^{N(v)}$)

and normalize the reciprocity map

$$\mathrm{rec}_F : \mathbf{A}_F^\times \to \mathrm{Gal}_F^{\mathrm{ab}} \quad \text{and} \quad \mathrm{rec}_K : \mathbf{A}_K^\times \to \mathrm{Gal}_K^{\mathrm{ab}}$$

accordingly.

2 The Galois group of $K[P^\infty]/K$

Fix a prime P of F with residue field $\mathbf{F} = \mathcal{O}_F/P$ of characteristic p and order $q = |\mathbf{F}|$. We have assembled here the basic facts we need pertaining to the infinite abelian extension $K[P^\infty] = \cup_{n\geq 0} K[P^n]$ of K. Recall that

$$G(n) = \mathrm{Gal}(K[P^n]/K) \quad \text{and} \quad G(\infty) = \mathrm{Gal}(K[P^\infty]/K) = \varprojlim G(n).$$

The first section describes $G(\infty)$ as a topological group: it is an extension of a free $\mathbf{Z}_p$-module of rank $[F_P : \mathbf{Q}_p]$ by a finite group G_0, the torsion subgroup of $G(\infty)$. The second section defines a filtration

$$\{1\} \subset G_2 \subset G_1 \subset G_0$$

which plays a crucial role in the proof (and statement) of our main results. Finally, the third section gives an explicit formula for a certain idempotent in the group algebra of $G(n)$.

2.1 The structure of $G(\infty)$

Lemma 2.1 *The reciprocity map induces an isomorphism of topological groups between $\widehat{K}^\times/K^\times U$ and $G(\infty)$ where*

$$U = \cap \widehat{\mathcal{O}}_{P^n}^\times = \{\lambda \in \widehat{\mathcal{O}}_K^\times,\ \lambda_P \in \mathcal{O}_{F,P}^\times\}.$$

Proof. We have to show that the natural continuous map

$$\phi : \widehat{K}^\times/K^\times U \to \varprojlim \widehat{K}^\times/K^\times \widehat{\mathcal{O}}_{P^n}^\times$$

is an isomorphism of topological groups. Put

$$X_n = K^\times \widehat{\mathcal{O}}_{P^n}^\times / K^\times U \simeq \mathcal{O}_{P^n,P}^\times / \mathcal{O}_{P^n}^\times \mathcal{O}_{F,P}^\times$$

so that $\ker(\phi) = \varprojlim X_n$ and $\mathrm{coker}(\phi) = \varprojlim^{(1)} X_n$. Note that $(\mathcal{O}_{P^n}^\times)_{n\geq 0}$ is a decreasing sequence of subgroups of $\mathcal{O}_K^\times$ with $\cap_{n\geq 0} \mathcal{O}_{P^n}^\times = \mathcal{O}_F^\times$: since $\mathcal{O}_F^\times$ has finite index in $\mathcal{O}_K^\times$, $\mathcal{O}_{P^n}^\times = \mathcal{O}_F^\times$ and $X_n = \mathcal{O}_{P^n,P}^\times/\mathcal{O}_{F,P}^\times$ for all $n \gg 0$. It follows that $\varprojlim X_n = \varprojlim^{(1)} X_n = \{1\}$, so that ϕ is indeed a group isomorphism. This also shows that $K^\times U \cap \widehat{\mathcal{O}}_{P^n}^\times = U$ for $n \gg 0$. In particular, $K^\times U$ is a locally closed, hence closed subgroup of $\widehat{K}^\times$. Being a separated quotient of the compact group $\widehat{K}^\times/\overline{K^\times}$, $\widehat{K}^\times/K^\times U$ is also compact. Being a continuous bijection between compact spaces, ϕ is an homeomorphism.

It easily follows that the open subgroup $\mathrm{Gal}(K[P^\infty]/K[1])$ of $G(\infty)$ is isomorphic to $\mathcal{O}_{K,P}^\times/\mathcal{O}_K^\times\mathcal{O}_{F,P}^\times$. Since $\mathcal{O}_K^\times/\mathcal{O}_F^\times$ is finite and $\mathcal{O}_{K,P}^\times/\mathcal{O}_{F,P}^\times$ contains an open subgroup topologically isomorphic to $\mathbf{Z}_p^{[F_P:\mathbf{Q}_p]}$, a classical result on profinite groups implies that

Corollary 2.2 *The torsion subgroup G_0 of $G(\infty)$ is finite and $G(\infty)/G_0$ is topologically isomorphic to $\mathbf{Z}_p^{[F_P:\mathbf{Q}_p]}$.*

2.2 A filtration of G_0

Let $G(\infty)'$ be the subgroup of $G(\infty)$ which is generated by the Frobeniuses of those primes of K which are not above P (these primes are unramified in $K[P^n]$ for all $n \geq 0$). In particular, $G(\infty)'$ is a *countable* but *dense* subgroup of $G(\infty)$.

Lemma 2.3 *The reciprocity map induces topological isomorphisms*

$$\begin{array}{rcl} (\widehat{K}^\times)^P/(S^{-1}\mathcal{O}_F)^\times(\widehat{\mathcal{O}}_K^\times)^P & \xrightarrow{\simeq} & G(\infty)' \\ \textit{and} \qquad K_P^\times/K^\times F_P^\times & \xrightarrow{\simeq} & G(\infty)/G(\infty)'. \end{array}$$

Here: $S = \mathcal{O}_F - P$ and $X^P = \{\lambda \in X,\, \lambda_P = 1\}$ for $X \subset \widehat{K}^\times$.

Proof. Class field theory tells us that $G(\infty)'$ is the image of $\mathrm{rec}_K((\widehat{K}^\times)^P)$ in $G(\infty)$. Both statements thus follow from Lemma 2.1.

The map $\lambda \mapsto \lambda\widehat{\mathcal{O}}_K \cap K^\times$ yields an isomorphism between $(\widehat{K}^\times)^P/(\widehat{\mathcal{O}}_K^\times)^P$ and the group $\mathcal{I}_K^P$ of all fractional ideals of K which are relatively prime to P. This bijection maps $(S^{-1}\mathcal{O}_F)^\times(\widehat{\mathcal{O}}_K^\times)^P/(\widehat{\mathcal{O}}_K^\times)^P$ to the group $\mathcal{P}_F^P$ of those ideals in $\mathcal{I}_K^P$ which are principal *and* generated by an element of $F^\times$ – which then necessarily belongs to $(S^{-1}\mathcal{O}_F)^\times$. We thus obtain a perhaps more enlightening description of $G(\infty)'$: it is isomorphic to $\mathcal{I}_K^P/\mathcal{P}_F^P$. The isomorphism sends the class of a prime $Q \nmid P$ of K to its Frobenius in $G(\infty)'$.

Definition 2.4 We denote by $G_1 \subset G_0$ the torsion subgroup of $G(\infty)'$.

There is an obvious finite subgroup in $\mathcal{I}_K^P/\mathcal{P}_F^P$. Indeed, let $\mathcal{I}_F^P$ be the group of all fractional ideals J in K for which $J = \mathcal{O}_K I$ for some fractional ideal I of F relatively prime to P. Then $\mathcal{P}_F^P \subset \mathcal{I}_F^P \subset \mathcal{I}_K^P$ and $\mathcal{I}_F^P/\mathcal{P}_F^P$ is finite. In fact,

$$\mathcal{I}_F^P/\mathcal{P}_F^P \simeq (\widehat{F}^\times)^P/(S^{-1}\mathcal{O}_F)^\times(\widehat{\mathcal{O}}_F^\times)^P \simeq \widehat{F}^\times/F^\times\widehat{\mathcal{O}}_F^\times \simeq \mathrm{Pic}(\mathcal{O}_F).$$

Definition 2.5 We denote by $G_2 \simeq \mathrm{Pic}(\mathcal{O}_F)$ the corresponding subgroup of G_1.

Note that G_2 is simply the image of $\mathrm{rec}_K(\widehat{F}^\times)$ in $G(\infty)$ and the isomorphism between G_2 and $\mathrm{Pic}(\mathcal{O}_F) \simeq \widehat{F}^\times/F^\times\widehat{\mathcal{O}}_F^\times$ is induced by the reciprocity map of K. By definition, G_1/G_2 is isomorphic to the torsion subgroup of $(\widehat{K}^\times)^P/(\widehat{F}^\times\widehat{\mathcal{O}}_K^\times)^P \simeq \mathcal{I}_K^P/\mathcal{I}_F^P$. We thus obtain:

Lemma 2.6 *G_1/G_2 is an $\mathbf{F}_2$-vector space with basis*

$$\{\sigma_Q \mod G_2;\, Q \mid \mathcal{D}'\}$$

where $\mathcal{D}'$ is the squarefree product of those primes $Q \neq P$ of F which ramify in K, and $\sigma_Q = \mathrm{Frob}_{\mathcal{Q}} \in G_1$ with $\mathcal{Q}^2 = Q\mathcal{O}_K$. In particular,

$$G_1/G_2 = \{\sigma_D \mod G_2;\, D \mid \mathcal{D}'\}$$

where $\sigma_D = \prod_{Q|D} \sigma_Q$ for $D \mid \mathcal{D}'$.

The following lemma is an easy consequence of the above discussion.

Lemma 2.7 *Let $Q \neq P$ be a prime of $\mathcal{O}_F$ which does not split in K and let $\mathcal{Q}$ be the unique prime of $\mathcal{O}_K$ above Q. Then the decomposition subgroup of $\mathcal{Q}$ in $G(\infty)$ is finite. More precisely, it is a subgroup of G_2 if $\mathcal{Q} = Q\mathcal{O}_K$ and a subgroup of G_1 not contained in G_2 if $\mathcal{Q}^2 = Q\mathcal{O}_K$.*

2.3 A formula

Let $\chi_0 : G_0 \to \mathbf{C}^\times$ be a fixed character of G_0. For $n > 0$, we say that a character $\chi : G(n) \to \mathbf{C}^\times$ is *primitive* if it does not factor through $G(n-1)$. We denote by $P(\chi_0, n)$ the set of primitive characters of $G(n)$ inducing χ_0 on G_0 and let $\mathbf{e}(\chi_0, n)$ be the sum of the orthogonal idempotents

$$\mathbf{e}_\chi \stackrel{\text{def}}{=} \frac{1}{|G(n)|} \sum_{\sigma \in G(n)} \overline{\chi}(\sigma) \cdot \sigma \in \mathbf{C}[G(n)], \quad \chi \in P(\chi_0, n).$$

Note that $\mathbf{e}(\chi_0, n)$ is yet another idempotent in $\mathbf{C}[G(n)]$.

Lemma 2.8 *For $n \gg 0$, we may identify G_0 with its image $G_0(n)$ in $G(n)$ and*

$$\mathbf{e}(\chi_0, n) = \frac{1}{q\,|G_0|} \cdot \left(q - \mathrm{Tr}_{Z(n)}\right) \cdot \sum_{\sigma \in G_0} \overline{\chi}_0(\sigma) \cdot \sigma \quad \textit{in } \mathbf{C}[G(n)].$$

Here, $\mathrm{Tr}_{Z(n)} \stackrel{\text{def}}{=} \sum_{\sigma \in Z(n)} \sigma$ *with* $Z(n) \stackrel{\text{def}}{=} \mathrm{Gal}(K[P^n]/K[P^{n-1}])$.

Proof. We denote by $G^\vee$ the group of characters of a given G. Write

$$\mathbf{e}(\chi_0, n) = \sum_{\sigma \in G(n)} \mathbf{e}_\sigma(\chi_0, n) \cdot \sigma \in \mathbf{C}[G(n)]$$

and put $H(n) = G(n)/G_0(n)$. If n is sufficiently large, (1) $G_0 \twoheadrightarrow G_0(n)$ is an isomorphism, (2) $G(n) \twoheadrightarrow H(n)$ induces an isomorphism from $Z(n) = \ker\left(G(n) \twoheadrightarrow G(n-1)\right)$ to the kernel of $H(n) \twoheadrightarrow H(n-1)$, and (3) the kernel $X(n)$ of $G(n) \to H(n-1)$ is the direct sum of $G_0(n)$ and $Z(n)$ in $G(n)$. In particular, there exists an element $\chi_0' \in G(n)^\vee$ inducing χ_0 on $G_0(n) \simeq G_0$ and 1 on $Z(n)$, so that

$$P(\chi_0, n) = H(n)^\vee \chi_0' - H(n-1)^\vee \chi_0' = \left(H(n)^\vee - H(n-1)^\vee\right) \chi_0'.$$

For $\sigma \in G(n)$, we thus obtain

$$\begin{aligned} |G(n)| \cdot \mathbf{e}_\sigma(\chi_0, n) &= \left(\sum_{\chi \in H(n)^\vee} \overline{\chi}(\sigma) - \sum_{\chi \in H(n-1)^\vee} \overline{\chi}(\sigma) \right) \cdot \overline{\chi}_0'(\sigma) \\ &= \left\{ \begin{array}{ll} 0 & \text{if} \quad \sigma \notin X(n) \\ -\,|H(n-1)| & \text{if} \quad \sigma \in X(n) \setminus G_0(n) \\ |H(n)| - |H(n-1)| & \text{if} \quad \sigma \in G_0(n) \end{array} \right\} \cdot \overline{\chi}_0'(\sigma). \end{aligned}$$

Since $X(n) = G_0(n) \oplus Z(n)$ with $\chi_0' = \chi_0$ on $G_0(n)$ and 1 on $Z(n)$,

$$\begin{aligned}
|G(n)| \cdot \mathbf{e}(\chi_0, n) &= \sum_{\sigma \in G_0(n)} \sum_{\tau \in Z(n)} |G(n)| \cdot \mathbf{e}_{\sigma\tau}(\chi_0, n) \cdot \sigma\tau \\
&= \sum_{\sigma \in G_0} \Big(|H(n)| - |H(n-1)| \cdot \mathrm{Tr}_{Z(n)} \Big) \cdot \overline{\chi}_0(\sigma)\sigma
\end{aligned}$$

This is our formula. Indeed,

$$|G(n)| = |G_0|\,|H(n)|\,, \quad |H(n)| = |Z(n)|\,|H(n-1)|\,,$$

and $|Z(n)| = |\mathbf{F}| = q$ by Lemma 2.9 below.

The reciprocity map induces an isomorphism between

$$K^\times \widehat{\mathcal{O}}^\times_{P^{n-1}} / K^\times \widehat{\mathcal{O}}^\times_{P^n} \simeq \mathcal{O}^\times_{P^{n-1},P} / \mathcal{O}^\times_{P^{n-1}} \mathcal{O}^\times_{P^n,P}$$

and $Z(n)$. For $n \gg 0$, $\mathcal{O}^\times_{P^{n-1}} = \mathcal{O}^\times_F$ is contained in $\mathcal{O}^\times_{P^n,P}$, so that $Z(n) \simeq \mathcal{O}^\times_{P^{n-1},P} / \mathcal{O}^\times_{P^n,P}$. On the other hand, for any $n \geq 1$, the $\mathbf{F}$-algebra $\mathcal{O}_{P^n}/P\mathcal{O}_{P^n}$ is isomorphic to $\mathbf{F}[\epsilon] = \mathbf{F}[X]/X^2\mathbf{F}[X]$, and the projection $\mathcal{O}_{P^n,P} \to \mathcal{O}_{P^n,P}/P\mathcal{O}_{P^n,P} \simeq \mathcal{O}_{P^n}/P\mathcal{O}_{P^n}$ induces an isomorphism between $\mathcal{O}^\times_{P^n,P}/\mathcal{O}^\times_{P^{n+1},P}$ and $\mathbf{F}[\epsilon]^\times/\mathbf{F}^\times \simeq \{1 + \alpha\epsilon;\, \alpha \in \mathbf{F}\}$. We thus obtain:

Lemma 2.9 *For $n \gg 0$, $Z(n) \simeq \mathbf{F}$ as a group.*

3 Shimura Curves

Let F be a totally real number field. To each finite set S of finite places of F such that $|S| + [F : \mathbf{Q}]$ is odd, we may attach a collection of Shimura curves over F. If K is a totally imaginary quadratic extension of F in which the primes of S do not split, these curves are provided with a systematic supply of CM points defined over the maximal abelian extension K^{ab} of K. As explained in the introduction, our aim in this paper (for the indefinite case) is to prove the non-triviality of certain cycles supported on these points.

This section provides some of the necessary background on Shimura curves, with [2] as our main reference. Further topics are discussed in Section 3 of [5]. To simplify the exposition, we require S to be nonempty if $F = \mathbf{Q}$. This rules out precisely the case where our Shimura curves are the classical modular curves over $\mathbf{Q}$. This assumption implies that our Shimura curves are complete – there are no cusps to be added, but then also no obvious way to embed the curves into their Jacobians.

3.1 Shimura curves

Let $\{\tau_1, \cdots, \tau_d\} = \mathrm{Hom}_{\mathbf{Q}}(F, \mathbf{R})$ be the set of real embeddings of F. We shall always view F as a subfield of $\mathbf{R}$ (or $\mathbf{C}$) through τ_1. Let B be a quaternion algebra over F which ramifies precisely at $S \cup \{\tau_2, \cdots, \tau_d\}$, a finite set of even order. Let G be the reductive group over $\mathbf{Q}$ whose set of points on a commutative $\mathbf{Q}$-algebra A is given by $G(A) = (B \otimes A)^\times$. Let Z be the center of G.

In particular, $G_{\mathbf{R}} \simeq G_1 \times \cdots \times G_d$ where $B_{\tau_i} = B \otimes_{F,\tau_i} \mathbf{R}$ and G_i is the algebraic group over $\mathbf{R}$ whose set of points on a commutative $\mathbf{R}$-algebra A is given by $G_i(A) = (B_{\tau_i} \otimes_{\mathbf{R}} A)^\times$. Fix $\epsilon \in \{\pm 1\}$ and let X be the $G(\mathbf{R})$-conjugacy class of the morphism from $\mathbb{S} \stackrel{\text{def}}{=} \mathrm{Res}_{\mathbf{C}/\mathbf{R}}(\mathbb{G}_{m,\mathbf{C}})$ to $G_{\mathbf{R}}$ which maps $z = x + iy \in \mathbb{S}(\mathbf{R}) = \mathbf{C}^\times$ to

$$\left[\begin{pmatrix} x & y \\ -y & x \end{pmatrix}^{\epsilon}, 1, \cdots, 1 \right] \in G_1(\mathbf{R}) \times \cdots \times G_d(\mathbf{R}) \simeq G(\mathbf{R}). \tag{7}$$

We have used an isomorphism of $\mathbf{R}$-algebras $B_{\tau_1} \simeq M_2(\mathbf{R})$ to identify G_1 and $\mathrm{GL}_2/\mathbf{R}$; the resulting conjugacy class X does not depend upon this choice, but it does depend on ϵ, cf. Section 3.3.1 of [5] and Remark 3.1 below.

It is well-known that X carries a complex structure for which the left action of $G(\mathbf{R})$ is holomorphic. For every compact open subgroup H of $G(\mathbf{A}_f)$, the quotient of $G(\mathbf{A}_f)/H \times X$ by the diagonal left action of $G(\mathbf{Q})$ is a *compact* Riemann surface

$$M_H^{\mathrm{an}} \stackrel{\text{def}}{=} G(\mathbf{Q}) \backslash \left(G(\mathbf{A}_f)/H \times X \right).$$

The *Shimura curve* M_H is Shimura's canonical model for M_H^{an}. It is a proper and smooth curve over F (the reflex field) whose underlying Riemann surface $M_H(\mathbf{C})$ equals M_H^{an}.

Remark 3.1 With notations as above, let $h : \mathbb{S} \to G_{\mathbf{R}}$ be the morphism defined by (7). There are $G(\mathbf{R})$-equivariant diffeomorphisms

$$\begin{array}{ccccc} X & \stackrel{\simeq}{\longleftarrow} & G(\mathbf{R})/H_\infty & \stackrel{\simeq}{\longrightarrow} & \mathbf{C} \setminus \mathbf{R} \\ ghg^{-1} & \longleftarrow\!\shortmid & g & \longmapsto & g \cdot \epsilon i \end{array}$$

where

$$\begin{aligned} H_\infty &= \mathrm{Stab}_{G(\mathbf{R})}(h) = \mathrm{Stab}_{G(\mathbf{R})}(\pm i) \\ &= \mathbf{R}^\times \mathrm{SO}_2(\mathbf{R}) \times G_2(\mathbf{R}) \times \cdots \times G_d(\mathbf{R}) \end{aligned}$$

with $G(\mathbf{R})$ acting on $\mathbf{C}\setminus\mathbf{R}$ through its first component $G_1(\mathbf{R})\simeq \mathrm{GL}_2(\mathbf{R})$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\cdot\lambda = \frac{a\lambda+b}{c\lambda+d}$. With these conventions, the derivative of $\lambda\mapsto gh(z)g^{-1}\cdot\lambda$ at $\lambda = g\cdot\epsilon i$ equals $z/\overline{z}$ (for $g\in G(\mathbf{R})$, $\lambda\in\mathbf{C}\setminus\mathbf{R}$ and $z\in\mathbf{C}^\times = \S(\mathbf{R})$). In other words, the above bijection between X and $\mathbf{C}\setminus\mathbf{R}$ is an *holomorphic* diffeomorphism.

This computation shows that the Shimura curves of the introduction, which are also those considered in [30] or [14], correspond to the case where $\epsilon = 1$. On the other hand, Carayol explicitly works with the $\epsilon=-1$ case in our main reference [2].

3.2 Connected components

We denote by

$$M_H \xrightarrow{c} \mathcal{M}_H \to \mathrm{Spec}(F)$$

the *Stein factorization* of the structural morphism $M_H\to\mathrm{Spec}(F)$, so that $\mathcal{M}_H \overset{\text{def}}{=} \mathrm{Spec}\,\Gamma(M_H,\mathcal{O}_{M_H})$ is a finite étale F-scheme and the F-morphism $c: M_H\to\mathcal{M}_H$ is proper and smooth with geometrically connected fibers. In particular,

$$\mathcal{M}_H(\overline{F}) \simeq \pi_0(M_H\times_F \overline{F}) \simeq \pi_0(M_H(\mathbf{C})) \simeq \pi_0(M_H^{\mathrm{an}}). \tag{8}$$

Remark 3.2 (1) In the notations of remark 3.1, let X^+ be the connected component of h in X, so that $X^+ = G(\mathbf{R})^+\cdot h\simeq \mathcal{H}_\epsilon$ where $G(\mathbf{R})^+$ is the neutral component of $G(\mathbf{R})$ and $\mathcal{H}_\epsilon \overset{\text{def}}{=} \{\lambda\in\mathbf{C};\ \epsilon\cdot\Re(\lambda)>0\}$. Put $G(\mathbf{Q})^+ \overset{\text{def}}{=} G(\mathbf{R})^+\cap G(\mathbf{Q})$. Then $\pi_0(M_H^{\mathrm{an}})\simeq G(\mathbf{Q})^+\backslash G(\mathbf{A}_f)/H$, corresponding to the decomposition

$$\begin{array}{ccccc} \coprod_\alpha \overline{\Gamma}_\alpha\backslash\mathcal{H}_\epsilon & \xleftarrow{\simeq} & \coprod_\alpha \Gamma_\alpha\backslash X^+ & \xrightarrow{\simeq} & M_H^{\mathrm{an}} \\ [g\cdot\epsilon i]\in\overline{\Gamma}_\alpha\backslash\mathcal{H}_\epsilon & \longleftarrow & [x = ghg^{-1}]\in\Gamma_\alpha\backslash X^+ & \longmapsto & [(\alpha,x)] \end{array} \tag{9}$$

where $\alpha\in G(\mathbf{A}_f)$ runs through a set of representatives of the finite set $G(\mathbf{Q})^+\backslash G(\mathbf{A}_f)/H$, Γ_α is the discrete subgroup $\alpha H\alpha^{-1}\cap G(\mathbf{Q})^+$ of $G(\mathbf{R})^+$ and $\overline{\Gamma}_\alpha\subset\mathrm{PGL}_2^+(\mathbf{R})$ is its image through the obvious map $G(\mathbf{R})^+\to\mathrm{GL}_2^+(\mathbf{R})\to\mathrm{PGL}_2^+(\mathbf{R})$.

(2) The strong approximation theorem [26, p. 81] and the norm theorem [26, p. 80] imply that the reduced norm $\mathrm{nr}:\widehat{B}^\times\to\widehat{F}^\times$ induces a bijection

$$\pi_0(M_H^{\mathrm{an}})\simeq G(\mathbf{Q})^+\backslash G(\mathbf{A}_f)/H \xrightarrow{\simeq} Z(\mathbf{Q})^+\backslash Z(\mathbf{A}_f)/\mathrm{nr}(H) \tag{10}$$

where $Z(\mathbf{Q})^+ = \mathrm{nr}(G(\mathbf{Q})^+)$ is the subgroup of totally positive elements in $Z(\mathbf{Q}) = F^\times$. Using also (8), we obtain a left action of Gal_F on the RHS of (10). The general theory of Shimura varieties implies that the latter action factors through $\mathrm{Gal}_F^{\mathrm{ab}}$, where it is given by the following *reciprocity law* (see Lemma 3.12 in [5]): for $\lambda \in \widehat{F}^\times$, the element $\sigma = \mathrm{rec}_F(\lambda)$ of $\mathrm{Gal}_F^{\mathrm{ab}}$ acts on the RHS of (10) as multiplication by λ^ϵ. In particular, this action is transitive and M_H is therefore a *connected* F-curve (although not a geometrically connected one).

3.3 Related group schemes

The *Jacobian* J_H of M_H is the identity component of the relative Picard scheme P_H of $M_H \to \mathrm{Spec}(F)$ and the *Néron-Severi* group NS_H of M_H is the quotient of P_H by J_H. By [12, VI], J_H is an abelian scheme over F while NS_H is a "separable discrete" F-group scheme. The canonical isomorphism [12, V.6.1] of F-group schemes

$$P_H \xrightarrow{\simeq} \mathrm{Res}_{\mathcal{M}_H/F}(\mathrm{Pic}_{M_H/\mathcal{M}_H})$$

induces an isomorphism between J_H and $\mathrm{Res}_{\mathcal{M}_H/F}(\mathrm{Pic}^0_{M_H/\mathcal{M}_H})$, so that

$$\mathrm{NS}_H \xrightarrow{\simeq} \mathrm{Res}_{\mathcal{M}_H/F}\left(\mathrm{Pic}_{M_H/\mathcal{M}_H}/\mathrm{Pic}^0_{M_H/\mathcal{M}_H}\right) \xrightarrow{\simeq} \mathrm{Res}_{\mathcal{M}_H/F}(\underline{\mathbf{Z}})$$

where we have identified $\mathrm{Pic}_{M_H/\mathcal{M}_H}/\mathrm{Pic}^0_{M_H/\mathcal{M}_H}$ with the constant $\mathcal{M}_H$-group scheme $\underline{\mathbf{Z}}$ using the degree map $\deg_H : \mathrm{Pic}_{M_H/\mathcal{M}_H} \to \underline{\mathbf{Z}}$. We denote by

$$\overline{\deg}_H : P_H \to \mathrm{NS}_H = P_H/J_H$$

the quotient map, so that $\overline{\deg}_H = \mathrm{Res}_{\mathcal{M}_H/F}(\deg_H)$ under the above identifications.

Remark 3.3 If $\mathcal{M}_H(\overline{F}) = \{s_\alpha\}$ with $s_\alpha : \mathrm{Spec}(\overline{F}) \to \mathcal{M}_H$,

$$\begin{array}{rclcrcl} M_H \times_F \mathrm{Spec}(\overline{F}) & = & \coprod_\alpha \mathcal{C}_\alpha, & & P_H \times_F \mathrm{Spec}(\overline{F}) & = & \prod_\alpha P_\alpha, \\ J_H \times_F \mathrm{Spec}(\overline{F}) & = & \prod_\alpha J_\alpha, & \text{and} & \mathrm{NS}_H \times_F \mathrm{Spec}(\overline{F}) & = & \prod_\alpha \underline{\mathbf{Z}}_\alpha \end{array}$$

where $\mathcal{C}_\alpha = c^{-1}(s_\alpha)$, $P_\alpha = \mathrm{Pic}(\mathcal{C}_\alpha)$, $J_\alpha = \mathrm{Pic}^0(\mathcal{C}_\alpha)$ and $\underline{\mathbf{Z}}_\alpha = s_\alpha^*(\underline{\mathbf{Z}})$ is isomorphic to $\underline{\mathbf{Z}}$ over $\overline{F}$. With these identifications, $\overline{\deg}_H$ maps $(p_\alpha) \in \prod_\alpha P_\alpha$ to $(\deg(p_\alpha)) \in \prod_\alpha \mathbf{Z}$.

3.4 Hecke operators

As H varies among the compact open subgroups of $G(\mathbf{A}_f)$, the Shimura curves $\{M_H\}_H$ form a projective system with finite flat transition maps which is equipped with a "continuous" right action of $G(\mathbf{A}_f)$. Specifically, for any element $g \in G(\mathbf{A}_f)$ and for any compact open subgroups H_1 and H_2 of $G(\mathbf{A}_f)$ such that $g^{-1}H_1 g \subset H_2$, multiplication on the right by g in $G(\mathbf{A}_f)$ defines a map $M_{H_1}^{\mathrm{an}} \to M_{H_2}^{\mathrm{an}}$ which descends to a finite flat F-morphism

$$[\cdot g] = [\cdot g]_{H_1,H_2} : M_{H_1} \to M_{H_2}.$$

We shall refer to such a map as *the degeneracy map* induced by g. Letting H_1 and H_2 vary, these degeneracy maps together define an automorphism $[\cdot g]$ of $\varprojlim \{M_H\}_H$.

There is a natural *left* action of the Hecke algebra

$$\mathbf{T}_H \stackrel{\mathrm{def}}{=} \mathrm{End}_{\mathbf{Z}[G(\mathbf{A}_f)]}\left(\mathbf{Z}[G(\mathbf{A}_f)/H]\right) \simeq \mathbf{Z}[H\backslash G(\mathbf{A}_f)/H]$$

on P_H, J_H and NS_H. We normalize these actions in the Albanese fashion (CoVa2riantly): for $\alpha \in G(\mathbf{A}_f)$, the Hecke operator $\mathcal{T}_H(\alpha) \in \mathbf{T}_H$ corresponding to the double class $H\alpha H$ acts by $\mathcal{T}_H(\alpha) = f'_* \circ [\cdot\alpha]_* \circ f'^*$ where f and f' are the obvious transition maps in the following diagram

$$\begin{tikzcd}
 & M_{H\cap\alpha H\alpha^{-1}} \arrow[r, "{[\cdot\alpha]}"] \arrow[dl, "f"'] & M_{\alpha^{-1}H\alpha\cap H} \arrow[dr, "f'"] & \\
M_H \arrow[rrr, dotted, "\mathcal{T}_H(\alpha)"] & & & M_H
\end{tikzcd}$$

The *degree* of $\mathcal{T}_H(\alpha)$ is the degree of f, namely the index of $H \cap \alpha H \alpha^{-1}$ in H. On the level of divisors, $\mathcal{T}_H(\alpha)$ maps $x = [g,h] \in M_H^{\mathrm{an}}$ to

$$\mathcal{T}_H(\alpha)(x) = \sum [g\alpha_i, h] \in \mathrm{Div} M_H^{\mathrm{an}}$$

where $H\alpha H = \coprod \alpha_i H$. If α belongs to the center of $G(\mathbf{A}_f)$, then $[\cdot\alpha] : M_H \to M_H$ is an automorphism of M_H/F and $\mathcal{T}_H(\alpha) = [\cdot\alpha]_*$.

Definition 3.4 We denote by θ_M and θ_J the induced left action of $Z(\mathbf{A}_f)$ on M_H and J_H: $\theta_M(\alpha) = [\cdot\alpha]$ and $\theta_J(\alpha) = [\cdot\alpha]_*$. These actions factor through $Z(\mathbf{Q})\backslash Z(\mathbf{A}_f)/Z(\mathbf{A}_f)\cap H$. When $H = \widehat{R}^\times$ for some $\mathcal{O}_F$-order $R \subset B$, $Z(\mathbf{A}_f) \cap H = \widehat{\mathcal{O}}_F^\times$ and we thus obtain left actions

$$\theta_M : \mathrm{Pic}(\mathcal{O}_F) \to \mathrm{Aut}_F M_H \quad \text{and} \quad \theta_J : \mathrm{Pic}(\mathcal{O}_F) \to \mathrm{Aut}_F J_H.$$

3.5 The Hodge class and the Hodge embedding

Let S be a scheme. To any commutative group scheme $\mathcal{G}$ over S, and indeed to any presheaf of abelian groups $\mathcal{G}$ on the category of S-scheme, we may attach a presheaf of $\mathbf{Q}$-vector spaces $\mathcal{G}\otimes\mathbf{Q}$ by the following rule: for any S-scheme X, $\mathcal{G}\otimes\mathbf{Q}(X)\stackrel{\text{def}}{=}\mathcal{G}(X)\otimes\mathbf{Q}$. To distinguish between the sections of $\mathcal{G}(X)$ and the sections of $\mathcal{G}\otimes\mathbf{Q}(X)$, we write $X\to\mathcal{G}$ for the former and $X\rightsquigarrow\mathcal{G}$ for the latter, but we will refer to both kind of sections as *morphisms*.

This construction is functorial in the sense that given two presheaves of abelian groups $\mathcal{G}_1$ and $\mathcal{G}_2$ over S, any element α of

$$\mathrm{Hom}^0_S(\mathcal{G}_1,\mathcal{G}_2)\stackrel{\text{def}}{=}\mathrm{Hom}_S(\mathcal{G}_1,\mathcal{G}_2)\otimes\mathbf{Q}$$

defines a morphism $\alpha:\mathcal{G}_1\rightsquigarrow\mathcal{G}_2$: choose $n\geq 1$ such that $n\alpha=\alpha_0\otimes 1$ for some $\alpha_0\in\mathrm{Hom}_S(\mathcal{G}_1,\mathcal{G}_2)$, set $\alpha(f)=\alpha_0(f)\otimes\frac{1}{n}\in\mathcal{G}_2(X)\otimes\mathbf{Q}$ for $f\in\mathcal{G}_1(X)$ and extend by linearity to $\mathcal{G}_1(X)\otimes\mathbf{Q}$. The resulting morphism does not depend upon the choice of n and α_0 and furthermore satisfies $\alpha(\lambda_1 f_1+\lambda_2 f_2)=\lambda_1\alpha(f_1)+\lambda_2\alpha(f_2)$ in $\mathcal{G}_2\otimes\mathbf{Q}(X)$ for any $\lambda_1,\lambda_2\in\mathbf{Q}$ and $f_1,f_2:X\rightsquigarrow\mathcal{G}_1$.

This said, the "Hodge embedding" is a morphism $\iota_H:M_H\rightsquigarrow J_H$ over F which we shall now define. To start with, consider the F-morphism $(\cdot)_H:M_H\to P_H$ which for any F-scheme X maps $x\in M_H(X)$ to the element $(x)\in P_H(X)$ which is represented by the effective relative Cartier divisor on $M_H\times_F X$ defined by x (viewing x as a section of $M_H\times_F X\to X$). Our morphism ι_H is the composite of this map with a retraction $P_H\rightsquigarrow J_H$ of $J_H\hookrightarrow P_H$. Defining the latter amounts to define a section $\mathrm{NS}_H\rightsquigarrow P_H$ of $\overline{\deg}_H:P_H\twoheadrightarrow\mathrm{NS}_H$ and since $\overline{\deg}_H=\mathrm{Res}_{\mathcal{M}_H/F}(\deg_H)$, we may as well search for a section $\underline{\mathbf{Z}}\rightsquigarrow\mathrm{Pic}_{M_H/\mathcal{M}_H}$ of $\deg_H:\mathrm{Pic}_{M_H/\mathcal{M}_H}\twoheadrightarrow\underline{\mathbf{Z}}$. In other word, we now want to construct an element (the *Hodge class*)

$$\delta_H\in\mathrm{Pic}_{M_H/\mathcal{M}_H}(\mathcal{M}_H)\otimes\mathbf{Q}=\mathrm{Pic}(M_H)\otimes\mathbf{Q}$$

such that $\deg(\delta_H)=1$ in $\underline{\mathbf{Z}}(\mathcal{M}_H)\otimes\mathbf{Q}=\mathbf{Q}$ (recall from remark 3.2 that $\mathcal{M}_H$ is connected). The resulting morphism $\iota_H:M_H\rightsquigarrow J_H$ will thus be given by

$$\iota_H(x)=(x)_H-s_H\circ\overline{\deg}_H(x)_H\quad\text{in } J_H\otimes\mathbf{Q}(X)$$

for any F-scheme X and $x\in M_H(X)$, with $s_H:\mathrm{NS}_H\rightsquigarrow P_H$ defined by

$$s_H\stackrel{\text{def}}{=}\mathrm{Res}_{\mathcal{M}_H/F}\left(\begin{array}{ccc}\underline{\mathbf{Z}} & \rightsquigarrow & \mathrm{Pic}_{M_H/\mathcal{M}_H}\\ n & \mapsto & n\cdot\delta_H\end{array}\right).$$

We may now proceed to the definition of the Hodge class δ_H. For a compact open subgroup H' of H, the *ramification divisor* of the transition map $f = \mathcal{T}_{H',H} : M_{H'} \to M_H$ is defined by

$$R_{H'/H} \stackrel{\text{def}}{=} \sum_x \text{length}_{\mathcal{O}_{M_{H'},x}} (\Omega_f)_x \cdot x \quad \text{in } \text{Div}(M_{H'})$$

where x runs through the finitely many closed points of $M_{H'}$ in the support of the sheaf of relative differentials $\Omega_f = \Omega_{M_{H'}/M_H}$. The *branch divisor* is the flat push-out of $R_{H'/H}$:

$$B_{H'/H} \stackrel{\text{def}}{=} f_* R_{H'/H} \quad \text{in } \text{Div}(M_H).$$

When H' is a normal subgroup of H, $M_H = M_{H'}/H$ and the branch divisor pulls-back to $f^* B_{H'/H} = \deg(f) \cdot R_{H'/H}$.

For $H'' \subset H' \subset H$, $f = \mathcal{T}_{H',H}$, $g = \mathcal{T}_{H'',H'}$ and $h = f \circ g = \mathcal{T}_{H'',H}$,

$$0 \to g^* \Omega_f \to \Omega_h \to \Omega_g \to 0$$

is an exact sequence of coherent sheaves on $M_{H''}$ (this may be proven using a variant of Proposition 2.1 of [13, Chapter IV], the flatness of g and the snake lemma). This exact sequence shows that

$$\begin{array}{rrcll} & R_{H''/H} & = & R_{H''/H'} + g^* R_{H'/H} & \text{in } \text{Div}(M_{H''}) \\ \text{and} & B_{H''/H} & = & f_* B_{H''/H'} + \deg(g) \cdot B_{H'/H} & \text{in } \text{Div}(M_H). \end{array} \tag{11}$$

If H' is sufficiently small, $R_{H''/H'} = 0$ and $B_{H''/H'} = 0$ for any $H'' \subset H'$ (see for instance [2, Corollaire 1.4.1.3]). In particular,

$$B_H \stackrel{\text{def}}{=} \frac{1}{\deg(f)} B_{H'/H} \in \text{Div}(M_H) \otimes \mathbf{Q}$$

does not depend upon H', provided that H' is sufficiently small. When H itself is sufficiently small, $B_H = 0$. In general:

Lemma 3.5 $f^* B_H = B_{H'} + R_{H'/H}$ *in* $\text{Div}(M_{H'}) \otimes \mathbf{Q}$.

Proof. Let H'' be a sufficiently small normal subgroup of H contained in H'. With notations as above, $g^* : \text{Div}(M_{H'}) \otimes \mathbf{Q} \to \text{Div}(M_{H''}) \otimes \mathbf{Q}$ is injective,

$$\begin{array}{rrcccl} & g^* f^* B_{H'} & = & \frac{1}{\deg(h)} h^* h_* R_{H''/H} & = & R_{H''/H}, \\ \text{and} & g^* B_{H'} & = & \frac{1}{\deg(g)} g^* g_* R_{H''/H'} & = & R_{H''/H'}. \end{array}$$

The lemma thus follows from (11).

On the other hand, Hurwitz formula [13, IV, Prop. 2.3] tells us that

$$\mathcal{K}_{H'} = f^*\mathcal{K}_H + \text{class of } R_{H'/H} \quad \text{in } \mathrm{Pic}(M_{H'})$$

where $\mathcal{K}_H$ is the canonical class on M_H, namely the class of $\Omega_{M_H/F}$. It follows that

$$\begin{array}{rcll} f^*(\mathcal{K}_H + B_H) & = & \mathcal{K}_{H'} + B_{H'} & \text{in } \mathrm{Pic}(M_{H'}) \otimes \mathbf{Q}, \\ f_*(\mathcal{K}_{H'} + B_{H'}) & = & \deg(f) \cdot (\mathcal{K}_H + B_H) & \text{in } \mathrm{Pic}(M_H) \otimes \mathbf{Q}. \end{array} \tag{12}$$

If H' is sufficiently small, $B_{H'} = 0$ and $\deg(\mathcal{K}_{H'}) > 0$. The above formulae therefore imply that $\deg(\mathcal{K}_H + B_H) > 0$ for *any* H, and we may thus define

$$\delta_H \stackrel{\text{def}}{=} \frac{1}{\deg(\mathcal{K}_H + B_H)} \cdot (\mathcal{K}_H + B_H) \in \mathrm{Pic}(M_H) \otimes \mathbf{Q}.$$

By construction: $\deg(\delta_H) = 1$,

$$\begin{array}{rrcll} & f^*\delta_H & = & \deg(f) \cdot \delta_{H'} & \text{in } \mathrm{Pic}(M_{H'}) \otimes \mathbf{Q} \\ \text{and} & f_*\delta_{H'} & = & \delta_H & \text{in } \mathrm{Pic}(M_H) \otimes \mathbf{Q}. \end{array} \tag{13}$$

Lemma 3.6 *For any $\alpha \in G(\mathbf{A}_f)$, $\mathcal{T}_H(\alpha)(\delta_H) = \deg(\mathcal{T}_H(\alpha)) \cdot \delta_H$.*

Proof. Given the definition of $\mathcal{T}_H(\alpha)$, this follows from (13) once we know that for any H and α, $[\cdot\alpha]_*\delta_H = \delta_{\alpha^{-1}H\alpha}$ in $\mathrm{Pic}(M_{\alpha^{-1}H\alpha}) \otimes \mathbf{Q}$. This is obvious if H is sufficiently small and the general case follows, using (13) again.

Remark 3.7 With notations as in remark 3.2, the restriction of B_H to $\overline{\Gamma}_\alpha \backslash \mathcal{H}_\epsilon$ equals $\sum_x (1 - e_x^{-1}) \cdot x$ where x runs through a set of representatives of $\overline{\Gamma}_\alpha \backslash \mathcal{H}_\epsilon$ in $\mathcal{H}_\epsilon$ and e_x is the order of its stabilizer in $\overline{\Gamma}_\alpha \subset \mathrm{PGL}_2^+(\mathbf{R})$. Compare with [9, section 23].

Remark 3.8 With the notations of remark 3.3, let $\delta_\alpha \in \mathrm{Pic}(\mathcal{C}_\alpha) \otimes \mathbf{Q}$ be the restriction to $\mathcal{C}_\alpha$ of the pull-back of δ_H to

$$\mathrm{Pic}(M_H \times_F \mathrm{Spec}(\overline{F})) \otimes \mathbf{Q} = \prod_\alpha \mathrm{Pic}(\mathcal{C}_\alpha) \otimes \mathbf{Q}.$$

Then $\deg(\delta_\alpha) = 1$ and the restriction of ι_H to $\mathcal{C}_\alpha$ maps $x \in \mathcal{C}_\alpha(\overline{F})$ to $(x) - \delta_\alpha \in J_\alpha(\overline{F}) \otimes \mathbf{Q}$. In particular, the image of ι_H on $M_H(\overline{F})$ spans $J_H(\overline{F}) \otimes \mathbf{Q}$ over $\mathbf{Q}$.

The terminology *Hodge Class* is due to S. Zhang. In his generalization of the Gross-Zagier formulae to the case of Shimura curves, the morphism $\iota_H : M_H \rightsquigarrow J_H$ plays the role of the embedding $x \mapsto (x) - (\infty)$ of a classical modular curve into its Jacobian. This is why we refer to ι_H as the Hodge "embedding". It *is* a finite morphism, in the sense that some nonzero multiple $n\iota_H$ of ι_H is a genuine finite morphism from M_H to J_H. More generally:

Lemma 3.9 *Let $\pi : J_H \rightsquigarrow A$ be a nonzero morphism of abelian varieties over F. Then $\alpha = \pi \circ \iota_H : M_H \rightsquigarrow A$ is finite (in the above sense).*

Proof. We may assume that $\pi : J_H \to A$ and $\alpha : M_H \to A$ are genuine morphisms. Since M_H is a connected complete curve over F, α is either finite or constant, and it can not constant by remark 3.8.

3.6 Differentials and automorphic forms

Let $\Omega_H = \Omega_{M_H/F}$ be the sheaf of differentials on M_H and denote by Ω_H^{an} the pull-back of Ω_H to M_H^{an}, so that Ω_H^{an} is the sheaf of holomorphic 1-forms on M_H^{an}. The *right* action of $G(\mathbf{A}_f)$ on the projective system $\{M_H\}_H$ induces a $\mathbf{C}$-linear *left* action of $G(\mathbf{A}_f)$ on the inductive system $\{\Gamma(\Omega_H^{\mathrm{an}})\}_H$ of global sections of these sheaves. We want to identify $\varinjlim \Gamma(\Omega_H^{\mathrm{an}})$, together with its $G(\mathbf{A}_f)$-action, with a suitable space $\mathcal{S}_2$ of automorphic forms on G.

Fix an isomorphism $G(\mathbf{R}) \simeq \mathrm{GL}_2(\mathbf{R}) \times G_2(\mathbf{R}) \times \cdots \times G_d(\mathbf{R})$ as in section 3.1 and let $\mathcal{S}_2$ be the complex vector space of all functions $F : G(\mathbf{A}) = G(\mathbf{A}_f) \times G(\mathbf{R}) \to \mathbf{C}$ with the following properties:

P1 F is left $G(\mathbf{Q})$-invariant.

P2 F is right invariant under $\mathbf{R}^* \times G_2(\mathbf{R}) \times \cdots \times G_d(\mathbf{R}) \subset G(\mathbf{R})$.

P3 F is right invariant under some compact open subgroup of $G(\mathbf{A}_f)$.

P4 For every $g \in G(\mathbf{A})$ and $\theta \in \mathbf{R}$,

$$F\left(g\left(\begin{pmatrix} \cos(\theta) & \epsilon\sin(\theta) \\ -\epsilon\sin(\theta) & \cos(\theta) \end{pmatrix}, 1, \cdots, 1\right)\right) = \exp(2i\theta)F(g).$$

P5 For every $g \in G(\mathbf{A})$, the function

$$z = x + iy \mapsto F(g, z) \stackrel{\mathrm{def}}{=} \frac{1}{y}F\left(g \times \left(\begin{pmatrix} \epsilon y & x \\ 0 & 1 \end{pmatrix}, 1, \cdots, 1\right)\right)$$

is holomorphic on $\mathcal{H}_\epsilon$.

There is a left action of $G(\mathbf{A}_f)$ on $\mathcal{S}_2$ given by $(g \cdot F)(x) = F(xg)$.

Proposition 3.10 *There is a $G(\mathbf{A}_f)$-equivariant bijection*

$$\varinjlim \Gamma(\Omega_H^{\mathrm{an}}) \xrightarrow{\sim} \mathcal{S}_2$$

which identifies $\Gamma(\Omega_H^{\mathrm{an}})$ with $\mathcal{S}_2^H$.

Proof. Recall from remark 3.2 that there is a decomposition

$$M_H^{\mathrm{an}} = \coprod_\alpha \overline{\Gamma}_\alpha \backslash \mathcal{H}_\epsilon$$

where α runs through a set of representatives of $G(\mathbf{Q})^+\backslash G(\mathbf{A}_f)/H$ and $\overline{\Gamma}_\alpha$ is the image of $\Gamma_\alpha = \alpha H \alpha^{-1} \cap G(\mathbf{Q})^+$ in $\mathrm{PGL}_2^+(\mathbf{R})$.

Let $\omega \in \Gamma(\Omega_H^{\mathrm{an}})$ be a global holomorphic 1-form on M_H^{an}. The restriction of ω to the connected component $\overline{\Gamma}_\alpha\backslash\mathcal{H}_\epsilon$ of M_H^{an} pulls back to a $\overline{\Gamma}_\alpha$-invariant holomorphic form on $\mathcal{H}_\epsilon$. The latter equals $f_\alpha(z)dz$ for some holomorphic function f_α on $\mathcal{H}_\epsilon$ such that $f_\alpha|\gamma = f_\alpha$ for all $\gamma \in \overline{\Gamma}_\alpha$, where

$$(f|\gamma)(z) = \det(\gamma)(cz+d)^{-2} f\left(\frac{az+b}{cz+d}\right)$$

for a function f on $\mathcal{H}_\epsilon$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{GL}_2^+(\mathbf{R})$. Since

$$G(\mathbf{A}) = \coprod_\alpha G(\mathbf{Q}) \cdot \left(\alpha H \times G(\mathbf{R})^+\right),$$

we may write any element $g \in G(\mathbf{A})$ as a product $g = g_{\mathbf{Q}}(\alpha h \times g_{\mathbf{R}}^+)$ for some α with $g_{\mathbf{Q}} \in G(\mathbf{Q})$, $h \in H$ and

$$g_{\mathbf{R}}^+ \in G(\mathbf{R})^+ = \mathrm{GL}_2^+(\mathbf{R}) \times G_2(\mathbf{R}) \times \cdots \times G_d(\mathbf{R}).$$

We put $F_\omega(g) = \left(f_\alpha|g_{\mathbf{R},1}^+\right)(\epsilon i)$ where $g_{\mathbf{R},1}^+$ is the first component of $g_{\mathbf{R}}^+$. If g also equals $g'^+_{\mathbf{R}}\, g'_{\mathbf{Q}}(\alpha' h' \times g'^+_{\mathbf{R}})$, then $\alpha = \alpha'$ and $g'^{-1}_{\mathbf{Q}} g_{\mathbf{Q}}$ belongs to $G^+(\mathbf{Q}) \cap \alpha H \alpha^{-1} = \Gamma_\alpha$, so that

$$f_{\alpha'}|g'^+_{\mathbf{R},1} = f_\alpha|g'^{-1}_{\mathbf{Q}} g_{\mathbf{Q}} g^+_{\mathbf{R},1} = f_\alpha|g^+_{\mathbf{R},1}.$$

It follows that F_ω is a well-defined complex valued function on $G(\mathbf{A})$. We leave it to the reader to check that F_ω belongs to $\mathcal{S}_2^H$. Conversely, any H-invariant element F in $\mathcal{S}_2$ defines an holomorphic differential 1-form ω_F on M_H^{an}: with notations as in **P5**, the restriction of ω_F to $\overline{\Gamma}_\alpha\backslash\mathcal{H}_\epsilon$ pulls back to $F(\alpha, z)dz$ on $\mathcal{H}_\epsilon$.

Remark 3.11 The complex cotangent space of J_H at 0 is canonically isomorphic to $\Gamma(\Omega_H^{\mathrm{an}})$, and therefore also to $\mathcal{S}_2^H$. With these identifications, the natural right action of $\mathbf{T}_H = \mathrm{End}_{\mathbf{Z}[G(\mathbf{A}_f)]}(\mathbf{Z}[G(\mathbf{A}_f)/H])$ on $\mathcal{S}_2^H \simeq \mathrm{Hom}_{\mathbf{Z}[G(\mathbf{A}_f)]}(\mathbf{Z}[G(\mathbf{A}_f)/H], \mathcal{S}_2)$ coincide with the right action induced on the cotangent space by the left action of $\mathbf{T}_H$ on J_H.

Remark 3.12 When $\epsilon = 1$, $\mathcal{S}_2$ is exactly the subspace of $\mathcal{S}$ which is defined in the introduction, given our choice of an isomorphism between G_1 and $\mathrm{GL}_2(\mathbf{R})$ (cf. remark 1.7 and definition 1.8). This follows from the relevant properties of lowest weight vectors, much as in the classical case [6, Section 11.5].

3.7 The P-new quotient

Let P be a prime of F where B is split, and consider a compact open subgroup H of $G(\mathbf{A}_f)$ which decomposes as $H = H^P R_P^\times$, where R_P is an Eichler order in $B_P \simeq M_2(F_P)$ and H^P is a compact open subgroup of $G(\mathbf{A}_f)^P = \{g \in G(\mathbf{A}_f);\, g_P = 1\}$.

Definition 3.13 The *P-new quotient* of J_H is the largest quotient $\pi : J_H \twoheadrightarrow J_H^{P-\mathrm{new}}$ of J_H such that for any Eichler order $R'_P \subset B_P$ strictly containing R_P, $\pi \circ f^* = 0$ where $H' = H^P R'^\times_P$ and $f^* : J_{H'} \to J_H$ is the morphism induced by the degeneracy map $f : M_H \to M_{H'}$.

The quotient map $\pi : J_H \twoheadrightarrow J_H^{P-\mathrm{new}}$ induces an embedding from the complex cotangent space of $J_H^{P-\mathrm{new}}$ at 0 into the complex cotangent space of J_H at 0. Identifying the latter space first with $\Gamma(\Omega_H^{\mathrm{an}})$ and then with the space of H-invariant elements in $\mathcal{S}_2$ (Proposition 3.10), we obtain the P-new subspace $\mathcal{S}_{2,P-\mathrm{new}}^H$ of $\mathcal{S}_2^H$. By construction,

$$\mathcal{S}_2^H = \mathcal{S}_{2,P-\mathrm{new}}^H \oplus \mathcal{S}_{2,P-\mathrm{old}}^H$$

where $\mathcal{S}_{2,P-\mathrm{old}}^H$ is the subspace of $\mathcal{S}_2^H$ spanned by the elements fixed by $R'^\times_P$ for some Eichler order $R'_P \subset B_P$ strictly containing R_P.

3.8 CM Points

Let K be a totally imaginary quadratic extension of F and put $T = \mathrm{Res}_{K/\mathbf{Q}}(\mathbb{G}_{m,K})$. Any ring homomorphism $K \hookrightarrow B$ induces an embedding $T \hookrightarrow G$. A morphism $h : \mathbb{S} \to G_{\mathbf{R}}$ in X is said to have *complex multiplication* by K if it factors through the morphism $T_{\mathbf{R}} \hookrightarrow G_{\mathbf{R}}$ which is induced by an F-algebra homomorphism $K \hookrightarrow B$. For a compact open subgroup H of $G(\mathbf{A}_f)$, we say that $x \in M_H(\mathbf{C})$ is a *CM point* if $x = [g, h] \in M_H^{\mathrm{an}}$ for some $g \in G(\mathbf{A}_f)$ and $h \in X$ with complex multiplication by K.

We assume that K splits B, which amounts to require that K_v is a field for every finite place v of F where B ramifies. Then, there exists an F-algebra homomorphism $K \hookrightarrow B$, and any two such homomorphisms are conjugated by an element of $B^\times = G(\mathbf{Q})$. We fix such an homomorphism and let $T \hookrightarrow G$ be the induced morphism.

In each of the two connected components of X, there is exactly one morphism $\mathbb{S} \to G_{\mathbf{R}}$ which factors through $T_{\mathbf{R}} \hookrightarrow G_{\mathbf{R}}$. These two morphisms are permuted by the normalizer of $T(\mathbf{Q})$ in $G(\mathbf{Q})$, and $T(\mathbf{Q})$ is their common stabilizer in $G(\mathbf{Q})$. They correspond respectively to

$$z \in \mathbb{S} \mapsto (z \text{ or } \bar{z}, 1, \cdots, 1) \in T_1 \times \cdots \times T_d \simeq T_{\mathbf{R}}$$

where $K_i = K \otimes_{F,\tau_i} \mathbf{R}$, $T_i = \mathrm{Res}_{K_i/\mathbf{R}}(\mathbb{G}_{m,K_i})$ for $1 \leq i \leq d$, and where we have chosen an extension $\tau_1 : K \hookrightarrow \mathbf{C}$ of $\tau_1 : F \hookrightarrow \mathbf{R}$ to identify K_1 with $\mathbf{C}$ and T_1 with $\mathbb{S}$. We choose τ_1 in such a way that the morphism $h_K : \mathbb{S} \to G_{\mathbf{R}}$ corresponding to $z \mapsto (z, 1, \cdots 1)$ belongs to the connected component of the morphism $h : \mathbb{S} \to G_{\mathbf{R}}$ which is defined by (7). The map

$$g \in G(\mathbf{A}_f) \mapsto [g, h_K] \in M_H^{\mathrm{an}} = G(\mathbf{Q}) \backslash (G(\mathbf{A}_f)/H \times X)$$

then induces a bijection between the set of CM points in M_H^{an} and the set $\mathrm{CM}_H = T(\mathbf{Q}) \backslash G(\mathbf{A}_f)/H$ of the introduction. In the sequel, we will use this identification without any further reference. In particular, we denote by $[g] \in M_H^{\mathrm{an}} = M_H(\mathbf{C})$ the CM point corresponding to $[g, h_K]$.

By Shimura's theory, the CM points are algebraic and defined over the maximal abelian extension K^{ab} of K. Moreover, for $\lambda \in T(\mathbf{A}_f)$ and $g \in G(\mathbf{A}_f)$, the action of $\sigma = \mathrm{rec}_K(\lambda) \in \mathrm{Gal}_K^{\mathrm{ab}}$ on $x = [g] \in \mathrm{CM}_H$ is given by the following reciprocity law (viewing K as a subfield of $\mathbf{C}$ through τ_1):

$$\sigma \cdot x = [\lambda^\epsilon g] \in \mathrm{CM}_H.$$

We refer to Sections 3 and 3.2 of [5] for a more detailed discussion of CM points on Shimura curves, including a proof of the above facts.

4 The Indefinite Case

To the data of F, B, K (and ϵ), we have attached a collection of Shimura curves $\{M_H\}$ equipped with a systematic supply of CM points defined over the maximal abelian extension K^{ab} of K. We now also fix a prime P of F where B is split, and restrict our attention to the CM points of P-power conductor in a given Shimura curve $M = M_H$, with $H = \widehat{R}^\times$ for some $\mathcal{O}_F$-order $R \subset B$. We assume that

(H1) R_P is an Eichler order in $B_P \simeq M_2(F_P)$.

(H2) For any prime $Q \neq P$ of F which ramifies in K, B is split at Q and R_Q is a maximal order in $B_Q \simeq M_2(F_Q)$.

We put $J = J_H$, $\mathrm{CM} = \mathrm{CM}_H$, $\iota = \iota_H$ and so on... We denote by $\mathrm{CM}(P^n) \subset M(K[P^n])$ the set of CM points of conductor P^n and put

$$\mathrm{CM}(P^\infty) = \cup_{n\geq 0}\mathrm{CM}(P^n) \subset M(K[P^\infty]).$$

Thanks to **(H1)**, we have the notion of *good* CM points, as defined in the introduction. Recall that all CM points are good when R_P is maximal; otherwise, the good CM points are those which are of type I or II, in the terminology of section 6.

In this section, we study the contribution of these points to the growth of the Mordell-Weil groups of suitable quotients A of J, as one ascends the abelian extension $K[P^\infty]$ of K.

4.1 Statement of the main results

We say that $\pi : J \rightsquigarrow A$ is a *surjective morphism* if some nonzero multiple of π is a genuine surjective morphism $J \to A$ of abelian varieties over F. We say that π is *P-new* if it factors through the P-new quotient of J, cf. definition 3.13. We say that π is $\mathrm{Pic}(\mathcal{O}_F)$-*equivariant* if A is endowed with an action θ_A of $\mathrm{Pic}(\mathcal{O}_F)$ such that

$$\forall \sigma \in \mathrm{Pic}(\mathcal{O}_F) : \quad \theta_A(\sigma) \circ \pi = \pi \circ \theta_J(\sigma) \quad \text{in } \mathrm{Hom}^0(J, A),$$

cf. definition 3.4.

We put $C \stackrel{\text{def}}{=} \mathbf{Z}[\mathrm{Pic}(\mathcal{O}_F)]$. For a character $\omega : \mathrm{Pic}(\mathcal{O}_F) \to \mathbf{C}^\times$, we let $\mathbf{Q}\{\omega\}$ be the cokernel of the induced morphism $\omega_* : C \otimes \mathbf{Q} \to \mathbf{C}$. Then

$\mathbf{Q}\{\omega\}$ only depends upon the $\mathrm{Aut}(\mathbf{C})$-conjugacy class $\{\omega\}$ of ω, and $\mathbf{C}\otimes\mathbf{Q}\simeq\prod_{\{\omega\}}\mathbf{Q}\{\omega\}$. If A is endowed with an action of $\mathrm{Pic}(\mathcal{O}_F)$, then $\mathrm{End}^0(A)$ is a $C\otimes\mathbf{Q}$-algebra and we write

$$A\simeq\oplus_{\{\omega\}}A\{\omega\}$$

for the corresponding decomposition in the category $\mathbf{Ab}_F^0$ of abelian varieties over F up to isogenies. If $A=A\{\omega\}$ for some ω, then C acts on A through its quotient $\mathbf{Z}\{\omega\}$, the image of C in $\mathbf{Q}\{\omega\}$.

Recall from section 2 that the torsion subgroup G_0 of $G(\infty)=\mathrm{Gal}(K[P^\infty]/K)$ contains a subgroup G_2 which is canonically isomorphic to $\mathrm{Pic}(\mathcal{O}_F)$. For a character χ of $G(\infty)$ or G_0, we denote by $\mathrm{Res}(\chi)$ the induced character on $\mathrm{Pic}(\mathcal{O}_F)$. For a character χ of $G(n)$, we denote by $\mathbf{e}_\chi\in\mathbf{C}[G(n)]$ the idempotent of χ. We say that χ is *primitive* if it does not factor through $G(n-1)$.

Theorem 4.1 *Suppose that $\pi: J\rightsquigarrow A$ is a surjective, $\mathrm{Pic}(\mathcal{O}_F)$-equivariant and P-new morphism. Fix a character χ_0 of G_0 such that $A\{\omega\}\neq 0$ where $\omega^\epsilon=\mathrm{Res}(\chi)$. Then: for any $n\gg 0$ and any good CM point $x\in\mathrm{CM}(P^n)$, there exists a primitive character χ of $G(n)$ inducing χ_0 on G_0 such that $\mathbf{e}_\chi\alpha(x)\neq 0$ in $A\otimes\mathbf{C}$.*

Replacing π by $\pi\{\omega\}: J\rightsquigarrow A\rightsquigarrow A\{\omega\}$ and using Lemma 4.6 below, one easily checks that Theorem 4.1 is in fact equivalent to the following variant, in which we use ω to embed $\mathbf{Z}\{\omega\}$ into $\mathbf{C}$.

Theorem 4.2 *Suppose that $\pi: J\rightsquigarrow A$ is a surjective, $\mathrm{Pic}(\mathcal{O}_F)$-equivariant and P-new morphism. Suppose also that $A=A\{\omega\}\neq 0$ for some character ω of $\mathrm{Pic}(\mathcal{O}_F)$. Fix a character χ_0 of G_0 inducing ω^ϵ on $G_2\simeq\mathrm{Pic}(\mathcal{O}_F)$. Then: for any $n\gg 0$ and any good CM point $x\in\mathrm{CM}(P^n)$, there exists a primitive character χ of $G(n)$ inducing χ_0 on G_0 such that $\mathbf{e}_\chi\alpha(x)\neq 0$ in $A\otimes_{\mathbf{Z}\{\omega\}}\mathbf{C}$.*

Remark 4.3 In the situation of Theorem 1.10, $\epsilon=1$ and

$$A=J/\mathrm{Ann}_{\mathbf{T}}(\mathbf{C}\cdot\Phi)J$$

where $\mathbf{C}\cdot\Phi=\mathcal{S}_2(\pi')^H$, π' is an automorphic representation of G with central character $\omega: Z(\mathbf{A}_f)\twoheadrightarrow\mathrm{Pic}(\mathcal{O}_F)\to\mathbf{C}^\times$ and $H=\widehat{R}^\times$ with R defined by (6) – see remarks 3.1, 3.11 and 3.12. The assumptions **(H1)** and **(H2)** are then satisfied, and the projection $J\to A$ is surjective and $\mathrm{Pic}(\mathcal{O}_F)$-equivariant (with $A = A\{\omega\}$). By construction, there is no Eichler order in B_P strictly containing R_P whose group of invertible elements fixes Φ. It thus follows from the discussion after definition 3.13 that $J \to A$ also factors through the P-new quotient of J. Since $\epsilon = 1$, Theorem 4.1 asserts that for any character χ_0 of G_0 such that $\chi_0 \cdot \omega = 1$ on $Z(\mathbf{A}_f)$, and any good CM point $x \in \mathrm{CM}(P^n)$ with n sufficiently large, there exists a character $\chi \in P(n, \chi_0)$ such that

$$\mathbf{a}(x, \chi) = \mathbf{e}_{\chi^{-1}}\alpha(x) \neq 0 \quad \text{in } A(K[P^n]) \otimes \mathbf{C}.$$

This is exactly the statement of Theorem 1.10.

4.2 An easy variant

As an introduction to this circle of ideas, we will first show that a weaker variant of Theorem 4.1 can be obtained by very elementary methods, in the spirit of [19]. Thus, let $\pi : J \rightsquigarrow A$ be a nonzero surjective morphism, and put $\alpha = \pi \circ \iota : M \rightsquigarrow A$ where $\iota : M \rightsquigarrow J$ is the "Hodge embedding" of section 3.5. Then:

Proposition 4.4 *For all $n \gg 0$ and all $x \in \mathrm{CM}(P^n)$,*

$$\alpha(x) \neq 0 \quad \textit{in } A(K[P^n]) \otimes \mathbf{Q}.$$

Proof. Using Lemma 3.9, we may assume that $\alpha : M \to A$ is a finite morphism. In particular, there exists a positive integer d such that $|\alpha^{-1}(x)| \leq d$ for any $x \in A(\mathbf{C})$. On the other hand, it follows from Lemma 2.7 that the torsion subgroup of $A(K[P^\infty])$ is finite, say of order $t > 0$. Then α maps at most dt points in $\mathrm{CM}(P^\infty)$ to torsion points in A, and the proposition follows.

Corollary 4.5 *There exists a character $\chi : G(n) \to \mathbf{C}^\times$ such that*

$$\mathbf{e}_\chi \alpha(x) \neq 0 \quad \textit{in } A(K[P^n]) \otimes \mathbf{C}.$$

Suppose moreover that π is a $\mathrm{Pic}(\mathcal{O}_F)$-equivariant morphism. On $A(K[P^n]) \otimes \mathbf{C}$, we then also have an action of $\mathrm{Pic}(\mathcal{O}_F)$. For a character $\omega : \mathrm{Pic}(\mathcal{O}_F) \to \mathbf{C}^\times$, let $\mathbf{e}_\chi^\omega \alpha(x)$ be the ω-component of $\mathbf{e}_\chi \alpha(x)$. For any $\sigma \in \mathrm{Aut}(\mathbf{C})$, the automorphism $1 \otimes \sigma$ of $A(K[P^n]) \otimes \mathbf{C}$ maps $\mathbf{e}_\chi^\omega \alpha(x)$ to $\mathbf{e}_{\sigma\circ\chi}^{\sigma\circ\omega} \alpha(x)$: if the former is nonzero, so is the latter.

Lemma 4.6 $\mathbf{e}_\chi^\omega \alpha(x) \neq 0$ *unless* $\mathrm{Res}(\chi) = \omega^\epsilon$ *on* $G_2 \simeq \mathrm{Pic}(\mathcal{O}_F)$.

Proof. Write $x = [g]$ for some $g \in G(\mathbf{A}_f)$. For $\lambda \in Z(\mathbf{A}_f)$, put

$$\mathrm{Pic}(\mathcal{O}_F) \ni [\lambda] = \sigma = \mathrm{rec}_K(\lambda) \in G_2.$$

If ρ denotes the Galois action, we find that

$$\theta_M(\sigma)(x) = [g\lambda] = [\lambda g] = \rho(\sigma^\epsilon)(x) \quad \text{in } M(K[P^n]).$$

It follows that

$$\begin{array}{rrcll} & \theta_J(\sigma)(\iota x) & = & \rho(\sigma^\epsilon)(\iota x) & \text{in } J(K[P^n]) \otimes \mathbf{Q}, \\ & \theta_A(\sigma)(\alpha(x)) & = & \rho(\sigma^\epsilon)(\alpha(x)) & \text{in } A(K[P^n]) \otimes \mathbf{Q}, \\ \text{and} & \omega(\sigma) \cdot \mathbf{e}_\chi^\omega \alpha(x) & = & \chi(\sigma^\epsilon) \cdot \mathbf{e}_\chi^\omega \alpha(x) & \text{in } A(K[P^n]) \otimes \mathbf{C}. \end{array} \tag{14}$$

In particular, $\mathbf{e}_\chi^\omega \alpha(x) = 0$ if $\omega(\sigma) \neq \chi(\sigma^\epsilon)$ for some $\sigma \in G_2$.

We thus obtain the following refinement of Proposition 4.4.

Proposition 4.7 *Let* $\omega : \mathrm{Pic}(\mathcal{O}_F) \to \mathbf{C}^\times$ *be any character such that* $A\{\omega\} \neq 0$. *Then for all* $n \gg 0$ *and all* $x \in \mathrm{CM}(P^n)$, *there exists a character* $\chi : G(n) \to \mathbf{C}^\times$ *inducing* ω^ϵ *on* G_2 *such that*

$$\mathbf{e}_\chi \alpha(x) \neq 0 \quad \text{in } A(K[P^n]) \otimes \mathbf{C}.$$

Proof. Applying Proposition 4.4 to $\pi\{\omega\} : J \rightsquigarrow A \rightsquigarrow A\{\omega\}$, we find a character χ' on $G(n)$ such that $\mathbf{e}_{\chi'} \alpha(x) \neq 0$ in $A\{\omega\}(K[P^n]) \otimes \mathbf{C}$. Lemma 4.6 then implies that $\mathrm{Res}(\chi') = \sigma \cdot \omega^\epsilon$ for some $\sigma \in \mathrm{Aut}(\mathbf{C})$, and we take $\chi = \sigma^{-1} \circ \chi'$.

Remark 4.8 In contrast to Theorem 4.1, this proposition does not require π to be P-new, nor x to be good. It holds true without the assumptions **(H1)** and **(H2)**. On the other hand, Theorem 4.1 yields a *primitive* character whose *tame part* χ_0 is fixed (but arbitrary). This seems to entail a significantly deeper assertion on the growth of the Mordell-Weil groups of A along $K[P^\infty]/K$.

4.3 Proof of Theorem 4.2.

To prove that $\mathbf{e}_\chi \alpha(x)$ is nonzero for *some* primitive character χ of $G(n)$ inducing χ_0 on G_0, it is certainly sufficient to show that the *sum* of these values is a nonzero element in $A(K[P^n]) \otimes_{\mathbf{Z}\{\omega\}} \mathbf{C}$. Provided that n is sufficiently large, Lemma 2.8 implies that this sum is equal to

$$\mathbf{e}(\chi_0, n) \cdot \alpha(x) = \frac{1}{q\,|G_0|} \cdot \pi \left(\sum_{\sigma \in G_0} \overline{\chi}_0(\sigma)\sigma \cdot d(x) \right) \tag{15}$$

where $d(x) = (q - \mathrm{Tr}_{Z(n)})(\iota x)$. When x is a *good* CM point, $d(x)$ may be computed using the distribution relations of section 6.

Lemma 4.9 *Let δ be the exponent of P in the level of the Eichler order $R_P \subset B_P$. If n is sufficiently large, the following relations hold in the P-new quotient $J^{P-new} \otimes \mathbf{Q}$ of $J \otimes \mathbf{Q}$.*

1. *If $\delta = 0$, $d(x) = q \cdot \iota x - T_P^l \cdot \iota x' + \iota x''$ where $T_P^l \in \mathbf{T}$ is a certain Hecke operator, $x' = \mathrm{pr}_u(x)$ belongs to $\mathrm{CM}(P^{n-1})$ and $x'' = \mathrm{pr}_l(x')$ belongs to $\mathrm{CM}(P^{n-2})$.*
2. *If $\delta = 1$, $d(x) = q \cdot \iota x + \iota x'$ with $x' = \mathrm{pr}(x)$ in $\mathrm{CM}(P^{n-1})$.*
3. *If $\delta \geq 2$ and x is a* good *CM point, $d(x) = q \cdot \iota x$.*

Proof. We refer the reader to section 6 for the notations and proofs. Strictly speaking, we do not show there that x' belongs to $\mathrm{CM}(P^{n-1})$ and x'' belongs to $\mathrm{CM}(P^{n-2})$. This however easily follows from the construction of these points. Also, lemmas 6.6, 6.11 and 6.14 compute formulas involving the image of $\sum_{\lambda \in \mathcal{O}_{n-1}^\times / \mathcal{O}_n^\times} \mathrm{rec}_K(\lambda) \cdot x$ in the P-new quotient of the free abelian group $\mathbf{Z}[\mathrm{CM}]$. To retrieve the above formulas, use the discussion preceding Lemma 2.9 and the compatibility of ι with the formation of the P-new quotients of $\mathbf{Z}[\mathrm{CM}]$ and J.

Since $\pi : J \rightsquigarrow A$ is P-new, these relations also hold in $A \otimes \mathbf{Q}$. In particular, for $\delta \geq 2$, part (3) of the above lemma implies that Theorem 4.2 is now a consequence of the following theorem, whose proof will be given in sections 4.5-4.6.

Theorem 4.10 *Suppose that $\pi : J \rightsquigarrow A$ is a surjective,* $\mathrm{Pic}(\mathcal{O}_F)$*-equivariant morphism such that $A = A\{\omega\} \neq 0$ for some character ω of* $\mathrm{Pic}(\mathcal{O}_F)$*. Fix a character χ of G_0 inducing ω^ϵ on $G_2 \simeq$* $\mathrm{Pic}(\mathcal{O}_F)$*. Then for all but finitely many $x \in$* $\mathrm{CM}(P^\infty)$*,*

$$\mathbf{e}_\chi \alpha(x) = \frac{1}{|G_0|} \sum_{\sigma \in G_0} \overline{\chi}(\sigma)\sigma \cdot \alpha(x) \neq 0 \quad \textit{in } A \otimes_{\mathbf{Z}\{\omega\}} \mathbf{C}.$$

Remark 4.11 This is the statement which is actually used in [14].

In the next subsection, we will show that theorem 4.2 also follows from the above theorem when $\delta = 0$ or 1, provided that we change the original P-new parameterization $\pi : J \rightsquigarrow A$ of A to a non-optimal parameterization $\pi^+ : J^+ \rightsquigarrow A$. Although this new parameterization will still satisfy to the assumptions **(H1)** and **(H2)**, the proof of Theorem 4.10 only requires **(H2)** to hold.

4.4 Changing the level (from $\delta = 0$ or 1 to $\delta = 2$)

Suppose first that $\delta = 0$. Let $R_P^+ \subset R_P$ be the Eichler order of level P^2 in B_P which is constructed in section 6.5. Put $H^+ = H^P R_P^+$, $M^+ = M_{H^+}$, $J^+ = J_{H^+}$ and so on. By Lemma 6.16, there exists degeneracy maps d_0, d_1 and $d_2 : M^+ \to M$ as well as an element $\vartheta \in C^\times$ with the property that for all $x \in \mathrm{CM}(P^n)$ with $n \geq 2$, there exists a CM point $x^+ \in \mathrm{CM}^+(P^n)$ such that

$$(d_0, d_1, d_2)(x^+) = (x, x', \vartheta^{-1}x'').$$

Combining this with part (1) of Lemma 4.9 (and using also the results of section 3.5, especially formula 13 and Lemma 3.6) we obtain:

$$d(x) = \left(q(d_0)_* - T_P^l(d_1)_* + \vartheta(d_2)_*\right)(\iota^+ x^+) \quad \text{in } J \otimes \mathbf{Q}$$

so that $\pi \circ d(x) = \alpha^+(x^+)$ in $A \otimes \mathbf{Q}$, where $\alpha^+ = \pi^+ \circ \iota^+$ with

$$\pi^+ = \pi \circ (q, -T_P^l, \vartheta) \circ \begin{pmatrix} d_0 \\ d_1 \\ d_2 \end{pmatrix}_* : J^+ \to J^3 \to J \rightsquigarrow A.$$

In particular, (15) becomes

$$\mathbf{e}(\chi_0, n) \cdot \alpha(x) = \frac{1}{q\,|G_0|} \cdot \sum_{\sigma \in G_0} \overline{\chi}_0(\sigma)\sigma \cdot \alpha^+(x^+) \quad \text{in } A \otimes_{\mathbf{Z}\{\omega\}} \mathbf{C}.$$

Theorem 4.2 for π thus follows from theorem 4.10 for π^+, once we know that our new parameterization $\pi^+ : J^+ \rightsquigarrow A$ is surjective and $\mathrm{Pic}(\mathcal{O}_F)$-equivariant (these are the assumptions of theorem 4.10). The $\mathrm{Pic}(\mathcal{O}_F)$-equivariance is straightforward. Since the second and third morphisms in the definition of π^+ are surjective, it remains to show that the first one is also surjective, which amounts to showing that the induced map on the (complex) cotangent spaces at 0 is an injection. In view of Proposition 3.10, this all boils down to the following lemma.

Lemma 4.12 *The kernel of* $(d_0^*, d_1^*, d_2^*) : (\mathcal{S}_2^H)^3 \to \mathcal{S}_2^{H^+}$ *is trivial.*

Proof. With notations as in section 6.5, the above map is given by

$$(F_0, F_1, F_2) \mapsto F_0' + F_1' + F_2'$$

where $F_i'(g) = d_i^* F_i(g) = F_i(g b_i)$ with $b_i \in B_P^\times$ such that $b_i L(0) = L(2-i)$. Here, $L = (L(0), L(2))$ is a 2-lattice in some simple left $B_P \simeq M_2(F_P)$-module $V \simeq F_P^2$ such that $R_P = \{\alpha \in B_P; \alpha L(0) \subset L(0)\}$. Put

$$R_i = \{\alpha \in B_P;\ \alpha L(2-i) \subset L(2-i)\}$$

so that $R_2 = R_P$ and $R_i^\times = b_i R_2^\times b_i^{-1}$ fixes F_i'. One easily checks that $R_0^\times \cap R_1^\times$ and $R_1^\times \cap R_2^\times$ generate $R_1^\times$ inside $B_P^\times$. By [21, Chapter 2, section 1.4], $R_0^\times$ and $R_1^\times$ (resp. $R_1^\times$ and $R_2^\times$) generate the subgroup $(B_P^\times)^0$ of all elements in $B_P^\times \simeq \mathrm{GL}_2(F_P)$ whose reduced norm (=determinant) belongs to $\mathcal{O}_{F,P}^\times \subset F_P^\times$.

Suppose that $F_0' + F_1' + F_2' \equiv 0$ on $G(\mathbf{A})$. Then $F_2' = -F_0' - F_1'$ is fixed by $R_2^\times$ and $R_0^\times \cap R_1^\times$ and therefore also by $(B_P^\times)^0$. Being continuous, left invariant under $G(\mathbf{Q})$ and right invariant under $(B_P^\times)^0 H^P$, the function $F_2' : G(\mathbf{A}) \to \mathbf{C}$ is then also left (and right) invariant under the kernel $G^1(\mathbf{A})$ of the reduced norm $\mathrm{nr} : G(\mathbf{A}) = \widehat{B}^\times \to \widehat{F}^\times$ by the strong approximation theorem [26, p. 81]. For any $g \in G(\mathbf{A})$ and $\theta \in \mathbf{R}$, we thus obtain (using the property **P4** of section 3.6)

$$F_2'(g) = F_2'\left(g \times \left(\begin{pmatrix} \cos\theta & \epsilon \sin\theta \\ -\epsilon \sin\theta & \cos\theta \end{pmatrix}, 1, \cdots, 1\right)\right) = e^{2i\theta} F_2'(g),$$

so that $F_2' \equiv 0$ on $G(\mathbf{A})$. Similarly, $F_0' \equiv 0$ on $G(\mathbf{A})$. It follows that $F_1' \equiv 0$, hence $F_0 \equiv F_1 \equiv F_2 \equiv 0$ on $G(\mathbf{A})$ and (d_0^*, d_1^*, d_2^*) is indeed injective.

Suppose next that $\delta = 1$. Using now Lemma 6.17, we find two degeneracy maps d_{01} and $d_{12} : M^+ \to M$, as well as an F-automorphism ϑ of M such that for all $x \in \mathrm{CM}(P^n)$ with $n \geq 2$, there exists a CM point $x^+ \in \mathrm{CM}^+(P^n)$ such that (x, x') equals

$$(d_{01}, \vartheta^{-1} d_{12})(x^+) \quad \text{or} \quad (d_{12}, \vartheta d_{01})(x^+).$$

Theorem 4.2 (with π) thus again follows from Theorem 4.10 with

$$\pi' = \pi \circ \left\{ \begin{array}{c} (q, \vartheta_*^{-1}) \\ \text{or} \\ (\vartheta_*, q) \end{array} \right\} \circ \left(\begin{array}{c} d_{01} \\ d_{12} \end{array} \right)_* : J^+ \to J^2 \to J \rightsquigarrow A$$

once we know that π' induces an injection on the complex cotangent spaces. Since π is P-new, this now amounts to the following lemma (see section 3.7).

Lemma 4.13 *The kernel of* $(d_{01}^*, d_{12}^*) : (\mathcal{S}_{2,P-new}^H)^2 \to \mathcal{S}_2^{H^+}$ *is trivial.*

Proof. With notations as in section 6.5, the above map is now given by

$$(F_{01}, F_{12}) \mapsto F'_{01} + F'_{12}$$

where $F'_{01} = F_{01}$ and $F'_{12}(g) = F_{12}(g b_{12})$ with $b_{12} \in B_P^\times$ such that $b_{12}(L(0), L(1)) = (L(1), L(2))$ for some 2-lattice $L = (L(0), L(2))$ in V such that $R_P^\times = R_0^\times \cap R_1^\times$ with

$$R_i = \{\alpha \in B_P;\, \alpha L(i) \subset L(i)\}.$$

If $F'_{01} + F'_{12} = 0$, $F'_{01} = -F'_{12}$ is fixed by $R_0^\times \cap R_1^\times$ and $R_1^\times \cap R_2^\times$. It is therefore also fixed by $R_1^\times$ so that F_{01} and F_{12} both belong to the P-old subspace of $\mathcal{S}_2^H$.

4.5 Geometric Galois action

We now turn to the proof of Theorem 4.10. Thus, let $\pi : J \rightsquigarrow A$ be a surjective and $\mathrm{Pic}(\mathcal{O}_F)$-equivariant morphism such that $A = A\{\omega\} \neq 0$ for some character ω of $\mathrm{Pic}(\mathcal{O}_F)$, and let χ be a fixed character of G_0 inducing ω^ϵ on $G_2 \simeq \mathrm{Pic}(\mathcal{O}_F)$.

For $0 \leq i \leq 2$, let C_i be the subring of $\mathbf{C}$ which is generated by the values of χ on G_i, so that $C_2 \subset C_1 \subset C_0$, C_1 is finite flat over C_2, and so is C_0 over C_1. Since χ induces ω^ϵ on G_2, the canonical factorization of

$\omega_* : C \to \mathbf{C}$ yields an isomorphism between $\mathbf{Z}\{\omega\}$ and C_2. Let A_i be the (nonzero) abelian variety over F which is defined by

$$A_i(X) = A(X) \otimes_C C_i = A(X) \otimes_{\mathbf{Z}\{\omega\}} C_i$$

for any F-scheme X (note that $A_2 \simeq A$).

Upon multiplying π by a suitable integer, we may assume that π and α are genuine morphisms. For any $x \in \mathrm{CM}(P^n)$, we may then view

$$\mathbf{a}(x) \stackrel{\mathrm{def}}{=} \sum_{\sigma \in G_0} \overline{\chi}(\sigma)\sigma \cdot \alpha(x)$$

as an element of

$$A_0(K[P^\infty]) = A_1(K[P^\infty]) \otimes_{C_1} C_0 = A(K[P^\infty]) \otimes_{\mathbf{Z}\{\omega\}} C_0,$$

and we now have to show that $\mathbf{a}(x)$ is a *nontorsion* element in this group, provided that n is sufficiently large.

Using formula (14), which applies thanks to the $\mathrm{Pic}(\mathcal{O}_F)$-equivariance of π, we immediately find that

$$\mathbf{a}(x) = |G_2| \sum_{\sigma \in \mathcal{R}'} \overline{\chi}(\sigma)\sigma \cdot \alpha(x)$$

where $\mathcal{R}' \subset G_0$ is the following set of representatives for G_0/G_2. We first choose a set of representatives $\mathcal{R} \subset G_0$ of G_0/G_1 containing 1, and then take

$$\mathcal{R}' = \{\tau\sigma_D^\epsilon;\ \tau \in \mathcal{R} \text{ and } D \mid \mathcal{D}'\}$$

where $\mathcal{D}' \subset \mathcal{O}_F$ and the σ_D's for $D \mid \mathcal{D}'$ were defined in Lemma 2.6. The next lemma will allow us to further simplify $\mathbf{a}(x)$.

Lemma 4.14 *There exists a Shimura curve M_1 and a collection of degeneracy maps $\{\mathbf{d}_D : M_1 \to M;\ D \mid \mathcal{D}'\}$ such that for all $n \geq 0$,*

$$\forall x \in \mathrm{CM}(P^n),\ \exists x_1 \in \mathrm{CM}_1(P^n) \text{ s.t. } \forall D \mid \mathcal{D}' : \quad \sigma_D^\epsilon x = \mathbf{d}_D(x_1).$$

Proof. Our assumption **(H2)** asserts that for any $Q \mid \mathcal{D}'$, R_Q is a maximal order in $B_Q \simeq M_2(F_Q)$. Let Γ_Q be the set of elements in R_Q whose reduced norm is a uniformizer in $\mathcal{O}_{F,Q}$, and choose some α_Q in Γ_Q. Then $\Gamma_Q = R_Q^\times \alpha_Q R_Q^\times$ and

$$R_{1,Q} \stackrel{\mathrm{def}}{=} R_Q \cap \alpha_Q R_Q \alpha_Q^{-1} \subset B_Q$$

is an Eichler order of level Q. Put $H_1 = \widehat{R}_1^\times$, where R_1 is the unique $\mathcal{O}_F$-order in B which agrees with R outside $\mathcal{D}'$, and equals $R_{1,Q}$ at $Q \mid \mathcal{D}'$. Put $M_1 = M_{H_1}$, $\mathrm{CM}_1 = \mathrm{CM}_{H_1}$ and so on. For $D \mid \mathcal{D}'$, put

$$\alpha_D \stackrel{\text{def}}{=} \prod_{Q \mid D} \alpha_Q \in G(\mathbf{A}_f). \tag{16}$$

Then $\alpha_D^{-1} H_1 \alpha_D \subset H$. Let $\mathbf{d}_D = [\cdot \alpha_D] : M_1 \to M$ be the corresponding degeneracy map.

Recall also that $\sigma_D = \prod_{Q \mid D} \sigma_Q$ for $D \mid \mathcal{D}'$, where $\sigma_Q \in G_1$ is the geometric Frobenius of the unique prime $\mathcal{Q}$ of K above Q (so that $\mathcal{Q}^2 = Q\mathcal{O}_K$). Let $\pi_Q \in \mathcal{O}_{K,Q}$ be a local uniformizer at $\mathcal{Q}$, and for $D \mid \mathcal{D}'$, put $\pi_D = \prod_{Q \mid D} \pi_Q$ in $\widehat{K}^\times$, so that σ_D is the restriction of $\mathrm{rec}_K(\pi_D)$ to $K[P^\infty]$.

Consider now some $x = [g] \in \mathrm{CM}(P^n)$, with $g \in G(\mathbf{A}_f)$ and $n \geq 0$. For each $Q \mid \mathcal{D}'$, $K_Q \cap g_Q R_Q g_Q^{-1} = \mathcal{O}_{K,Q}$. In particular, π_Q belongs to $g_Q R_Q g_Q^{-1}$ and $g_Q^{-1} \pi_Q g_Q$ belongs to Γ_Q: there exists $r_{1,Q}$ and $r_{2,Q}$ in $R_Q^\times$ such that $\pi_Q g_Q = g_Q r_{1,Q} \alpha_Q r_{2,Q}$. Put $r_i = \prod_{Q \mid \mathcal{D}'} r_{i,Q} \in H$ and $x_1 = [g r_1] \in \mathrm{CM}_1$. For $D \mid \mathcal{D}'$, we find that

$$\mathbf{d}_D(x_1) = [g r_1 \alpha_D] = [g r_1 \alpha_D r_2] = [\pi_D g_Q] = \sigma_D^\epsilon x.$$

Finally, x_1 belongs to $\mathrm{CM}_1(P^n)$, because $\widehat{K}^\times \cap g r_1 H_1 (g r_1)^{-1}$ is obviously equal to $\widehat{K}^\times \cap g H g^{-1}$ away from $\mathcal{D}'$, and

$$\begin{aligned}
&K_Q \cap g_Q r_{1,Q} R_{1,Q}^\times (g_Q r_{1,Q})^{-1} = \\
&\quad = \quad K_Q \cap g_Q r_{1,Q} R_Q^\times (g_Q r_{1,Q})^{-1} \cap g_Q r_{1,Q} \alpha_Q R_Q^\times \alpha_Q^{-1} (g_Q r_{1,Q})^{-1} \\
&\quad = \quad \left(K_Q \cap g_Q R_Q^\times g_Q^{-1}\right) \cap \left(K_Q \cap \pi_Q g_Q R_Q^\times g_Q^{-1} \pi_Q^{-1}\right) \\
&\quad = \quad K_Q \cap g_Q R_Q^\times g_Q^{-1}
\end{aligned}$$

for $Q \mid \mathcal{D}'$. This finishes the proof of Lemma 4.14.

Put $J_1 = \mathrm{Pic}^0 M_1$ and let $\iota_1 : M_1 \rightsquigarrow J_1$ be the corresponding "Hodge embedding". With notations as above, we find that $\mathbf{a}(x) = |G_2|\, \mathbf{b}(x_1)$, where for any CM point $y \in \mathrm{CM}_1(P^\infty)$,

$$\mathbf{b}(y) \stackrel{\text{def}}{=} \sum_{\tau \in \mathcal{R}} \overline{\chi}(\tau) \tau \cdot \alpha_1(y) \quad \text{in } A_1(K[P^\infty]) \otimes_{C_1} C_0.$$

Here, $\alpha_1 \stackrel{\mathrm{def}}{=} \pi_1 \circ \iota_1$ with $\pi_1 : J_1 \to A_1$ defined by

$$\begin{array}{ccccccc} J_1 & \longrightarrow & J^{\{D|\mathcal{D}\}} & \xrightarrow{\pi} & A^{\{D|\mathcal{D}\}} & \longrightarrow & A_1 = A \otimes_C C_1 \\ j & \longmapsto & ((\mathbf{d}_D)_*(j))_{D|\mathcal{D}} & & (a_D)_{D|\mathcal{D}} & \longmapsto & \sum_{D|\mathcal{D}} \overline{\chi}(\sigma_D^\epsilon) a_D \end{array}$$

Indeed, Lemma 4.14 (together with the formula (13)) implies that

$$\alpha_1(x_1) = \sum_{D|\mathcal{D}} \overline{\chi}(\sigma_D^\epsilon)\sigma_D^\epsilon \cdot \alpha(x) \quad \text{in } A(K[P^\infty]) \otimes_C C_1 = A_1(K[P^\infty]).$$

We now have to show that for all $x \in \mathrm{CM}_1(P^n)$ with $n \gg 0$,

$$\mathbf{b}(x) \neq \text{torsion} \quad \text{in } A_1(K[P^\infty]) \otimes_{C_1} C_0.$$

We will need to know that our new parameterization $\pi_1 : J_1 \to A_1$ is still surjective. As before (lemmas 4.12 and 4.13), this amounts to the following lemma.

Lemma 4.15 *The kernel of* $\sum_{D|\mathcal{D}'} \mathbf{d}_D^* : (\mathcal{S}_2^H)^{\{D|\mathcal{D}'\}} \to \mathcal{S}_2^{H_1}$ *is trivial.*

Proof. We retain the notations of the proof of Lemma 4.14. The map under consideration is given by

$$(F_D)_{D|\mathcal{D}'} \mapsto \textstyle\sum_{D|\mathcal{D}'} \alpha_D \cdot F_D$$

where $(\alpha_D \cdot F)(g) = F(g\alpha_D)$ for any $F : G(\mathbf{A}) \to \mathbf{C}$ and $g \in G(\mathbf{A})$. We show that it is injective by induction on the number of prime divisors of $\mathcal{D}'$. There is nothing to prove if $\mathcal{D}' = \mathcal{O}_F$. Otherwise, let Q be a prime divisor of $\mathcal{D}'$. We put $\mathcal{D}'_Q = \mathcal{D}'/Q$ and $H'_1 = \widehat{R'_1}^\times$, where R'_1 is the unique $\mathcal{O}_F$-order in B which agrees with R_1 outside Q, and equals R_Q at Q. The functions

$$\mathcal{F}_0 = \sum_{D|\mathcal{D}'_Q} \alpha_D \cdot F_D \quad \text{and} \quad \mathcal{F}_1 = \sum_{D|\mathcal{D}'_Q} \alpha_D \cdot F_{DQ}$$

then belong to $\mathcal{S}_2^{H'_1}$, and

$$\sum_{D|\mathcal{D}'} \alpha_D \cdot F_D = \mathcal{F}_0 + \alpha_Q \cdot \mathcal{F}_1.$$

If this function is trivial on $G(\mathbf{A})$, $\mathcal{F}_0 = -\alpha_Q \cdot \mathcal{F}_1$ is fixed by $R_Q^\times$ and $\alpha_Q R_Q^\times \alpha_Q^{-1}$. Arguing as in the proof of Lemma 4.12, we obtain $\mathcal{F}_0 \equiv \mathcal{F}_1 \equiv 0$ on $G(\mathbf{A})$. By induction, $F_D \equiv 0$ on $G(\mathbf{A})$ for all $D \mid \mathcal{D}'$.

4.6 Chaotic Galois action

We still have to show that for all but finitely many x in $\mathrm{CM}_1(P^\infty)$, $\mathbf{b}(x)$ is a nontorsion point in $A_0(K[P^\infty])$. Two proofs of this fact may be extracted from the results of [5]. These proofs are both based upon the following elementary observations:

- The torsion submodule of $A_0(K[P^\infty])$ is *finite*.

This easily follows from Lemma 2.7. We thus want $\mathbf{b}(x)$ to land away from a given finite set, provided that x belongs to $\mathrm{CM}_1(P^n)$ with $n \gg 0$.

- The map $x \mapsto \mathbf{b}(x)$ may be decomposed as follows:

$$\begin{array}{ccccccc} \mathrm{CM}_1(P^\infty) & \xrightarrow{\Delta} & M_1^{\mathcal{R}} & \xrightarrow{\alpha_1} & A_1^{\mathcal{R}} & \xrightarrow{\Sigma} & A_0 \\ x & \longmapsto & (\sigma x)_{\sigma\in\mathcal{R}} & & (a_\sigma)_{\sigma\in\mathcal{R}} & \mapsto & \sum \overline{\chi}(\sigma) a_\sigma \end{array}$$

In this decomposition, the second and third maps are algebraic morphisms defined over F. Moreover: Σ is surjective (this easily follows from the definitions) and α_1 is finite (by lemmas 4.15 and 3.9). In some sense, this decomposition separates the geometrical and arithmetical aspects in the definition of $\mathbf{b}(x)$.

First proof (using a proven case of the André-Oort conjecture).

Suppose that $\mathbf{b}(x)$ is a torsion point in $A_0(K[P^\infty])$ for infinitely many $x \in \mathrm{CM}_1(P^\infty)$. We may then find some element a_0 in $A_0(\mathbf{C})$ such that $\mathcal{E} = \mathbf{b}^{-1}(a_0)$ is an infinite subset of $\mathrm{CM}_1(P^\infty)$. Since $\Sigma \circ \alpha_1 : M_1^{\mathcal{R}} \to A_0$ is an algebraic morphism, it is continuous for the Zariski topology and $(\Sigma\circ\alpha_1)^{-1}(a_0)$ contains the Zariski closure $\overline{\Delta(\mathcal{E})}^{\mathrm{Zar}}$ of $\Delta(\mathcal{E})$ in $M_1^{\mathcal{R}}(\mathbf{C})$. As explained in Remark 3.21 of [5], a proven case of the André-Oort conjecture implies that $\overline{\Delta(\mathcal{E})}^{\mathrm{Zar}}$ contains a connected component of $M_1^{\mathcal{R}}(\mathbf{C})$.

Remark 4.16 The above reference requires $\mathcal{E}$ to be an infinite collection of *P-isogenous* CM points, where two CM points x and x' are said to be P-isogenous if they can be represented by g and $g' \in G(\mathbf{A}_f)$ with $g_v = g'_v$ for all $v \neq P$. Now, if a P-isogeny class contains a CM point of conductor P^n for some $n \geq 0$, it is actually contained in $\mathrm{CM}_1(P^\infty)$ and any other P-isogeny class in $\mathrm{CM}_1(P^\infty)$ also contains a point of conductor P^n. Since $\mathrm{CM}_1(P^n)$

is finite, we thus see that $\mathrm{CM}_1(P^\infty)$ is the disjoint union of *finitely* many P-isogeny classes, and one of them at least has infinite intersection with our infinite set $\mathcal{E}$.

We thus obtain a collection of connected components $(\mathcal{C}_\sigma)_{\sigma\in\mathcal{R}}$ of $M_1(\mathbf{C})$ with the property that for all $(x_\sigma)_{\sigma\in\mathcal{R}}$ in $\prod_{\sigma\in\mathcal{R}}\mathcal{C}_\sigma$,

$$\sum_{\sigma\in\mathcal{R}} \overline{\chi}(\sigma)\alpha_1(x_\sigma) = a_0.$$

It easily follows that α_1 should then be constant on $\mathcal{C}_1$. Being defined over F on the *connected* curve M_1 (cf. remark 3.2), α_1 would then be constant on M_1, a contradiction.

Second proof (using a theorem of M. Ratner).

Let U be a nonempty open subscheme of $\mathrm{Spec}(\mathcal{O}_F)$ such that for every closed point $v \in U$, $v \neq P$, $B_v \simeq M_2(F_v)$ and $R_{1,v}$ is maximal. Shrinking U if necessary, we may assume that M_1 has a proper and smooth model $\mathbf{M}_1$ over U, which agrees locally with the models considered in [5], and $\alpha_1 : M_1 \to A_1$ extends uniquely to a *finite* morphism $\alpha_1 : \mathbf{M}_1 \to \mathbf{A}_1$, where $\mathbf{A}_1$ is the Néron model of A_1 over U. In the Stein factorization

$$\mathbf{M}_1 \xrightarrow{c} \boldsymbol{\mathcal{M}}_{H_1} \to U$$

of the stuctural morphism $\mathbf{M}_1 \to U$, the *scheme of connected component* $\boldsymbol{\mathcal{M}}_1$ is then a finite and étale cover of U, and the fibers of c are geometrically connected. For each closed point $v \in U$, we choose a place $\overline{v}$ of $\overline{F} \subset \mathbf{C}$ above v, with valuation ring $\mathcal{O}(\overline{v})$ and residue field $\mathbf{F}(\overline{v})$, an algebraic closure of the residue field $\mathbf{F}(v)$ of v. We thus obtain the following diagram of reduction maps

$$\begin{array}{rccccc} \mathrm{red}_v: & M_1(\overline{F}) & \xleftarrow{\simeq} & \mathbf{M}_1(\mathcal{O}(\overline{v})) & \longrightarrow & \mathbf{M}_1(\mathbf{F}(\overline{v})) \\ & \downarrow c & & \downarrow c & & \downarrow c \\ \mathrm{red}_v: & \boldsymbol{\mathcal{M}}_1(\overline{F}) & \xleftarrow{\simeq} & \boldsymbol{\mathcal{M}}_1(\mathcal{O}(\overline{v})) & \xrightarrow{\simeq} & \boldsymbol{\mathcal{M}}_1(\mathbf{F}(\overline{v})). \end{array}$$

We denote by $\mathcal{C} \mapsto \mathcal{C}(v)$ the induced bijection between the sets of geometrical connected components in the generic and special fibers, and we denote by $\mathcal{C}_x$ the connected component of $x \in M_1(\overline{F})$. In particular, $\mathcal{C}_x(v) = c^{-1}(\mathrm{red}_v c(x))$ is the connected component of $\mathrm{red}_v(x)$. Inside $\mathbf{M}_1(\mathbf{F}(\overline{v}))$, there is a finite collection of distinguished points, namely the *supersingular points* as described in Section 3.1.3 of [5]. We denote by $\mathcal{C}^{\mathrm{ss}}(v)$ the set of supersingular points inside $\mathcal{C}(v)$.

We let $d > 0$ be a uniform upper bound on the number of geometrical points in the fibers of $\alpha_1 : \mathbf{M}_1 \to \mathbf{A}_1$ (such a bound does exist, thanks to the generic flatness theorem, see for instance [11, Corollaire 6.9.3]). We let $t > 0$ be the order of the torsion subgroup of $A_0(K[P^\infty])$. One easily checks that the order of $\mathcal{C}^{\mathrm{ss}}(v)$ goes to infinity with the order of the residue field $\mathbf{F}(v)$ of v. Shrinking U if necessary, we may therefore assume that for all $\mathcal{C}$ and v,

$$|\mathcal{C}^{\mathrm{ss}}(v)| > td.$$

Now, let v be a closed point of U which is *inert* in K (there are infinitely many such points). Then Lemma 3.1 of [5] states that any CM point $x \in \mathrm{CM}_1$ reduces to a supersingular point $\mathrm{red}_v(x) \in \mathcal{C}_x^{\mathrm{ss}}(v)$ and we have the following crucial result.

Proposition 4.17 *For all but finitely many x in $\mathrm{CM}_1(P^\infty)$, the following property holds. For any $(z_\sigma)_{\sigma\in\mathcal{R}}$ in $\prod_{\sigma\in\mathcal{R}} \mathcal{C}_{\sigma x}^{\mathrm{ss}}(v)$, there exists some $\gamma \in \mathrm{Gal}_K^{\mathrm{ab}}$ such that*

$$\forall \sigma \in \mathcal{R} : \quad \mathrm{red}_v(\gamma\sigma \cdot x) = z_\sigma \quad \textit{in } \mathbf{M}_1(\mathbf{F}(\overline{v})).$$

Proof. This is a special case of Corollary 2.10 of [5], except that the latter deals with *P-isogeny classes* of CM points instead of the set of all CM points of P-power conductor that we consider here. However, we have already observed in remark 4.16 that $\mathrm{CM}_1(P^\infty)$ is the disjoint union of finitely many such P-isogeny classes, and the proposition follows.

Corollary 4.18 *For all but finitely many x in $\mathrm{CM}_1(P^\infty)$, the following property holds. For any z in $\mathcal{C}_x^{\mathrm{ss}}(v)$, there is a $\gamma \in \mathrm{Gal}_K^{\mathrm{ab}}$ such that*

$$\mathrm{red}_v(\gamma \cdot \mathbf{b}(x)) - \mathrm{red}_v(\mathbf{b}(x)) = \alpha_1(z) - \alpha_1(z_1)$$

in $\mathbf{A}_0(\mathbf{F}(\overline{v})) = \mathbf{A}_1(\mathbf{F}(\overline{v})) \otimes_{C_1} C_0$, with $z_1 = \mathrm{red}_v(x) \in \mathcal{C}_x^{\mathrm{ss}}(v)$.

Proof. Take $z_\sigma = \mathrm{red}_v(\sigma x)$ for $\sigma \neq 1$ in $\mathcal{R}$.

This finishes the proof of Theorem 4.10. Indeed, the Galois orbit of any torsion point in $A_0(K[P^\infty])$ has at most t elements, while the above corollary implies that for all but finitely many $x \in \mathrm{CM}_1(P^\infty)$,

$$\left|\mathrm{Gal}_K^{\mathrm{ab}} \cdot \mathbf{b}(x)\right| \geq \left|\mathrm{red}_v\left(\mathrm{Gal}_K^{\mathrm{ab}} \cdot \mathbf{b}(x)\right)\right| \geq \left|\alpha_1\left(\mathcal{C}_x^{\mathrm{ss}}(v)\right)\right| \geq \frac{1}{d}\left|\mathcal{C}_x^{\mathrm{ss}}(v)\right| > t.$$

5 The definite case

Suppose now that B is a *definite* quaternion algebra, so that $B \otimes \mathbf{R} \simeq \mathbf{H}^{[F:\mathbf{Q}]}$. Let K be a totally imaginary quadratic extension of F contained in B. We put $G = \mathrm{Res}_{F/\mathbf{Q}}(B^\times)$, $T = \mathrm{Res}_{F/\mathbf{Q}}(K^\times)$ and $Z = \mathrm{Res}_{F/\mathbf{Q}}(F^\times)$ as before.

5.1 Automorphic forms and representations

We denote by $\mathcal{S}_2$ the space of all *weight* 2 *automorphic forms* on G, namely the space of all smooth (=locally constant) functions

$$\theta : G(\mathbf{Q})\backslash G(\mathbf{A}_f) \to \mathbf{C}.$$

There is an admissible left action of $G(\mathbf{A}_f)$ on $\mathcal{S}_2$, given by right translations: for $g \in G(\mathbf{A}_f)$ and $x \in G(\mathbf{Q})\backslash G(\mathbf{A}_f)$,

$$(g \cdot \theta)(x) = \theta(xg).$$

This representation is semi-simple, and $\mathcal{S}_2$ is the algebraic direct sum of its irreducible subrepresentations. An irreducible representation π' of $G(\mathbf{A}_f)$ is *automorphic* if it occurs in $\mathcal{S}_2$. It then occurs with multiplicity one, and we denote by $\mathcal{S}_2(\pi')$ the corresponding subspace of $\mathcal{S}_2$, so that

$$\mathcal{S}_2 = \oplus_{\pi'} \mathcal{S}_2(\pi').$$

If π' is finite dimensional, it is of dimension 1 and corresponds to a smooth character χ of $G(\mathbf{A}_f)$. Then χ is trivial on $G(\mathbf{Q})$, $\mathcal{S}_2(\pi')$ equals $\mathbf{C} \cdot \chi$, and χ factors through the reduced norm $G(\mathbf{A}_f) \to Z(\mathbf{A}_f)$. A function $\theta \in \mathcal{S}_2$ is said to be *Eisenstein* if it belongs to the subspace spanned by these finite dimensional subrepresentations of $\mathcal{S}_2$. Equivalently, θ is an Eisenstein function if and only if it factors through the reduced norm (because any such function spans a finite dimensional $G(\mathbf{A}_f)$-invariant subspace of $\mathcal{S}_2$).

We say that π' is *cuspidal* if its representation space has infinite dimension. The space of (weight 2) *cuspforms* $\mathcal{S}_2^0$ is the $G(\mathbf{A}_f)$-invariant subspace of $\mathcal{S}_2$ which is spanned by its irreducible cuspidal subrepresentations. Thus, $\theta = 0$ is the only cuspform which is also Eisenstein.

5.2 The exceptional case

The Jacquet-Langlands correspondence assigns, to every cuspidal representation π' of $G(\mathbf{A}_f)$ as above, an irreducible automorphic representation $\pi = \mathrm{JL}(\pi')$ of GL_2/F, of weight $(2, \cdots, 2)$. We say that (π', K) is *exceptional* if (π, K) is exceptional. Thus, (π', K) is exceptional if and only if $\pi \simeq \pi \otimes \eta$, where η is the quadratic character attached to K/F. We want now to describe a simple characterization of these exceptional cases.

Write $\pi = \otimes \pi_v$, $\pi' = \otimes \pi'_v$ and let $\mathcal{N}$ be the conductor of π. For every finite place v of F not dividing $\mathcal{N}$,

$$\pi_v \simeq \pi'_v \simeq \pi(\mu_{1,v}, \mu_{2,v}) \simeq \pi(\mu_{2,v}, \mu_{1,v})$$

for some unramified characters $\mu_{i,v} : F_v^\times \to \mathbf{C}^\times$, $i = 1, 2$. These characters are uniquely determined by $\beta_{i,v} = \mu_{i,v}(\varpi_v)$, where ϖ_v is any local uniformizer at v, and by the strong multiplicity one theorem, the knowledge of all but finitely many of the unordered pairs $\{\beta_{1,v}, \beta_{2,v}\}$ uniquely determines π and π'. On the other hand, the representation space $\mathcal{S}(\pi_v)$ of π_v contains a unique line $\mathbf{C} \cdot \phi_v$ of vectors which are fixed by the maximal compact open subgroup $H_v = \mathrm{GL}_2(\mathcal{O}_{F,v})$ of $G_v = \mathrm{GL}_2(F_v) \simeq B_v^\times$. The spherical Hecke algebra

$$\mathrm{End}_{\mathbf{Z}[G_v]}(\mathbf{Z}[G_v/H_v]) \simeq \mathbf{Z}[H_v \backslash G_v / H_v]$$

acts on $\mathbf{C} \cdot \phi_v$, and the eigenvalues of the Hecke operators

$$T_v = \left[H_v \left(\begin{smallmatrix} \varpi_v & 0 \\ 0 & 1 \end{smallmatrix}\right) H_v\right] \quad \text{and} \quad S_v = \left[H_v \left(\begin{smallmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{smallmatrix}\right) H_v\right]$$

are respectively given by

$$a_v = (Nv)^{1/2}(\beta_{1,v} + \beta_{2,v}) \quad \text{and} \quad s_v = \beta_{1,v} \cdot \beta_{2,v}$$

where Nv is the order of the residue field of v. Note that $s_v = \omega_v(\varpi_v)$ where $\omega_v = \mu_{1,v}\mu_{2,v}$ is the local component of the central character ω of π and π'. Therefore, the knowledge of ω and all but finitely many of the a_v's uniquely determines π and π'.

Proposition 5.1 *(π, K) and (π', K) are exceptional if and only if $a_v = 0$ for all but finitely many of the v's which are inert in K.*

Proof. With notations as above, the v-component of $\pi \otimes \eta$ is equal to $\pi(\mu_{1,v}\eta_v, \mu_{2,v}\eta_v)$ where η_v is the local component of η. Therefore, $\pi \simeq \pi \otimes \eta$

if and only if $\{\beta_{1,v}, \beta_{2,v}\} = \{\beta_{1,v}\eta(v), \beta_{2,v}\eta(v)\}$ for almost all v, where $\eta(v) = \eta_v(\varpi_v)$ equals 1 if v splits in K, and -1 if v is inert in K. The proposition easily follows.

Remark 5.2 It is well-known that the field $E_\pi \subset \mathbf{C}$ generated by the a_v's and the values of ω is a number field. Moreover, for any finite place λ of E_π with residue characteristic ℓ, there exists a unique (up to isomorphism) continuous representation

$$\rho_{\pi,\lambda} : \mathrm{Gal}_F \to \mathrm{GL}_2(E_{\pi,\lambda})$$

such that that for every finite place $v \nmid \ell\mathcal{N}$, $\rho_{\pi,\lambda}$ is unramified at v and the characteristic polynomial of $\rho_{\pi,\lambda}(\mathrm{Frob}_v)$ equals

$$X^2 - a_v X + Nv \cdot s_v \in E_\pi[X] \subset E_{\pi,\lambda}[X].$$

See [22] and the reference therein.

Put $\mathcal{E} = E_{\pi,\lambda}$ and let $V = \mathcal{E}^2$ be the representation space of $\rho = \rho_{\pi,\lambda}$. If $\pi \simeq \pi \otimes \eta$, then $\rho \simeq \rho \otimes \eta$ (viewing now η has a Galois character). In particular, there exists $\theta \in \mathrm{GL}(V)$ such that

$$\theta \circ \rho = \eta \cdot \rho \circ \theta \quad \text{on } \mathrm{Gal}_F.$$

Since ρ is absolutely irreducible (this follows from the proof of Proposition 3.1 in [22]), θ^2 is a scalar in $\mathcal{E}^\times$ but θ is not. Let $\mathcal{E}'$ be a quadratic extension of $\mathcal{E}$ containing a square root c of θ^2. Put $V' = V \otimes_{\mathcal{E}} \mathcal{E}'$. Then $V' = V'_+ \oplus V'_-$ where $\theta = \pm c$ on $V'_\pm$, and $\dim_{\mathcal{E}'} V'_\pm = 1$. Moreover,

$$\forall \sigma \in \mathrm{Gal}_F : \quad (\rho \otimes \mathbf{Id}_{\mathcal{E}'})(\sigma)(V'_\pm) = V'_{\pm\eta(\sigma)}.$$

It easily follows that $\rho \otimes \mathbf{Id}_{\mathcal{E}'} \simeq \mathrm{Ind}_{\mathrm{Gal}_K}^{\mathrm{Gal}_F}(\alpha)$, where $\alpha : \mathrm{Gal}_K \to \mathcal{E}'^\times$ is the continuous character giving the action of Gal_K on V'_+.

Conversely, suppose that the base change of ρ to an algebraic closure $\overline{\mathcal{E}}$ of $\mathcal{E}$ is isomorphic to $\mathrm{Ind}_{\mathrm{Gal}_K}^{\mathrm{Gal}_F}(\alpha)$ for some character $\alpha : \mathrm{Gal}_K \to \overline{\mathcal{E}}^\times$. Then $\pi \simeq \pi \otimes \eta$ by Proposition 5.1, since

$$a_v = \mathrm{Tr}(\rho(\mathrm{Frob}_v)) = 0$$

for almost all v's that are inert in K.

5.3 CM points and Galois actions

Given a compact open subgroup H of $G(\mathbf{A}_f)$, we say that $\theta \in \mathcal{S}_2$ has *level* H if it is fixed by H. The space $\mathcal{S}_2^H$ of all such functions may thus be identified with the finite dimensional space of all complex valued function on the *finite* set

$$M_H \stackrel{\text{def}}{=} G(\mathbf{Q})\backslash G(\mathbf{A}_f)/H.$$

In particular, any such θ yields a function $\psi = \theta \circ \mathrm{red}$ on

$$\mathrm{CM}_H \stackrel{\text{def}}{=} T(\mathbf{Q})\backslash G(\mathbf{A}_f)/H,$$

where $\mathrm{red} : \mathrm{CM}_H \to M_H$ is the obvious map.

Also, θ is an Eisenstein function if and only if it factors through the map $c : M_H \to N_H$ which is induced by the reduced norm, where

$$N_H \stackrel{\text{def}}{=} Z(\mathbf{Q})^+\backslash Z(\mathbf{A}_f)/\mathrm{nr}(H)$$

and $Z(\mathbf{Q})^+ = \mathrm{nr}(G(\mathbf{Q}))$ is the subgroup of totally positive elements in $Z(\mathbf{Q}) = F^\times$. We will need to consider a somewhat weaker condition.

Recall from the introduction that the set CM_H of *CM points*, is endowed with the following *Galois action*: for $x = [g] \in \mathrm{CM}_H$ and $\sigma = \mathrm{rec}_K(\lambda) \in \mathrm{Gal}_K^{\mathrm{ab}}$ (with $g \in G(\mathbf{A}_f)$ and $\lambda \in T(\mathbf{A}_f)$),

$$\sigma \cdot x = [\lambda g] \in \mathrm{CM}_H.$$

The Galois group $\mathrm{Gal}_F^{\mathrm{ab}}$ similarly acts on N_H, and we thus obtain an action of $\mathrm{Gal}_K^{\mathrm{ab}}$ on N_H: for $x = [z] \in N_H$ and $\sigma = \mathrm{rec}_K(\lambda) \in \mathrm{Gal}_K^{\mathrm{ab}}$ (with $z \in Z(\mathbf{A}_f)$ and $\lambda \in T(\mathbf{A}_f)$),

$$\sigma \cdot x = [\mathrm{nr}(\lambda) z] \in N_H.$$

By construction, the composite map

$$\mathrm{CM}_H \xrightarrow{\mathrm{red}} M_H \xrightarrow{c} N_H$$

is $\mathrm{Gal}_K^{\mathrm{ab}}$-equivariant (and surjective).

Definition 5.3 We say that $\theta : M_H \to \mathbf{C}$ is *exceptional* (with respect to K) if there exists some $z_0 \in N_H$ with the property that θ is constant on $c^{-1}(z)$, for all z in $\mathrm{Gal}_K^{\mathrm{ab}} \cdot z_0$.

Remark 5.4 Since K is quadratic over F, there are at most two $\mathrm{Gal}_K^{\mathrm{ab}}$-orbits in N_H. If there is just one, $\theta : M_H \to \mathbf{C}$ is exceptional if and only if it is Eisenstein: this occurs for instance whenever $\mathrm{nr}(H)$ is the maximal compact open subgroup of $Z(\mathbf{A}_f)$, provided that K/F ramifies at some finite place. On the other hand, if there are two $\mathrm{Gal}_K^{\mathrm{ab}}$-orbits in N_H, there might be exceptional θ's which are not Eisenstein.

Lemma 5.5 *Let π' be any cuspidal representation of $G(\mathbf{A}_f)$. Suppose that $\mathcal{S}_2(\pi')$ contains a nonzero θ which is exceptional with respect to K. Then (π', K) is exceptional.*

Proof. Let $H = \prod_v H_v$ be a compact open subgroup of $G(\mathbf{A}_f)$ such that θ is right invariant under H. Since π is cuspidal, θ is not Eisenstein and the above remark shows that there must be *exactly* two $\mathrm{Gal}_K^{\mathrm{ab}}$-orbits in N_H, say X and Y, with θ constant on $c^{-1}(z)$ for all z in X, but $\theta(x_1) \neq \theta(x_2)$ for some x_1 and x_2 in M_H with $c(x_1) = c(x_2) = y \in Y$.

For all but finitely many v's, $H_v = R_v^\times$ where $R_v \simeq M_2(\mathcal{O}_{F,v})$ is a maximal order in $B_v \simeq M_2(F_v)$. For any such v, we know that

$$\theta | T_v = a_v \theta$$

where for $x = [g] \in M_H$ with $g \in G(\mathbf{A}_f)$,

$$(\theta | T_v)(x) = \sum_{i \in I_v} \theta(x_{v,i}) \quad \text{with } x_{v,i} = [g\gamma_{v,i}] \in M_H.$$

Here, $H_v \left(\begin{smallmatrix} \varpi_v & 0 \\ 0 & 1 \end{smallmatrix}\right) H_v = \coprod_{i \in I_v} \gamma_{v,i} H_v$ with ϖ_v a local uniformizer in F_v. Note that for x in $c^{-1}(y)$, the $x_{v,i}$'s all belong to $c^{-1}(\mathrm{Frob}_v \cdot y)$. If v is inert in K, $\mathrm{Frob}_v \cdot y$ belongs to X and θ is constant on its fiber, say $\theta(x') = \theta(v, y)$ for all $x' \in c^{-1}(\mathrm{Frob}_v \cdot y)$. For such v's, we thus obtain

$$a_v \theta(x_1) = |I_v| \, \theta(v, y) = a_v \theta(x_2).$$

Since $\theta(x_1) \neq \theta(x_2)$ by construction, $a_v = 0$ whenever v is inert in K. The lemma now follows from Proposition 5.1.

5.4 Main results

To prove our main theorems in the definite case, we shall proceed backwards, starting from the analog of Proposition 4.17, and ending with our target result, the analog of Theorem 4.1.

Thus, let $\mathcal{R}' \subset G_0$ be the set of representatives for G_0/G_2 which we considered in section 4.5. Recall that

$$\mathcal{R}' = \{\tau\sigma_D;\ \tau \in \mathcal{R} \text{ and } D \mid \mathcal{D}'\}$$

where $\mathcal{R} \subset G_0$ is a set of representatives for G_0/G_1 containing 1, while $\mathcal{D}'$ and the σ_D's for $D \mid \mathcal{D}'$ were defined in Lemma 2.6. Suppose that $H = \widehat{R}^\times$ for some $\mathcal{O}_F$-order $R \subset B$, and consider the following maps:

$$\begin{array}{ccc} \mathrm{CM}_H(P^\infty) & \xrightarrow{\mathrm{RED}} & M_H^{\mathcal{R}} \\ x & \longmapsto & (\mathrm{red}(\tau \cdot x))_{\tau\in\mathcal{R}} \end{array} \quad \text{and} \quad \begin{array}{ccc} M_H^{\mathcal{R}} & \xrightarrow{C} & N_H^{\mathcal{R}} \\ (a_\tau)_{\tau\in\mathcal{R}} & \longmapsto & (c(a_\tau))_{\tau\in\mathcal{R}} \end{array}$$

If we endow $N_H^{\mathcal{R}}$ with the diagonal Galois action, the composite

$$C \circ \mathrm{RED} : \mathrm{CM}_H(P^\infty) \to N_H^{\mathcal{R}}$$

becomes a $G(\infty)$-equivariant map. In particular, for any $x \in \mathrm{CM}_H(P^\infty)$,

$$\mathrm{RED}(G(\infty) \cdot x) \subset C^{-1}(G(\infty) \cdot C(x)).$$

The following key result is our initial input from [5].

Proposition 5.6 *For all but finitely many $x \in \mathrm{CM}_H(P^\infty)$,*

$$\mathrm{RED}(G(\infty) \cdot x) = C^{-1}(G(\infty) \cdot C \circ \mathrm{RED}(x)).$$

Proof. Given the definition of G_1, this is just a special case of Corollary 2.10 of [5] except that the latter deals with P-isogeny classes of CM points instead of the set of all CM points of P-power conductor which we consider here. Nevertheless, $\mathrm{CM}_H(P^\infty)$ breaks up as the disjoint union of *finitely* many such P-isogeny classes, cf. remark 4.16. The proposition follows.

Corollary 5.7 *Let θ be any non-exceptional function on M_H, and let $\psi = \theta \circ \mathrm{red}$ be the induced function on CM_H. Let χ be any character of G_0. Then, for any CM point $x \in \mathrm{CM}_H(P^n)$ with n sufficiently large, there exists some $y \in G(\infty) \cdot x$ such that*

$$\sum_{\tau\in\mathcal{R}} \chi(\tau)\psi(\tau \cdot y) \neq 0.$$

Proof. Replacing x by $\sigma \cdot x$ for some $\sigma \in G(\infty)$, we may assume that $\theta(p_1) \neq \theta(p_2)$ for some p_1 and p_2 in $c^{-1}(c \circ \mathrm{red}(x))$. If n is sufficiently large, the proposition then produces y_1 and y_2 in $G(\infty) \cdot x$ such that $\mathrm{red}(y_1) = p_1$, $\mathrm{red}(y_2) = p_2$ and

$$\mathrm{red}(\tau \cdot y_1) = \mathrm{red}(\tau \cdot x) = \mathrm{red}(\tau \cdot y_2)$$

for any $\tau \neq 1$ in $\mathcal{R}$. If $\mathbf{a}(y) = \sum_{\tau \in \mathcal{R}} \chi(\tau)\psi(\tau \cdot y)$, we thus obtain

$$\mathbf{a}(y_1) - \mathbf{a}(y_2) = \theta(p_1) - \theta(p_2) \neq 0,$$

and one at least of $\mathbf{a}(y_1)$ or $\mathbf{a}(y_2)$ is nonzero.

Suppose now that we are given an irreducible cuspidal representation π' of $G(\mathbf{A}_f)$, with (unramified) central character ω. We still consider a level structure of the form $H = \widehat{R}^\times$ for some $\mathcal{O}_F$-order $R \subset B$, but we now also require the following condition.

(H2) For any prime $Q \neq P$ of F which ramifies in K, B is split at Q and R_Q is a maximal order in $B_Q \simeq M_2(F_Q)$.

Our next result is the analog of Theorem 4.10.

Proposition 5.8 *Suppose that θ is a nonzero function in $\mathcal{S}_2(\pi')^H$, and let ψ be the induced function on CM_H. Let χ be any character of G_0 such that $\chi \cdot \omega = 1$ on $\mathbf{A}_F^\times$. Then, for all $x \in \mathrm{CM}(P^n)$ with n sufficiently large, there exists some $y \in G(\infty) \cdot x$ such that*

$$\mathbf{a}(y) \stackrel{\text{def}}{=} \sum_{\sigma \in G_0} \chi(\sigma)\psi(\sigma \cdot y) \neq 0.$$

Proof. Since $Z(\mathbf{A}_f)$ acts on $\mathcal{S}_2(\pi') \ni \theta$ through ω, we find that

$$\begin{aligned} \mathbf{a}(y) &= |G_2| \sum_{\sigma \in \mathcal{R}'} \chi(\sigma)\psi(\sigma \cdot y). \\ &= |G_2| \sum_{\tau \in \mathcal{R}} \chi(\tau) \sum_{D | \mathcal{D}'} \chi(\sigma_D)\psi(\sigma_D \tau \cdot y). \end{aligned}$$

Using Lemma 5.9 below, we obtain

$$\mathbf{a}(y) = |G_2| \sum_{\tau \in \mathcal{R}} \chi(\tau)\psi_1(\tau \cdot y_1)$$

where $\psi_1 : \mathrm{CM}_{H_1} \to \mathbf{C}$ is induced by a nonzero function θ_1 of level $H_1 = \widehat{R}_1^\times$ in $\mathcal{S}_2(\pi')$, and $y \mapsto y_1$ is a Galois equivariant map from $\mathrm{CM}_H(P^n)$ to $\mathrm{CM}_{H_1}(P^n)$. Since θ_1 is non-exceptional by Lemma 5.5, we may apply Corollary 5.7 to θ_1 and $x_1 \in \mathrm{CM}_{H_1}(P^n)$, thus obtaining some $y_1 = \gamma \cdot x_1$ in $G(\infty) \cdot x_1$ such that

$$\sum_{\tau \in \mathcal{R}} \chi(\tau)\psi_1(\tau \cdot y_1) = |G_2|^{-1}\,\mathbf{a}(y) \neq 0,$$

with $y = \gamma \cdot x \in G(\infty) \cdot x$.

Lemma 5.9 *There exists an $\mathcal{O}_F$-order $R_1 \subset R$, a nonzero function θ_1 of level $H_1 = \widehat{R}_1^\times$ in $\mathcal{S}_2(\pi')$, and for each $n \geq 0$, a Galois equivariant map $x \mapsto x_1$ from $\mathrm{CM}_H(P^n)$ to $\mathrm{CM}_{H_1}(P^n)$ such that*

$$\sum_{D|\mathcal{D}'} \chi(\sigma_D)\psi(\sigma_D \cdot x) = \psi_1(x_1), \tag{17}$$

where $\psi = \theta \circ \mathrm{red}$ and $\psi_1 = \theta_1 \circ \mathrm{red}$ as usual.

Proof. The proof is very similar to that of Lemma 4.14. We put

$$R_1 = R \cap \alpha_{\mathcal{D}'} R \alpha_{\mathcal{D}'}^{-1} \subset B$$

where for any prime divisor Q of $\mathcal{D}'$, Γ_Q is the set of elements in $R_Q \simeq M_2(\mathcal{O}_{F,Q})$ whose reduced norm (=determinant) is a uniformizer in $\mathcal{O}_{F,Q}$, α_Q is a chosen element in Γ_Q, and $\alpha_D = \prod_{Q|D} \alpha_Q$ for any divisor D of $\mathcal{D}'$. We then define

$$\theta_1 = \sum_{D|\mathcal{D}'} \chi(\sigma_D)(\alpha_D \cdot \theta).$$

Thus, θ_1 is a function of level $H_1 = \widehat{R}_1^\times$ in $\mathcal{S}_2(\pi')$.

Consider now some $x = [g] \in \mathrm{CM}_H(P^n)$, with $g \in G(\mathbf{A}_f)$ and $n \geq 0$. For each $Q \mid \mathcal{D}'$, we know that $K_Q \cap g_Q R_Q g_Q^{-1} = \mathcal{O}_{K,Q}$. If π_Q denotes a fixed generator of the maximal ideal of $\mathcal{O}_{K,Q}$, we thus find that $g_Q^{-1}\pi_Q g_Q$ belongs to Γ_Q. Since $\Gamma_Q = R_Q^\times \alpha_Q R_Q^\times$, there exists $r_{1,Q}$ and $r_{2,Q}$ in $R_Q^\times$ such that $g_Q^{-1}\pi_Q g_Q = r_{1,Q}\alpha_Q r_{2,Q}$. For $i \in \{1,2\}$, we put $r_i = \prod_{Q|\mathcal{D}'} r_{i,Q}$ and view it as an element of $H \subset G(\mathbf{A}_f)$. One easily checks, as in the proof of Lemma 4.14, that the CM point $x_1 = [gr_1]$ in CM_{H_1} has conductor P^n, and we claim that

1. the map $x \mapsto x_1$ is well-defined and Galois equivariant;

2. formula (17) holds for all $x \in \mathrm{CM}_H(P^\infty)$.

For (1), suppose that we replace g by $g' = \lambda g h$ for some $\lambda \in T(\mathbf{A}_f)$ and $h \in H$. For $Q \mid \mathcal{D}'$, let $r'_{1,Q}$ and $r'_{2,Q}$ be elements of $R_Q^\times$ such that

$$g'^{-1}_Q \pi_Q g'_Q = r'_{1,Q} \alpha_Q r'_{2,Q}.$$

Since $g'_Q = \lambda_Q g_Q h_Q$ and $\pi_Q \lambda_Q = \lambda_Q \pi_Q$ in $B_Q^\times$, we find that

$$r_{1,Q} \alpha_Q r_{2,Q} = g_Q^{-1} \pi_Q g_Q = h_Q r'_{1,Q} \alpha_Q r'_{2,Q} h_Q^{-1}.$$

In particular, $r_{1,Q}^{-1} h_Q r'_{1,Q}$ equals $\alpha_Q r_{2,Q} h_Q r'^{-1}_{2,Q} \alpha_Q^{-1}$, and thus belongs to $R_{1,Q}^\times = R_Q^\times \cap \alpha_Q R_Q^\times \alpha_Q^{-1}$. It follows that for $r'_1 = \prod_{Q|\mathcal{D}'} r'_{1,Q} \in H$, $[g' r'_1]$ equals $[\lambda g r_1]$ in $\mathrm{CM}_{H_1}(P^n)$, and this finishes the proof of (1).

For (2), we simply have to observe that for any divisor D of $\mathcal{D}'$, if ψ_D denotes the function on CM_{H_1} which is induced by $\alpha_D \cdot \theta \in \mathcal{S}_2(\pi')$,

$$\psi_D(x_1) = \theta(g r_1 \alpha_D) = \theta(g r_1 \alpha_D r_2) = \theta(\pi_D g) = \psi(\sigma_D \cdot x)$$

where $\pi_D = \prod_{Q|D} \pi_Q$, so that $\sigma_D = \mathrm{rec}_K(\pi_D)$.

To complete the proof of the lemma, it remains to show that θ_1 is nonzero. This may be proved by induction, exactly as in Proposition 5.3 of [24], or Lemma 4.15 above in the indefinite case. The final step of the argument runs as follows: if $\vartheta + \rho(\pi_Q)(\vartheta') = 0$ for some ϑ and ϑ' in $\mathcal{S}_2(\pi')$ that are fixed by $R_Q^\times$, then $\vartheta = -\rho(\pi_Q)(\vartheta')$ is fixed by the group spanned by $R_Q^\times$ and $\pi_Q R_Q^\times \pi_Q^{-1}$. This group contains the kernel of the reduced norm $B_P^\times \to F_P^\times$, and the strong approximation theorem then implies that ϑ is Eisenstein, hence zero.

Finally, suppose moreover that the following condition holds.

(H1) R_P is an Eichler order in $B_P \simeq M_2(F_P)$.

We then have the notion of *good* CM points. We say that $\theta \in \mathcal{S}_2(\pi')$ is P*-new* if it is fixed by $R_P^\times$, and π' contains no nonzero vectors which are fixed by ${R'_P}^\times$ for some Eichler order $R'_P \subset B_P$ strictly containing R_P. The following is the analog of Theorem 4.1.

Theorem 5.10 *Suppose that θ is a nonzero function in $\mathcal{S}_2(\pi')^H$. Suppose moreover that θ is P-new, and let ψ be the induced function on* CM_H*. Let*

χ_0 be any character of G_0 such that $\chi_0 \cdot \omega = 1$ on $\mathbf{A}_F^\times$. Then, for any good CM point $x \in \mathrm{CM}_H(P^n)$ with n sufficiently large, there exists a primitive character χ of $G(n)$ inducing χ_0 on G_0 such that

$$\mathbf{a}(x, \chi) \stackrel{\text{def}}{=} \sum_{\sigma \in G(n)} \chi(\sigma)\psi(\sigma \cdot x) \neq 0.$$

Proof. Since $\mathbf{a}(\gamma \cdot x, \chi) = \chi^{-1}(\gamma)\mathbf{a}(x, \chi)$ for any $\gamma \in G(n)$, it suffices to show that for some y in the Galois orbit of x, the average of the $\mathbf{a}(y, \chi)$'s is nonzero (with χ running through the set $P(n, \chi_0)$ of primitive characters of $G(n)$ inducing χ_0 on G_0). By Lemma 2.8, this amounts to showing that

$$\sum_{\sigma \in G_0} \chi_0(\sigma)\psi_*(\sigma \cdot d(y)) \neq 0$$

for some $y \in G(\infty) \cdot x$, where $\psi_* : \mathbf{Z}[\mathrm{CM}_H] \to \mathbf{C}$ is the natural extension of ψ and

$$d(y) \stackrel{\text{def}}{=} q \cdot y - \mathrm{Tr}_{Z(n)}(y) \in \mathbf{Z}[\mathrm{CM}_H].$$

Since θ is P-new, ψ_* factors through the P-new quotient $\mathbf{Z}[\mathrm{CM}_H]^{P-\mathrm{new}}$ of $\mathbf{Z}[\mathrm{CM}_H]$. In the latter, the image of $d(y)$ may be computed using the distribution relations of the appendix, provided that y (or x) is a *good* CM point of conductor P^n with n sufficiently large. We find that

$$\psi_*(d(y)) = \psi^+(y^+)$$

where $H^+ = \widehat{R^+}^\times$ for some $\mathcal{O}_F$-order $R^+ \subset B$, θ^+ is a function of level H^+ in $\mathcal{S}_2(\pi')$, ψ^+ is the induced function on CM_{H^+}, and y^+ belongs to $\mathrm{CM}_{H^+}(P^n)$. Moreover, the map $y \mapsto y^+$ commutes with the action of $\mathrm{Gal}_K^{\mathrm{ab}}$, so that y^+ belongs to $G(\infty) \cdot x^+ \subset \mathrm{CM}_{H_1}(P^n)$ and

$$\psi_*(\sigma \cdot d(y)) = \psi_*(d(\sigma y)) = \psi^+((\sigma y)^+) = \psi^+(\sigma \cdot y^+)$$

for any $\sigma \in G_0$. We now have to show that

$$\mathbf{a}(y^+) \stackrel{\text{def}}{=} \sum_{\sigma \in G_0} \chi_0(\sigma)\psi^+(\sigma \cdot y^+) \neq 0$$

for some $y^+ \in G(\infty) \cdot x^+$, provided that n is sufficiently large.

When $\delta \geq 2$, $R^+ = R$, $x^+ = x$ and $\theta^+ = q \cdot \theta$ with $q = |\mathcal{O}_F/P|$. Otherwise, R^+ is the unique $\mathcal{O}_F$-order in B which agrees with R outside P, and whose localization R_P^+ at P is the Eichler order of level P^2 constructed in section 6.5. In particular, R^+ satisfies to **(H2)**. Moreover:

$$\theta^+ = \begin{cases} b_0 \cdot \theta_0 + b_1 \cdot \theta_1 + b_2 \cdot \theta_2 & \text{if } \delta = 0 \\ b_{01} \cdot \theta_{01} + b_{12} \cdot \theta_{12} & \text{if } \delta = 1 \end{cases}$$

where the b_*'s are the elements of $B_P^\times$ defined in section 6.5, while the θ_*'s are the elements of $\mathcal{S}_2(\pi')^H$ which are respectively given by

$$\begin{array}{rrcll} & (\theta_0, \theta_1, \theta_2) & = & (q \cdot \theta, -T \cdot \theta, \gamma \cdot \theta) & \text{if } \delta = 0 \\ \text{and} & (\theta_{01}, \theta_{12}) & = & (q \cdot \theta, \gamma^{-1} \cdot \theta) & \\ \text{or} & (\theta_{01}, \theta_{12}) & = & (\gamma \cdot \theta, q \cdot \theta) & \text{if } \delta = 1. \end{array}$$

In the above formulas, T and γ are certain Hecke operators in $\mathbf{T}_H$, with $\gamma \in \mathbf{T}_H^\times$. The argument that we already used several times shows that $\theta^+ \neq 0$ in all (four) cases, and we may therefore apply Proposition 5.8 to conclude the proof of our theorem.

6 Appendix: Distribution Relations

Fix a number field F, a quadratic extension K of F and a quaternion algebra B over F containing K. Let $\mathcal{O}_F$ and $\mathcal{O}_K$ be the ring of integers in F and K. Let H be a compact open subgroup of $\widehat{B}^\times$ and put $\mathrm{CM}_H = K_+^\times \backslash \widehat{B}^\times / H$ where $K_+^\times$ is the subgroup of $K^\times$ which consists of those elements which are positive at every real place of K.

The Galois group $\mathrm{Gal}_K^{\mathrm{ab}} \simeq \overline{K_+^\times} \backslash \widehat{K}^\times$ acts on CM_H by $\sigma \cdot x = [\lambda^\epsilon b]$ for $\sigma = \mathrm{rec}_K(\lambda)$ and $x = [b]$ ($\lambda \in \widehat{K}^\times$, $b \in \widehat{B}^\times$). Here, ϵ is a fixed element in $\{\pm 1\}$. We extend this action by linearity to the free abelian group $\mathbf{Z}[\mathrm{CM}_H]$ generated by CM_H. On the latter, we also have a Galois equivariant left action of the Hecke algebra

$$\mathbf{T}_H \stackrel{\text{def}}{=} \mathrm{End}_{\mathbf{Z}[\widehat{B}^\times]}(\mathbf{Z}[\widehat{B}^\times / H]) \simeq \mathbf{Z}[H \backslash \widehat{B}^\times / H].$$

An element $[\alpha] \in H \backslash \widehat{B}^\times / H$ acts on $\mathbf{Z}[\widehat{B}^\times / H]$ or $\mathbf{Z}[\mathrm{CM}_H]$ by

$$[b] \mapsto \sum_{i=1}^n [b\alpha_i] \quad \text{for } H\alpha H = \coprod_{i=1}^n \alpha_i H \text{ and } b \in \widehat{B}^\times.$$

A *distribution relation* is an expression relating these two actions. The aim of this section is to establish some of these relations when $H = H^P R_P^\times$ where

P is a prime of F where B is split, H^P is any compact open subgroup of $(\widehat{B}^\times)^P = \{b \in \widehat{B}; b_P = 1\}$ and $R_P \subset B_P$ is an *Eichler order* of level P^δ for some $\delta \geq 0$.

More precisely, we shall relate the action of the "decomposition group at P" to the action of the local Hecke algebra

$$\mathbf{T}(R_P^\times) = \mathrm{End}_{\mathbf{Z}[B_P^\times]}(\mathbf{Z}[B_P^\times/R_P^\times]) \simeq \mathbf{Z}[R_P^\times \backslash B_P^\times/R_P^\times] \subset \mathbf{T}(H)$$

on CM_H. This naturally leads us to the study of the left action of $K_P^\times$ and $\mathbf{T}(R_P^\times)$ on $B_P^\times/R_P^\times$. For any $x = [b] \in B_P^\times/R_P^\times$, the stabilizer of x in $K_P^\times$ equals $\mathcal{O}(x)^\times$ where $\mathcal{O}(x) = K_P \cap bR_Pb^{-1}$ is an $\mathcal{O}_{F,P}$-order in K_P. On the other hand, any $\mathcal{O}_{F,P}$-order $\mathcal{O} \subset K_P$ is equal to

$$\mathcal{O}_n \stackrel{\text{def}}{=} \mathcal{O}_{F,P} + P^n\mathcal{O}_{K,P}$$

for a unique integer $n = \ell_P(\mathcal{O})$ (cf. section 6.1 below). For x as above, we put $\ell_P(x) \stackrel{\text{def}}{=} \ell_P(\mathcal{O}(x))$. This function on $B_P^\times/R_P^\times$ obviously factors through $K_P^\times \backslash B_P^\times/R_P^\times$. Using the decomposition

$$\mathrm{Gal}_K^{\mathrm{ab}} \backslash \mathrm{CM}_H \simeq \widehat{K}^\times \backslash \widehat{B}^\times/H \simeq (\widehat{K}^\times)^P \backslash (\widehat{B}^\times)^P/H^P \times K_P^\times \backslash B_P^\times/R_P^\times$$

we thus obtain a Galois invariant fibration $\ell_P : \mathrm{CM}_H \to \mathbf{N}$ with the property that for any $x \in \mathrm{CM}_H$ with $n = \ell_P(x)$, x is fixed by the closed subgroup $\mathrm{rec}_K(\mathcal{O}_n^\times)$ of $\mathrm{Gal}_K^{\mathrm{ab}}$. If $n \geq 1$, we put

$$\mathrm{Tr}(x) \stackrel{\text{def}}{=} \sum_{\lambda \in \mathcal{O}_{n-1}^\times/\mathcal{O}_n^\times} \mathrm{rec}_K(\lambda) \cdot x \in \mathbf{Z}[\mathrm{CM}_H].$$

This is, on the Galois side, the expression that we will try to compute in terms of the action of the local Hecke algebra.

When $\delta \geq 1$ (so that R_P is not a maximal order), our formulas simplify in the P-new quotient $\mathbf{Z}[\mathrm{CM}_H]^{P-\mathrm{new}}$ of $\mathbf{Z}[\mathrm{CM}_H]$. The latter is the quotient of $\mathbf{Z}[\mathrm{CM}_H]$ by the $\mathbf{Z}$-submodule which is spanned by the elements of the form $\sum[b\alpha_i]$ where $b \in \widehat{B}^\times$ and $\{\alpha_i\} \subset B_P^\times$ is a set of representatives of ${R'}_P^\times/R_P^\times$ for some Eichler order $R_P \subset R'_P \subset B_P$ of level $P^{\delta'}$ with $\delta' < \delta$.

We start this section with a review on the arithmetic of $\mathcal{O}_n$. The next three sections establish the distribution relations for $\mathrm{Tr}(x)$ when $\delta = 0$, $\delta = 1$ and $\delta \geq 2$ respectively. The final section explains how the various points that are involved in the formulas for $\delta = 0$ or $\delta = 1$ may all be retrieved from a single CM point of higher level $\delta = 2$.

To fix the notation, we put $\mathbf{F} = \mathcal{O}_F/P \simeq \mathcal{O}_{F,P}/P\mathcal{O}_{F,P}$ and let $\mathbf{F}[\epsilon] = \mathbf{F}[X]/X^2\mathbf{F}[X]$ be the infinitesimal deformation $\mathbf{F}$-algebra. We choose a local uniformizer ϖ_P of F at P. We set $\varepsilon_P = -1$, 0 or 1 depending upon whether P is inert, ramifies or splits in K. We denote by $\mathcal{P}$ (resp. $\mathcal{P}$ and $\mathcal{P}^*$) the primes of K above P and let $\sigma_{\mathcal{P}}$ (resp. $\sigma_{\mathcal{P}}$ and $\sigma_{\mathcal{P}^*}$) be the corresponding *geometric* Frobeniuses.

6.1 Orders.

Since $\mathcal{O}_{K,P}/\mathcal{O}_{F,P}$ is a torsionfree rank one $\mathcal{O}_{F,P}$-module, we may find an $\mathcal{O}_{F,P}$-basis $(1, \alpha_P)$ of $\mathcal{O}_{K,P}$. Let $\mathcal{O}$ be any $\mathcal{O}_{F,P}$-order in K_P. The projection of $\mathcal{O} \subset \mathcal{O}_{K,P} = \mathcal{O}_{F,P} \oplus \mathcal{O}_{F,P}\alpha_P$ to the second factor equals $P^n\mathcal{O}_{F,P}\alpha_P$ for a well-defined integer $n = \ell_P(\mathcal{O}) \geq 0$. Since $\mathcal{O}_{F,P} \subset \mathcal{O}$,

$$\mathcal{O} = \mathcal{O}_{F,P} \oplus P^n\mathcal{O}_{F,P}\alpha_P = \mathcal{O}_{F,P} + P^n\mathcal{O}_{K,P}.$$

Conversely, $\forall n \geq 0$, $\mathcal{O}_n \overset{\text{def}}{=} \mathcal{O}_{F,P} + P^n\mathcal{O}_{K,P}$ is an $\mathcal{O}_{F,P}$-order in K_P.

Since any $\mathcal{O}_n$-ideal is generated by at most two elements (it is already generated by two elements as an $\mathcal{O}_{F,P}$-module), $\mathcal{O}_n$ is Gorenstein ring for any $n \geq 0$ [1]. For $n = 0$, $\mathcal{O}_0 = \mathcal{O}_{K,P}$ and the $\mathbf{F}$-algebra $\mathcal{O}_0/P\mathcal{O}_0$ is a degree 2 extension of $\mathbf{F}$ if $\varepsilon_P = -1$, is isomorphic to $\mathbf{F}[\epsilon]$ if $\varepsilon_P = 0$ and to $\mathbf{F}^2$ if $\varepsilon_P = 1$. For $n > 0$, $\mathcal{O}_n$ is a local ring with maximal ideal $P\mathcal{O}_{n-1}$ and $\mathcal{O}_n/P\mathcal{O}_n$ is again isomorphic to $\mathbf{F}[\epsilon]$.

Lemma 6.1 *For any $n \geq 0$, the left action of $\mathcal{O}_n^\times$ on $\mathbf{P}^1(\mathcal{O}_n/P\mathcal{O}_n)$ factors through $\mathcal{O}_n^\times/\mathcal{O}_{n+1}^\times$. Its set of fixed points is given by the following formula*

$$\mathbf{P}^1(\mathcal{O}_n/P\mathcal{O}_n)^{\mathcal{O}_n^\times} = \begin{cases} \emptyset & \text{if } n = 0 \text{ and } \varepsilon_P = -1, \\ \{\mathcal{P}\mathcal{O}_0/P\mathcal{O}_0\} & \text{if } n = 0 \text{ and } \varepsilon_P = 0, \\ \{\mathcal{P}\mathcal{O}_0/P\mathcal{O}_0, \mathcal{P}^*\mathcal{O}_0/P\mathcal{O}_0\} & \text{if } n = 0 \text{ and } \varepsilon_P = 1, \\ \{P\mathcal{O}_{n-1}/P\mathcal{O}_n\} & \text{if } n > 0. \end{cases}$$

The remaining points are permuted faithfully and transitively by $\mathcal{O}_n^\times/\mathcal{O}_{n+1}^\times$.

Proof. This easily follows from the above discussion together with the observation that the quotient map $\mathcal{O}_n \to \mathcal{O}_n/P\mathcal{O}_n$ induces a bijection between $\mathcal{O}_n^\times/\mathcal{O}_{n+1}^\times$ and $(\mathcal{O}_n/P\mathcal{O}_n)^\times/\mathbf{F}^\times$.

6.2 The $\delta = 0$ case

Let V be a simple left B_P-module, so that $V \simeq F_P^2$ as an F_P-vector space. The embedding $K_P \hookrightarrow B_P$ endows V with the structure of a (left) K_P-module for which V is free of rank one. Let $\mathcal{L}$ be the set of $\mathcal{O}_{F,P}$-lattices in V and pick $L_0 \in \mathcal{L}$ such that $\{\alpha \in B_P; \alpha L \subset L\} = R_P$. Then $b \mapsto bL_0$ yields a bijection between $B_P^\times / R_P^\times$ and $\mathcal{L}$. The induced left actions of $K_P^\times$ and $\mathbf{T}(R_P^\times)$ on $\mathbf{Z}[\mathcal{L}]$ are respectively given by

$$(\lambda, L) \mapsto \lambda L \quad \text{and} \quad [R_P^\times \alpha R_P^\times](L) = \sum_{i=1}^{n} bL_i$$

for $\lambda \in K_P^\times$, $L = bL_0 \in \mathcal{L}$, $\alpha \in B_P^\times$, $R_P^\times \alpha R_P^\times = \coprod_{i=1}^n \alpha_i R_P^\times$ and $L_i = \alpha_i L_0$. The function ℓ_P on $\mathcal{L}$ maps a lattice L to the unique integer $n = \ell_P(L)$ such that $\{x \in K_P^\times; xL \subset L\}$ equals $\mathcal{O}_n$.

Lemma 6.2 *The function ℓ_P defines a bijection between $K_P^\times \backslash B_P^\times / R_P^\times \simeq K_P^\times \backslash \mathcal{L}$ and $\mathbf{N}$.*

Proof. Fix a K_P-basis e of V. For any $n \geq 0$, $\ell_P(\mathcal{O}_n e) = n$ – this shows that ℓ_P is surjective. Conversely, let L be a lattice with $\ell_P(L) = n$. Then L is a free (rank one) $\mathcal{O}_n$-module by [1, Proposition 7.2]. In particular, there exists an element $\lambda \in K_P^\times$ such that $L = \mathcal{O}_n \lambda e = \lambda \mathcal{O}_n e$. This shows that $\ell_P : K_P^\times \backslash \mathcal{L} \to \mathbf{N}$ is also injective.

Definition 6.3 Let $L \subset V$ be a lattice.

1. The *lower (resp. upper) neighbors* of L are the lattices $L' \subset L$ (resp. $L \subset L'$) such that $L/L' \simeq \mathbf{F}$ (resp. $L'/L \simeq \mathbf{F}$).

2. The *lower (resp. upper) Hecke operator* T_P^l (resp. T_P^u) on $\mathbf{Z}[\mathcal{L}]$ maps L to the sum of its lower (resp. upper) neighbors.

3. If $n = \ell_P(L) \geq 1$, the *lower (resp. upper) predecessor* of L is defined by
$$\mathrm{pr}_l(L) \stackrel{\text{def}}{=} P\mathcal{O}_{n-1}L \quad (\text{resp. } \mathrm{pr}_u(L) \stackrel{\text{def}}{=} \mathcal{O}_{n-1}L).$$

Remark 6.4 T_P^l and T_P^u are the local Hecke operators corresponding to respectively $R_P^\times \alpha R_P^\times$ and $R_P^\times \alpha^{-1} R_P^\times$ where α is any element of $R_P \simeq M_2(\mathcal{O}_{F,P})$ whose reduced norm (= determinant) is a uniformizer in F_P.

Lemma 6.5 *Let L be a lattice in V and put $n = \ell_P(L)$.*

1. *If $n = 0$, there are exactly $1 + \varepsilon_P$ lower neighbors L' of L for which $\ell_P(L') = 0$, namely $L' = \mathcal{P}L$ if $\varepsilon_P = 0$ and $L' = \mathcal{P}L$ or $\mathcal{P}^*L$ if $\varepsilon_P = 1$.*

2. *If $n > 0$, there is a unique lower neighbor L' of L for which $\ell_P(L') \leq n$, namely $L' = \mathrm{pr}_l(L)$ for which $\ell_P(L') = n - 1$.*

3. *In both cases, the remaining lower neighbors have $\ell_P = n+1$. They are permuted faithfully and transitively by $\mathcal{O}_n^\times/\mathcal{O}_{n+1}^\times$ and L is their common upper predecessor.*

Proof. This is a straightforward consequence of Lemma 6.1, together with the fact already observed in the proof of Lemma 6.2 that any lattice L with $n = \ell_P(L)$ is free of rank one over $\mathcal{O}_n$.

We leave it to the reader to formulate and prove an "upper" variant of this lemma. The function $L \mapsto \mathrm{pr}_l(L)$ (resp. $\mathrm{pr}_u(L)$) commutes with the action of $K_P^\times$, and so does the induced function on $\{[b] \in B_P^\times/R_P^\times, \ell_P(b) \geq 1\}$. The latter function extends to a $\widehat{K}^\times$-equivariant function on $\{[b] \in \widehat{B}^\times/H, \ell_P(b_p) \geq 1\}$ with values in $\widehat{B}^\times/H$ (take the identity on $(\widehat{B}^\times)^P/H^P$). Dividing by $K_+^\times$, we finally obtain Galois equivariant functions pr_l and pr_u on $\{x \in \mathrm{CM}_H, \ell_P(x) \geq 1\}$ with values in CM_H. These functions do not depend upon the various choices that we made (V and L_0).

Corollary 6.6 *For $x \in \mathrm{CM}_H$ with $\ell_P(x) = n \geq 1$,*

$$\mathrm{Tr}(x) = T_P^l(x') - x''$$

where $x' = \mathrm{pr}_u(x)$, $x'' = \mathrm{pr}_l(x')$ if $n \geq 2$ and

$$x'' = \begin{cases} 0 & (\varepsilon_P = -1) \\ \sigma_{\mathcal{P}}^{\epsilon} x' & (\varepsilon_P = 0) \\ (\sigma_{\mathcal{P}}^{\epsilon} + \sigma_{\mathcal{P}^*}^{\epsilon})x' & (\varepsilon_P = 1) \end{cases} \qquad \text{if } n = 1.$$

Note that if $\ell_P(x) = 1$, $\ell_P(x') = 0$ and x' is indeed defined over an abelian extension of K which is unramified above P.

6.3 The $\delta = 1$ case

With V as above, $B_P^\times/R_P^\times$ may now be identified with the set $\mathcal{L}_1$ of all pair of lattices $L = (L(0), L(1)) \in \mathcal{L}^2$ such that $L(1) \subset L(0)$ with $L(0)/L(1) \simeq \mathbf{F}$. Indeed, $B_P^\times \simeq \mathrm{GL}(V)$ acts transitively on $\mathcal{L}_1$ and there exists some $L_0 = (L_0(0), L_0(1)) \in \mathcal{L}_1$ whose stabilizer equals $R_P^\times$.

To each $L \in \mathcal{L}_1$, we may now attach two integers, namely

$$\ell_{P,0}(L) = \ell_P(L(0)) \quad \text{and} \quad \ell_{P,1}(L) = \ell_P(L(1)).$$

For $L \in \mathcal{L}_1$, $\ell_P(L) = \max(\ell_{P,0}(L), \ell_{P,1}(L))$ and exactly one of the following three situations occurs (see Lemma 6.5).

Definition 6.7 We say that

- L is of **type I** if $\ell_{P,0}(L) = n - 1 < \ell_{P,1}(L) = n$. The *leading vertex* of L equals $L(1)$ and if $n \geq 2$, we define the *predecessor* of L by

$$\mathrm{pr}(L) = (L(0), \mathrm{pr}_l L(0)) = (L(0), P\mathcal{O}_{n-2}L(0)).$$

- L is of **type II** if $\ell_{P,0}(L) = n > \ell_{P,1}(L) = n - 1$. Then $L(0)$ is the *leading* vertex and for $n \geq 2$, the *predecessor* of L is defined by

$$\mathrm{pr}(L) = (\mathrm{pr}_u L(1), L(1)) = (\mathcal{O}_{n-2}L(1), L(1)).$$

- L is of **type III** if $\ell_{P,0}(L) = n = \ell_{P,1}(L)$ (in which case $n = 0$, $\varepsilon_P = 0$ or 1 and $L(1) = \mathcal{P}L(0)$ or $L(1) = \mathcal{P}^*L(0)$). As a convention, we define the *leading* vertex of L to be $L(0)$.

Remark 6.8 The type of L together with the integer $n = \ell_P(L)$ almost determines the $K_P^\times$-homothety class of $\mathcal{L}$. Indeed, Lemma 6.2 implies that we can move the leading vertex of L to $\mathcal{O}_n e$ ($\{e\}$ is a K_P-basis of V). Then $L = (\mathcal{O}_{n-1}e, \mathcal{O}_n e)$ if L is of type I, $L = (\mathcal{O}_n e, P\mathcal{O}_{n-1}e)$ if L is of type II and $L = (\mathcal{O}_n e, \mathcal{P}\mathcal{O}_n e)$ or $(\mathcal{O}_n e, \mathcal{P}^*\mathcal{O}_n e)$ if L is of type III (in which case $n = 0$).

Definition 6.9 The lower (resp. upper) Hecke operator T_P^l (resp. T_P^u) on $\mathbf{Z}[\mathcal{L}_1]$ maps $L \in \mathcal{L}_1$ to the sum of all elements $L' \in \mathcal{L}_1$ such that $L'(1) = L(1)$ but $L'(0) \neq L(0)$ (resp. $L'(0) = L(0)$ but $L'(1) \neq L(1)$).

The P-new quotient $\mathbf{Z}[\mathcal{L}_1]^{P-\text{new}}$ of $\mathbf{Z}[\mathcal{L}_1]$ is the quotient of $\mathbf{Z}[\mathcal{L}_1]$ by the $\mathbf{Z}$-submodule which is spanned by the elements of the form $\sum_{L'(0)=M} L'$ or $\sum_{L'(1)=M} L'$ with M a lattice in V. By construction,

$$T_P^l \equiv T_P^u \equiv -\mathbf{1} \quad \text{on } \mathbf{Z}[\mathcal{L}_1]^{P-\text{new}}.$$

Remark 6.10 For $i \in \{0,1\}$, put $R(i) = \{b \in B_P;\, bL_0(i) \subset L_0(i)\}$ so that $R_P = R(0) \cap R(1)$. Then T_P^l and T_P^u are the local Hecke operators corresponding to respectively $R_P^\times \alpha R_P^\times$ and $R_P^\times \beta R_P^\times$, for any α in $R(1)^\times \backslash R(0)^\times$ and β in $R(0)^\times \setminus R(1)^\times$. Also, $R(0)^\times = R_P^\times \coprod R_P^\times \beta R_P^\times$ and $R(1)^\times = R_P^\times \coprod R_P^\times \alpha R_P^\times$.

For $L \in \mathcal{L}_1$ and $\lambda \in K_P^\times$, L and λL have the same type and $\mathrm{pr}(\lambda L) = \lambda \mathrm{pr}(L)$ (if $\ell_P(L) \geq 2$). We thus obtain a Galois invariant notion of *type* on CM_H and a Galois equivariant map $x \mapsto \mathrm{pr}(x)$ on $\{x \in \mathrm{CM}_H;\, \ell_P(x) \geq 2\}$ with values in CM_H. The following is then an easy consequence of Lemma 6.5.

Lemma 6.11 *For $x \in \mathrm{CM}_H$ with $\ell_P(x) \geq 2$,*

$$\mathrm{Tr}(x) = \begin{cases} T_P^u(\mathrm{pr}(x)) & \text{if } x \text{ is of type I} \\ T_P^l(\mathrm{pr}(x)) & \text{if } x \text{ is of type II} \end{cases}$$

In the P-new quotient of $\mathbf{Z}[\mathrm{CM}_H]$, these relations simplify to:

$$\mathrm{Tr}(x) = -\mathrm{pr}(x).$$

Remark 6.12 In contrast to the $\delta = 0$ case, the above constructions do depend upon the choice of L_0. More precisely, our definition of types on CM_H are sensible to the choice of an orientation on R_P: changing $L_0 = (L_0(0), L_0(1))$ to $L_0' = (L_0(1), PL_0(0))$ exchanges type I and type II points.

6.4 The $\delta \geq 2$ case

We now have $B_P^\times / R_P^\times \simeq \mathcal{L}_\delta$ where $\mathcal{L}_\delta$ is the set of all pairs of lattices $L = (L(0), L(\delta))$ in V such that $L(\delta) \subset L(0)$ with $L(0)/L(\delta) \simeq O_F/P^\delta$. We refer to such pairs as δ-lattices. To each $L \in \mathcal{L}_\delta$, we may attach the sequence of intermediate lattices

$$L(\delta) \subsetneq L(\delta - 1) \subsetneq \cdots \subsetneq L(1) \subsetneq L(0)$$

and the sequence of integers $\ell_{P,i}(L) \stackrel{\text{def}}{=} \ell_P(L(i))$, for $0 \le i \le \delta$. The function ℓ_P corresponds to

$$\ell_P(L) = \max(\ell_{P,i}(L)) = \max(\ell_{P,0}(L), \ell_{P,\delta}(L)).$$

Using Lemma 6.5, one easily checks that the sequence $\ell_{P,i}(L)$ satisfies the following property: there exists integers $0 \le i_1 \le i_2 \le \delta$ such that $\ell_{P,i+1}(L) - \ell_{P,i}(L)$ equals -1 for $0 \le i < i_1$, 0 for $i_1 \le i < i_2$ and 1 for $i_2 \le i < \delta$. Moreover, $\ell_{P,i}(L) = 0$ for all $i_1 \le i \le i_2$ if $i_2 \ne i_1$ in which case $\varepsilon_P = 0$ or 1, and $i_2 - i_1 \le 1$ if $\varepsilon_P = 0$. For our purposes, we only need to distinguish between three types of δ-lattices.

Definition 6.13 We say that $L \in \mathcal{L}_\delta$ is of **type I** if $\ell_{P,0}(L) < \ell_{P,\delta}(L)$, of **type II** if $\ell_{P,0}(L) > \ell_{P,\delta}(L)$ and of **type III** if $\ell_{P,0}(L) = \ell_{P,\delta}(L)$.

The P-new quotient $\mathbf{Z}[\mathcal{L}_\delta]^{P-\text{new}}$ of $\mathbf{Z}[\mathcal{L}_\delta]$ is the quotient of $\mathbf{Z}[\mathcal{L}_\delta]$ by the $\mathbf{Z}$-submodule which is spanned by the elements of the form

$$\textstyle\sum_{(L'(1),L'(\delta))=M} L' \quad \text{or} \quad \sum_{(L'(0),L'(\delta-1))=M} L'$$

with $M \in \mathcal{L}_{\delta-1}$. It easily follows from Lemma 6.5 that for any $L \in \mathcal{L}_\delta$ which is not of type III, $\mathrm{Tr}(L) = 0$ in $\mathbf{Z}[\mathcal{L}_\delta]^{P-\text{new}}$ where $\mathrm{Tr}(L) = \sum_{\lambda \in \mathcal{O}_{n-1}^\times/\mathcal{O}_n} \lambda L$ for $n = \ell_P(L)$. Indeed,

$$\mathrm{Tr}(L) = \begin{cases} \sum_{(L'(0),L'(\delta-1))=(L(0),L(\delta-1))} L' & \text{if } L \text{ is of type I,} \\ \sum_{(L'(1),L'(\delta))=(L'(1),L'(\delta))} L' & \text{if } L \text{ is of type II.} \end{cases}$$

Extending the notion of types to CM_H as in the previous section, we obtain:

Lemma 6.14 *For any $x \in \mathrm{CM}_H$ which is not of type III,*

$$\mathrm{Tr}(x) = 0 \quad \textit{in } \mathbf{Z}[\mathrm{CM}_H]^{P-new}.$$

Remark 6.15 If δ is odd, ℓ_P is bounded on the set of type III points in $\mathcal{L}_\delta$ or CM_H. If δ is even, there are type III points with $\ell_P = n$ for any $n \ge \delta/2$. In both cases, there are type I and type II points with $\ell_P = n$ for any $n > \delta/2$.

6.5 Predecessors and degeneracy maps.

Suppose first that $\delta = 0$ and let $L_0(0)$ be a lattice in V such that $R_P = \{\alpha \in B_P; \alpha L_0(0) \subset L_0(0)\}$. Choose a lattice $L_0(2) \subset L_0(0)$ such that

$L_0 = (L_0(0), L_0(2))$ is a 2-lattice and let

$$R_P^+ = \{\alpha \in B_P;\ \alpha L_0(0) \subset L_0(0) \text{ and } \alpha L_0(2) \subset L_0(2)\} \tag{18}$$

be the corresponding Eichler order (of level P^2). Put $H^+ = H^P (R_P^+)^\times$.

To a 2-lattice L, we may attach three lattices:

$$d_0(L) = L(2), \quad d_1(L) = L(1) \quad \text{and } d_2(L) = L(0).$$

Conversely, to each lattice L with $n = \ell_P(L) \geq 2$, we may attach a unique 2-lattice $L^+ = (\mathcal{O}_{n-2}L, L)$ with the property that

$$(d_0, d_1, d_2)(L^+) = (L, \mathrm{pr}_u L, \mathrm{pr}_u \circ \mathrm{pr}_u L) = (L, L', P^{-1}L'')$$

where $L' = \mathrm{pr}_u L$ and $L'' = \mathrm{pr}_l L'$. Being $K_P^\times$-equivariant, these constructions have Galois equivariant counterparts on suitable spaces of CM points. More precisely:

- Choose $b_i \in B_P^\times$ such that $b_i L_0(0) = L_0(2-i)$.
- Define $d_i : \mathrm{CM}_{H^+} \to \mathrm{CM}_H$ by $d_i([b]) = [bb_i]$ for $b \in \widehat{B}^\times$.
- Define $\vartheta : \mathrm{CM}_H \to \mathrm{CM}_H$ by $\vartheta([b]) = [b\varpi_P]$ for $b \in \widehat{B}^\times$.
- Use the identifications $B_P^\times / R_P^\times \leftrightarrow \mathcal{L}$ and $B_P^\times/(R_P^+)^\times \leftrightarrow \mathcal{L}_2$ to define the $K_P^\times$-equivariant map $x \mapsto x^+$ on $\{[b] \in B_P^\times/R_P^\times;\ \ell_P(bL_0(0)) \geq 2\}$ with values in $B_P^\times/(R_P^+)^\times$ which corresponds to $L \mapsto L^+$ on the level of lattices.
- Using the decomposition $\widehat{B}^\times/H = (\widehat{B}^\times)^P/H^P \times B_P^\times/R_P^\times$ (and similarly for $\widehat{B}^\times/H^+$), extend $x \mapsto x^+$ to a $\widehat{K}^\times$-equivariant map defined on the suitable subset of $\widehat{B}^\times/H$ with values in $\widehat{B}^\times/H^+$ (take the identity on $(\widehat{B}^\times)^P$).
- Dividing out by $K_+^\times$, we thus obtain a Galois equivariant map $x \mapsto x^+$ on $\{x \in \mathrm{CM}_H;\ \ell_P(x) \geq 2\}$ with values in CM_{H^+}.

By construction:

Lemma 6.16 *($\delta = 0$) For any $x \in \mathrm{CM}_H$ with $\ell_P(x) \geq 2$,*

$$(d_0, d_1, d_2)(x^+) = (x, x', \vartheta^{-1}x'') \quad \textit{in } \mathrm{CM}_H^3$$

where $x' = \mathrm{pr}_u(x)$ and $x'' = \mathrm{pr}_l(x')$.

The $\delta = 1$ case is only slightly more difficult. Fix a 1-lattice $(L_0(0), L_0(1))$ whose stabilizer equals $R_P^\times$ and let $L_0(2)$ be a sublattice of $L_0(1)$ such that $L_0 = (L_0(0), L_0(2))$ is a 2-lattice. Define R_P^+ by the same formula (18), so that R_P^+ is again an Eichler order of level P^2. Put $H^+ = H^P(R_P^+)^\times$.

To each 2-lattice L we may attach two 1-lattices, namely $d_{01}(L) = (L(0), L(1))$ and $d_{12}(L) = (L(1), L(2))$. Conversely suppose that L is a 1-lattice with $n = \ell_P(L) \geq 2$. If L is of type I, $L^+ = (\mathcal{O}_{n-2}L(0), L(1))$ is a 2-lattice and

$$(d_{01}, d_{12})(L^+) = (\vartheta^{-1}\mathrm{pr}(L), L)$$

where ϑ is now the permutation of $\mathcal{L}_1$ which maps $(L(0), L(1))$ to $(L(1), PL(0))$. If L is of type II, $L^+ = (L(0), P\mathcal{O}_{n-2}L(1))$ is a 2-lattice and

$$(d_{01}, d_{12})(L^+) = (L, \vartheta\mathrm{pr}(L)).$$

These constructions are again equivariant with respect to the action of $K_P^\times$, and may thus be extended to $\mathrm{Gal}_K^{\mathrm{ab}}$-equivariant constructions on CM points. More precisely,

- Choose $b_{01} = 1$ and $b_{12} \in B_P^\times$ such that $b_{12}(L_0(0), L_0(1)) = (L_0(1), L_0(2))$. Define d_{01} and $d_{12} : \mathrm{CM}_{H^+} \to \mathrm{CM}_H$ by $d_{01}([b]) = [bb_{01}]$, $d_{12}([b]) = [bb_{12}]$ for $b \in \widehat{B}^\times$.
- Choose ω in $B_P^\times$ such that $\omega(L_0(0), L_0(1)) = (L_0(1), PL_0(0))$ and define $\vartheta : \mathrm{CM}_H \to \mathrm{CM}_H$ by $\vartheta([b]) = [b\omega]$ for $b \in \widehat{B}^\times$.
- Proceeding as above in the $\delta = 0$ case, extend $L \mapsto L^+$ to a Galois equivariant function $x \mapsto x^+$ defined on $\{x \in \mathrm{CM}_H;\, \ell_P(x) \geq 2\}$ with values in CM_{H^+}.

With these notations, we obtain:

Lemma 6.17 *($\delta = 1$) For any $x \in \mathrm{CM}_H$ with $\ell_P(x) \geq 2$,*

$$(x, \mathrm{pr}(x)) = \begin{cases} (d_{12}, \vartheta d_{01})(x^+) & \text{if } x \text{ is of type I,} \\ (d_{01}, \vartheta^{-1} d_{12})(x^+) & \text{if } x \text{ is of type II.} \end{cases}$$

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