CM points and quaternion algebras

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November 18, 2004

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1 Introduction

Let F be a totally real number field of degree d, and let B denote a quaternion algebra over F. For the purposes of this introduction, we assume that either:

- B is *definite*, meaning that $B_v = B \otimes F_v$ is non-split for all real places v of F, or

- B is *indefinite*, meaning that B_v is split for precisely one real v.

We shall write G to denote the algebraic group over $\mathbf{Q}$ whose points over a $\mathbf{Q}$-algebra A are the set $(B \otimes A)^\times$.

Now let K be an imaginary quadratic extension of F. We suppose that there is given an embedding $K \to B$. Then associated to the data of B and K, one can define a collection of points, the so-called CM points. The natural habitat for these points depends on whether B is definite or indefinite: in the former case, the CM points are just an infinite discrete set, whereas in the latter, they inhabit certain canonical algebraic curves, the *Shimura curves*, associated to the indefinite algebra B. Our goal in this paper is to study the distribution of these CM points in certain auxiliary spaces. The main result proven here is the key ingredient in our proof in [**?**] of certain non-vanishing theorems for certain automorphic L-functions over F and their derivatives. The theorems of [**?**] may be regarded as generalizations of Mazur's conjectures in [**?**] when $F = \mathbf{Q}$.

Our original intention was simply to write a single paper proving the non-vanishing theorems for the L-functions, using the connection between L-functions and CM points, and proving a basic nontriviality theorem for the latter. However, in the course of doing this, we realized that although the CM points in the definite and indefinite cases are *a priori* very different, the proof of the main nontriviality result on CM points runs along parallel lines. In light of this, it seemed somewhat artificial to give essentially the same arguments twice, once in each of the two cases. The present paper therefore presents a rather general result about CM points on quaternion algebras, which allows us to obtain information about CM points in both the definite and indefinite cases. The former case follows trivially, but the latter requires us to develop a certain amount of foundational material on Shimura curves, their various models, and the associated CM points.

Since this paper is neccessarily rather technical, we want to give an overview of the contents. The first part deals with the abstract results. The main theorems are given in Theorem 2.9 and Corollary 2.10. Although the statements are somewhat complicated, they are not hard to prove, in view of our earlier results [**?**], [**?**], [**?**], where all the main ideas are already present. As before, the basic ingredient is Ratner's theorem on unipotent flows on p-adic Lie groups.

The second part is concerned with the applications of the abstract result to CM points on Shimura curves. We start with basic theory of Shimura curves, especially their integral models and reduction. In Section 3.1.1, we define the CM points and supersingular points, and establish the basic fact that the reduction of a CM point at an inert prime is a supersingular point. The basic result on CM points on Shimura curves is stated in Theorem 3.5. Section 3.2 gives a series of group theoretic descriptions of the various sets and maps which appear in Theorem 3.5, thus reducing its proof to a purely group theoretical statement, which may be deduced from the results in the first part of this paper.

The final two sections in the paper are meant to shed some light on related topics: section 3.3.1 investigates the dependence of Shimura curves on a certain parameter $\epsilon = \pm 1$, while section 3.3.2 provides some insight on a certain subgroup of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ which plays a prominent role in the statements of Theorem 3.5 and also appears in the André-Oort conjecture.

In conclusion, we mention that a fuller discussion of the circle of ideas and theorems that are the excuse for this paper may be found in the introduction of [**?**], where the main arithmetical applications are also spelled out.

2 CM points on quaternion algebras

2.1 CM points, special points and reduction maps

We keep the following notations: F is a totally real number field, K is a totally imaginary quadratic extension of F and B is *any* quaternion algebra over F which is split by K. At this point we make no assumption on B at infinity. We fix once and for all, an F-embedding $\iota : K \hookrightarrow B$ and a prime P of F where B is split. We denote by $\varpi_P \in F_P^\times$ a local uniformizer at P.

For any quaternion algebra B' over F, we denote by $\mathrm{Ram}(B')$, $\mathrm{Ram}_f(B')$ and $\mathrm{Ram}_\infty(B')$ the set of places (resp. finite places, resp. archimedean places) of F where B' ramifies.

2.1.1 Quaternion algebras.

Let S be a finite set of finite places of F such that

S1 $\forall v \in S$, B is unramified at v.

S2 $|S| + |\mathrm{Ram}_f(B)| + [F : \mathbf{Q}]$ is even.

S3 $\forall v \in S$, v is inert or ramifies in K.

The first two assumptions imply that there exists a totally definite quaternion algebra B_S over F such that $\mathrm{Ram}_f(B_S) = \mathrm{Ram}_f(B) \cup S$. The third assumption implies that there exists an F-embedding $\iota_S : K \to B_S$. We choose such a pair (B_S, ι_S).

2.1.2 Algebraic groups.

We put

$G = \mathrm{Res}_{F/\mathbf{Q}}(B^\times)$, $G_S = \mathrm{Res}_{F/\mathbf{Q}}(B_S^\times)$, $T = \mathrm{Res}_{F/\mathbf{Q}}(K^\times)$ and $Z = \mathrm{Res}_{F/\mathbf{Q}}(F^\times)$.

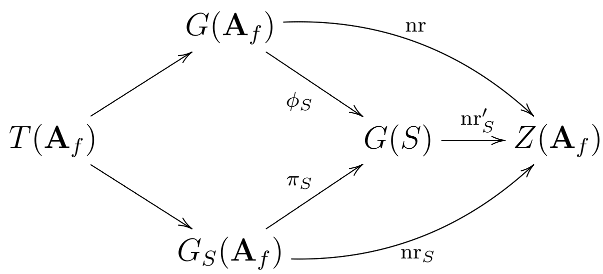
These are algebraic groups over $\mathbf{Q}$. We identify Z with the center of G and G_S. We use ι and ι_S to embed T as a maximal subtorus in G and G_S. We denote by $\mathrm{nr} : G \to Z$ and $\mathrm{nr}_S : G_S \to Z$ the algebraic group homomorphisms induced by the reduced norms $\mathrm{nr} : B^\times \to F^\times$ and $\mathrm{nr}_S : B_S^\times \to F^\times$.

2.1.3 Adelic groups.

We shall consider the following locally compact, totally discontinuous groups:

- $G(\mathbf{A}_f) = (B \otimes_{\mathbf{Q}} \mathbf{A}_f)^\times$, $G_S(\mathbf{A}_f) = (B_S \otimes_{\mathbf{Q}} \mathbf{A}_f)^\times$, $T(\mathbf{A}_f) = (K \otimes_{\mathbf{Q}} \mathbf{A}_f)^\times$ and $Z(\mathbf{A}_f) = (F \otimes_{\mathbf{Q}} \mathbf{A}_f)^\times$ with their usual topology.
- $G(S) = \prod_{v \notin S} B_{S,v}^\times \times \prod_{v \in S} F_v^\times$ where $\prod_{v \notin S} B_{S,v}^\times$ is the restricted product of the $B_{S,v}^\times$'s over all finite places of F not in S, with respect to the compact subgroups $R_v^\times \subset B_{S,v}^\times$, where R_v is the closure in $B_{S,v}$ of some fixed $\mathcal{O}_F$-order R in B_S.

These groups are related by a commutative diagram of continuous morphisms:



In this diagram,

- $T(\mathbf{A}_f) \to G(\mathbf{A}_f)$ and $T(\mathbf{A}_f) \to G_S(\mathbf{A}_f)$ are the closed embeddings induced by ι and ι_S.
- $\mathrm{nr} : G(\mathbf{A}_f) \to Z(\mathbf{A}_f)$ and $\mathrm{nr}_S : G_S(\mathbf{A}_f) \to Z(\mathbf{A}_f)$ are the continuous, open and surjective group homomorphisms induced by nr and nr_S.

- $\mathrm{nr}'_S : G(S) \to Z(\mathbf{A}_f)$ is the continuous, open and surjective group homomorphism induced by $\mathrm{nr}_{S,v} : B_{S,v}^\times \to F_v^\times$ for $v \notin S$ and by the identity on the remaining factors.

- $\pi_S : G_S(\mathbf{A}_f) = \prod_{v \notin S} B_{S,v}^\times \times \prod_{v \in S} B_{S,v}^\times \to G(S) = \prod_{v \notin S} B_{S,v}^\times \times \prod_{v \in S} F_v^\times$ is the continuous, open and surjective group homomorphism induced by the identity on $\prod_{v \notin S} B_{S,v}^\times$ and by the reduced norms $\mathrm{nr}_{S,v} : B_{S,v}^\times \to F_v^\times$ on the remaining factors. It induces an isomorphism of topological groups between $G_S(\mathbf{A}_f)/\ker(\pi_S)$ and $G(S)$. Since $\ker(\pi_S) \simeq \prod_{v \in S} B_{S,v}^1$ is compact, π_S is also a closed map.

The definition of

$$\phi_S : G(\mathbf{A}_f) = \prod_{v \notin S} B_v^\times \times \prod_{v \in S} B_v^\times \to G(S) = \prod_{v \notin S} B_{S,v}^\times \times \prod_{v \in S} F_v^\times$$

is more involved. By construction, B_v and $B_{S,v}$ are isomorphic for $v \notin S$. We shall construct a collection of isomorphisms $(\phi_v : B_v \to B_{S,v})_{v \notin S}$ such that (1) $\forall v \notin S$, $\phi_v \circ \iota = \iota_S$ on K_v, and (2) the product of the ϕ_v's yields a continuous isomorphism between $\prod_{v \notin S} B_v^\times$ and $\prod_{v \notin S} B_{S,v}^\times$. Note that any two such families are conjugated by an element of $\prod_{v \notin S} K_v^\times$. Once such a family has been chosen, we may define the morphism ϕ_S by taking $\prod_{v \notin S} \phi_v$ on $\prod_{v \notin S} B_v^\times$ and $\mathrm{nr}_v : B_v^\times \to F_v^\times$ on the remaining factors. It is then a continuous, open and surjective group homomorphism which makes the above diagram commute.

We first fix a maximal $\mathcal{O}_F$-order R in B (respectively R_S in B_S). For all but finitely many v's, (a) $R_v \simeq M_2(\mathcal{O}_{F_v}) \simeq R_{S,v}$ and (b) $\iota^{-1}(R_v)$ and $\iota_S^{-1}(R_{S,v})$ are the maximal order of K_v. For such v's we may choose the isomorphism $\phi_v : R_v \xrightarrow{\simeq} R_{S,v}$ in such a way that $\phi_v \circ \iota = \iota_S$ on K_v. Indeed, starting with any isomorphism $\phi_v^? : R_v \to R_{S,v}$, we obtain two optimal embeddings $\phi_v^? \circ \iota$ and ι_S of $\mathcal{O}_{K_v}$ in $R_{S,v}$. By [**?**, Théorème 3.2 p. 44], any two such embeddings are conjugated by an element of $R_{S,v}^\times$: the corresponding conjugate of $\phi_v^?$ has the required property.

For those v's that satisfy (a) and (b), we thus obtain an isomorphism $\phi_v : B_v \to B_{S,v}$ such that $\phi_v(R_v) = R_{S,v}$ and $\phi_v \circ \iota = \iota_S$ on K_v. For the remaining v's not in S, we only require the second condition: $\phi_v \circ \iota = \iota_S$ on K_v. Such ϕ_v's do exists by the Skolem-Noether theorem [**?**, Théorème 2.1 p. 6]. The resulting collection $(\phi_v)_{v \notin S}$ satisfies (1) and (2).

2.1.4 Main objects.

Definition 2.1 We define the space CM of *CM points* , the space $\mathcal{X}(S)$ of *special points at* S and the space $\mathcal{Z}$ of *connected components* by

$$\begin{aligned} \mathrm{CM} &= \overline{T(\mathbf{Q})}\backslash G(\mathbf{A}_f) \\ \mathcal{X}(S) &= \overline{G(S,\mathbf{Q})}\backslash G(S) \\ \mathcal{Z} &= \overline{Z(\mathbf{Q})^+}\backslash Z(\mathbf{A}_f) \end{aligned}$$

where $\overline{T(\mathbf{Q})}$ is the closure of $T(\mathbf{Q})$ in $T(\mathbf{A}_f)$, $\overline{G(S,\mathbf{Q})}$ is the closure of $G(S,\mathbf{Q}) = \pi_S(G_S(\mathbf{Q}))$ in $G(S)$ and $\overline{Z(\mathbf{Q})^+}$ is the closure of $Z(\mathbf{Q})^+ = F^{>0}$ in $Z(\mathbf{A}_f)$.

These are locally compact totally discontinuous Hausdorff spaces equipped with a right, continuous and transitive action of $G(\mathbf{A}_f)$ (with $G(\mathbf{A}_f)$ acting on $\mathcal{X}(S)$ through ϕ_S and on $\mathcal{Z}$ through nr). By [**?**, Théorème 1.4 p. 61], $\mathcal{X}(S)$ and $\mathcal{Z}$ are *compact* spaces.

Definition 2.2 The *reduction map* RED_S *at* S, the *connected component map* c_S and their composite

$$c : \mathrm{CM} \xrightarrow{\mathrm{RED}_S} \mathcal{X}(S) \xrightarrow{c_S} \mathcal{Z}.$$

are respectively induced by

$$\mathrm{nr} : G(\mathbf{A}_f) \xrightarrow{\phi_S} G(S) \xrightarrow{\mathrm{nr}'_S} Z(\mathbf{A}_f).$$

Remark 2.3 Since $\phi_S(T(\mathbf{Q})) = \pi_S(T(\mathbf{Q})) \subset \pi_S(G_S(\mathbf{Q})) = G(S,\mathbf{Q})$, ϕ_S maps $\overline{T(\mathbf{Q})}$ to $\overline{G(S,\mathbf{Q})}$ and indeed induces a map $\mathrm{CM} \to \mathcal{X}(S)$. Similarly, c_S is well-defined since $\mathrm{nr}'_S(G(S,\mathbf{Q})) = \mathrm{nr}_S(G_S(\mathbf{Q})) = Z(\mathbf{Q})^+$ (by the norm theorem [**?**, Théorème 4.1 p.80]).

It follows from the relevant properties of nr, ϕ_S and nr'_S that c, RED_S and c_S are continuous, open and surjective $G(\mathbf{A}_f)$-equivariant maps. Since $\mathcal{X}(S)$ is compact, c_S is also a closed map.

Remark 2.4 The terminology *CM points*, *special points* and *connected components* is motivated by the example of Shimura curves: see the second part of this paper, especially section 3.2.

2.1.5 Galois actions.

The profinite commutative group $\overline{T(\mathbf{Q})}\backslash T(\mathbf{A}_f)$ acts continuously on CM, by multiplication on the *left*. This action is faithful and commutes with the right action of $G(\mathbf{A}_f)$. Using the inverse of Artin's reciprocity map $\mathrm{rec}_K : \overline{T(\mathbf{Q})}\backslash T(\mathbf{A}_f) \xrightarrow{\simeq} \mathrm{Gal}_K^{\mathrm{ab}}$, we obtain a continuous, $G(\mathbf{A}_f)$-equivariant and faithful action of $\mathrm{Gal}_K^{\mathrm{ab}}$ on CM.[1]

Similarly, Artin's reciprocity map $\mathrm{rec}_F : \overline{Z(\mathbf{Q})^+}\backslash Z(\mathbf{A}_f) \xrightarrow{\simeq} \mathrm{Gal}_F^{\mathrm{ab}}$ allows one to view $\mathcal{Z}$ as a principal homogeneous $\mathrm{Gal}_F^{\mathrm{ab}}$-space. From this point of view, $c : \mathrm{CM} \to \mathcal{Z}$ is a $\mathrm{Gal}_K^{\mathrm{ab}}$-equivariant map in the sense that for $x \in \mathrm{CM}$ and $\sigma \in \mathrm{Gal}_K^{\mathrm{ab}}$,

$$c(\sigma \cdot x) = \sigma \mid_{F^{\mathrm{ab}}} \cdot c(x).$$

2.1.6 Further objects

For technical purposes, we will also need to consider the following objects:

- $\mathcal{X}_S = \overline{G_S(\mathbf{Q})}\backslash G_S(\mathbf{A}_f)$, where $\overline{G_S(\mathbf{Q})}$ is the closure of $G_S(\mathbf{Q})$ in $G_S(\mathbf{A}_f)$.
- $q_S : \mathcal{X}_S \to \mathcal{X}(S)$ is induced by $\pi_S : G_S(\mathbf{A}_f) \to G(S)$.

The composite map $c_S \circ q_S : \mathcal{X}_S \to \mathcal{Z}$ is induced by $\mathrm{nr}_S : G_S(\mathbf{A}_f) \to Z(\mathbf{A}_f)$.

By [**?**, Théorème 1.4 p. 61], $\mathcal{X}_S$ is compact. Note that q_S is indeed well defined since $\pi_S(G_S(\mathbf{Q})) = G(S, \mathbf{Q})$. In fact, $\pi_S(\overline{G_S(\mathbf{Q})}) = \overline{G(S, \mathbf{Q})}$ since π_S is a closed map: the fibers of q_S are the $\ker(\pi_S)$-orbits in $\mathcal{X}_S$. In particular, q_S yields a $G(S)$-equivariant homeomorphism between $\mathcal{X}_S/\ker(\pi_S)$ and $\mathcal{X}(S)$.

2.1.7 Measures

The group $G^1(S) = \ker(\mathrm{nr}'_S)$ (resp. $G_S^1(\mathbf{A}_f) = \ker(\mathrm{nr}_S)$) acts on the fibers of c_S (resp. $c_S \circ q_S$). In section 2.4.1 below, we shall prove the following proposition. Recall that a Borel probability measure on a topological space is a measure defined on its Borel subsets which assigns voume 1 to the total space.

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[1] This action extends to a continuous, $G(\mathbf{A}_f)$-equivariant action of $\mathrm{Gal}(K^{\mathrm{ab}}/F)$ as follows. By the Skolem-Noether theorem, there exists an element $b \in B^\times$ such that $x \mapsto x^b = b^{-1}xb$ induces the non-trivial F-automorphism of K. In particular, b^2 belongs to $T(\mathbf{Q})$. Multiplication on the left by b induces an involution ι on CM such that for all $x \in \mathrm{CM}$ and $\sigma \in \mathrm{Gal}_K^{\mathrm{ab}}$, $\iota(\sigma x) = \sigma^\iota \iota x$ where $\sigma \mapsto \sigma^\iota$ is the involution on $\mathrm{Gal}_K^{\mathrm{ab}}$ which is induced by the nontrivial element of $\mathrm{Gal}(K/F)$.

Proposition 2.5 *The above actions are transitive and for each $z \in \mathcal{Z}$,* (1) *there exists a unique $G^1_S(\mathbf{A}_f)$-invariant Borel probability measure μ_z on $(c_S \circ q_S)^{-1}(z)$, and* (2) *there exists a unique $G^1(S)$-invariant Borel probability measure μ_z on $c_S^{-1}(z)$.*

The uniqueness implies that these two measures are compatible, in the sense that the (proper) map $q_S : (c_S \circ q_S)^{-1}(z) \to c_S^{-1}(z)$ maps one to the other: this is why we use the same notation[2] μ_z for both measures. Similarly, for any $g \in G^1_S(\mathbf{A}_f)$ (resp. $G(S)$), the measure $\mu_{z \cdot g}(\star g)$ equals μ_z on $(c_S \circ q_S)^{-1}(z)$ (resp. on $c_S^{-1}(z)$).

2.1.8 Level structures

For a compact open subgroup H of $G(\mathbf{A}_f)$, we denote by CM_H, $\mathcal{X}_H(S)$ and $\mathcal{Z}_H$ the quotients of CM, $\mathcal{X}(S)$ and $\mathcal{Z}$ by the right action of H. We still denote by c, RED_S and c_S the induced maps on these quotient spaces:

$$c : \mathrm{CM}_H \xrightarrow{\mathrm{RED}_S} \mathcal{X}_H(S) \xrightarrow{c_S} \mathcal{Z}_H.$$

Note that $\mathcal{X}_H(S)$ and $\mathcal{Z}_H$ are *finite* spaces, being discrete and compact. We have

$$\mathcal{Z}_H = \mathcal{Z}/\mathrm{nr}(H) \quad \text{and} \quad \mathcal{X}_H(S) = \mathcal{X}(S)/H(S) \simeq \mathcal{X}_S/H_S$$

where $H(S) = \phi_S(H) \subset G(S)$ and $H_S = \pi_S^{-1}(H(S)) \subset G_S$. The Galois group $\mathrm{Gal}_K^{\mathrm{ab}}$ still acts continuously on the (now discrete) spaces CM_H and $\mathcal{Z}_H$, and c is a $\mathrm{Gal}_K^{\mathrm{ab}}$-equivariant map.

2.2 Main theorems: the statements

2.2.1 Simultaneous reduction maps.

Let $\mathfrak{S}$ be a nonempty finite collection of finite sets of non-archimedean places of F not containing P and satisfying conditions **S1** to **S3** of section 2.1.1. That is: each element of $\mathfrak{S}$ is a finite set S of finite places of F such that $\forall v \in S$, v is not equal to P, K_v is a field, and B_v is split, and $|S| + |\mathrm{Ram}_f(B)| + [F : \mathbf{Q}]$ is even. For each S in $\mathfrak{S}$, we choose a totally definite quaternion algebra B_S over F with $\mathrm{Ram}_f(B_S) = \mathrm{Ram}_f(B) \cup S$, an embedding $\iota_S : K \to B_S$ and a collection of isomorphisms $(\phi_v : B_v \to B_{S,v})_{v \notin S}$ as in section 2.1.3.

[2] explain the 'star' notation: it looks weird to me.

For each S in $\mathfrak{S}$, we thus obtain (among other things) an algebraic group G_S over $\mathbf{Q}$, two locally compact and totally discontinuous adelic groups $G_S(\mathbf{A}_f)$ and $G(S)$, a commutative diagram of continuous homomorphisms as in Section 2.1.3, a *special set* $\mathcal{X}(S) = \overline{G(S,\mathbf{Q})}\backslash G(S)$, a *reduction map* $\mathrm{RED}_S : \mathrm{CM} \to \mathcal{X}(S)$ and a *connected component* map $c_S : \mathcal{X}(S) \to \mathcal{Z}$ with the property that each fiber $c_S^{-1}(z)$ of c_S has a unique Borel probability measure μ_z which is right invariant under $G^1(S) = \ker(\mathrm{nr}'_S)$ (we refer the reader to section 2.1 for all notations).

Let $\mathfrak{R}$ be a nonempty finite subset of $\mathrm{Gal}_K^{\mathrm{ab}}$ and consider the sequence

$$\mathrm{CM} \xrightarrow{\textsc{Red}} \mathcal{X}(\mathfrak{S},\mathfrak{R}) \xrightarrow{C} \mathcal{Z}(\mathfrak{S},\mathfrak{R})$$

where

- $\mathcal{X}(\mathfrak{S}) = \prod_{S\in\mathfrak{S}} \mathcal{X}(S)$ and $\mathcal{X}(\mathfrak{S},\mathfrak{R}) = \prod_{\sigma\in\mathfrak{R}} \mathcal{X}(\mathfrak{S}) = \prod_{S,\sigma} \mathcal{X}(S)$;
- $\mathcal{Z}(\mathfrak{S}) = \prod_{S\in\mathfrak{S}} \mathcal{Z}$ and $\mathcal{Z}(\mathfrak{S},\mathfrak{R}) = \prod_{\sigma\in\mathfrak{R}} \mathcal{Z}(\mathfrak{S}) = \prod_{S,\sigma} \mathcal{Z}$;
- $C : \mathcal{X}(\mathfrak{S},\mathfrak{R}) \to \mathcal{Z}(\mathfrak{S},\mathfrak{R})$ maps $x = (x_{S,\sigma})$ to $C(x) = (c_S(x_{S,\sigma}))$;
- $\textsc{Red} : \mathrm{CM} \to \mathcal{X}(\mathfrak{S},\mathfrak{R})$ is the *simultaneous reduction map* which sends x to $\textsc{Red}(x) = (\mathrm{RED}_S(\sigma\cdot x))$.

We also put $G(\mathfrak{S},\mathfrak{R}) = \prod_{S,\sigma} G(S)$ and $G^1(\mathfrak{S},\mathfrak{R}) = \prod_{S,\sigma} G^1(S)$, so that $G(\mathfrak{S},\mathfrak{R})$ acts on $\mathcal{X}(\mathfrak{S},\mathfrak{R})$ and $\mathcal{Z}(\mathfrak{S},\mathfrak{R})$, C is equivariant for these actions and its fibers are the $G^1(\mathfrak{S},\mathfrak{R})$-orbits in $\mathcal{X}(\mathfrak{S},\mathfrak{R})$. For $z = (z_{S,\sigma})$ in $\mathcal{Z}(\mathfrak{S},\mathfrak{R})$, the measure $\mu_z = \prod_{S,\sigma} \mu_{z_{S,\sigma}}$ is a $G^1(\mathfrak{S},\mathfrak{R})$-invariant Borel probability measure on $C^{-1}(z) = \prod_{S,\sigma} c_S^{-1}(z_{S,\sigma})$. If $g \in G(\mathfrak{S},\mathfrak{R})$ and $z \in \mathcal{Z}(\mathfrak{S},\mathfrak{R})$, $\mu_{z\cdot g}(\star g) = \mu_z$ on $C^{-1}(z)$.

The Galois group $\mathrm{Gal}_K^{\mathrm{ab}}$ acts diagonally on $\mathcal{Z}(\mathfrak{S},\mathfrak{R}) = \prod_{S,\sigma} \mathcal{Z}$ (through its quotient $\mathrm{Gal}_F^{\mathrm{ab}}$) and the composite map $C \circ \textsc{Red} : \mathrm{CM} \to \mathcal{Z}(\mathfrak{S},\mathfrak{R})$ is $\mathrm{Gal}_K^{\mathrm{ab}}$-equivariant. For $x \in \mathrm{CM}$, we shall frequently write $\bar{x} = C \circ \textsc{Red}(x)$. Explicitly:

$$\bar{x} = C \circ \textsc{Red}(x) = (\sigma \cdot c(x))_{S,\sigma} \in \mathcal{Z}(\mathfrak{S},\mathfrak{R}) = \prod_{S,\sigma} \mathcal{Z}.$$

2.2.2 Main theorem.

In this section, we state the main results, without proofs. The proofs are long, and will be given later.

Definition 2.6 A *P-isogeny class* of CM points is a $B_P^\times$-orbit in CM. If $\mathcal{H} \subset \mathrm{CM}$ is a P-isogeny class and f is a $\mathbf{C}$-valued function on CM, we say that $f(x)$ *goes to* $a \in \mathbf{C}$ *as* x *goes to infinity in* $\mathcal{H}$ if the following holds: for any $\epsilon > 0$, there exists a compact subset $C(\epsilon)$ of CM such that $|f(x) - a| \leq \epsilon$ for all $x \in \mathcal{H} \setminus C(\epsilon)$.

Remark 2.7 This definition can be somewhat clarified if we introduce the Alexandroff "one point" compactification $\widehat{\mathrm{CM}} = \mathrm{CM} \cup \{\infty\}$ of the locally compact space CM. It is easy to see that the point $\infty \in \widehat{\mathrm{CM}}$ lies in the closure of any P-isogeny class $\mathcal{H}$ (simply because P-isogeny classes are not relatively compact in CM). Our definition of "$f(x)$ goes to $a \in \mathbf{C}$ as x goes to infinity in $\mathcal{H}$" is then equivalent to the assertion that the limit of $f\,|_{\mathcal{H}}$ at ∞ exists and equals a.

Definition 2.8 An element $\sigma \in \mathrm{Gal}_K^{\mathrm{ab}}$ is *P-rational* if $\sigma = \mathrm{rec}_K(\lambda)$ for some $\lambda \in \widehat{K}^\times$ whose P-component λ_P belongs to the subgroup $K^\times \cdot F_P^\times$ of $K_P^\times$. We denote by $\mathrm{Gal}_K^{P-\mathrm{rat}} \subset \mathrm{Gal}_K^{\mathrm{ab}}$ the subgroup of all P-rational elements.

In the above definition, $\mathrm{rec}_K : \widehat{K}^\times \twoheadrightarrow \mathrm{Gal}_K^{\mathrm{ab}}$ is Artin's reciprocity map. We normalize the latter by specifying that it sends local uniformizers to *geometric* Frobeniuses.

Theorem 2.9 *Suppose that the finite subset $\mathfrak{R}$ of $\mathrm{Gal}_K^{\mathrm{ab}}$ consists of elements which are pairwise distinct modulo $\mathrm{Gal}_K^{P-\mathrm{rat}}$. Let $\mathcal{H} \subset \mathrm{CM}$ be a P-isogeny class and let $\mathcal{G}$ be a compact open subgroup of $\mathrm{Gal}_K^{\mathrm{ab}}$ with Haar measure dg. Then for every continuous function $f : \mathcal{X}(\mathfrak{S}, \mathfrak{R}) \to \mathbf{C}$,*

$$x \mapsto \int_{\mathcal{G}} f \circ \mathrm{R{\scriptstyle ED}}(g \cdot x) dg - \int_{\mathcal{G}} dg \int_{C^{-1}(g \cdot \bar{x})} f d\mu_{g \cdot \bar{x}}$$

goes to 0 *as* x *goes to infinity in* $\mathcal{H}$.

2.2.3 Surjectivity.

Let H be a compact open subgroup of $G(\mathbf{A}_f)$. Replacing CM, $\mathcal{X}$ and $\mathcal{Z}$ by CM_H, $\mathcal{X}_H$ and $\mathcal{Z}_H$ in the constructions of section 2.2.1, we obtain a sequence

$$\mathrm{CM}_H \xrightarrow{\textsc{Red}} \mathcal{X}_H(\mathfrak{S},\mathfrak{R}) \xrightarrow{C} \mathcal{Z}_H(\mathfrak{S},\mathfrak{R})$$

where

- $\mathcal{X}_H(\mathfrak{S},\mathfrak{R}) = \prod_{S,\sigma} \mathcal{X}_H(S) = \mathcal{X}(\mathfrak{S},\mathfrak{R})/H(\mathfrak{S},\mathfrak{R})$ and
- $\mathcal{Z}_H(\mathfrak{S},\mathfrak{R}) = \prod_{S,\sigma} \mathcal{Z}_H = \mathcal{Z}(\mathfrak{S},\mathfrak{R})/H(\mathfrak{S},\mathfrak{R})$ with
- $H(\mathfrak{S},\mathfrak{R}) = \prod_{S,\sigma} H(S)$, a compact open subgroup of $G(\mathfrak{S},\mathfrak{R})$.

Applying the main theorem to the characteristic functions of the (finitely many) elements of $\mathcal{X}_H(\mathfrak{S},\mathfrak{R})$, we obtain the following surjectivity result. Let $\overline{\mathcal{H}}$ be the image of $\mathcal{H}$ in CM_H.

Corollary 2.10 *For all but finitely many x in $\overline{\mathcal{H}}$,*

$$\textsc{Red}(\mathcal{G}\cdot x) = C^{-1}(\mathcal{G}\cdot \bar{x}) \quad \text{in } \mathcal{X}_H(\mathfrak{S},\mathfrak{R})$$

where $\bar{x} = C \circ \textsc{Red}(x) \in \mathcal{Z}_H(\mathfrak{S},\mathfrak{R})$.

2.2.4 Equidistribution.

When $H = \widehat{R}^\times$ for some Eichler order R in B, we can furthermore specify the asymptotic behavior (as x varies inside $\overline{\mathcal{H}}$) of

$$\mathrm{Prob}\left\{\textsc{Red}(\mathcal{G}\cdot x) = s\right\} = \frac{1}{|\mathcal{G}\cdot x|}\left|\{g\cdot x;\ \textsc{Red}(g\cdot x) = s,\ g\in\mathcal{G}\}\right|$$

where s is a fixed point in $\mathcal{X}_H(\mathfrak{S},\mathfrak{R})$. To state our result, we first need to define a few constants.

Let $\mathcal{N} = \prod_Q Q^{n_Q}$ be the level of R. By construction, the compact open subgroup $H_S = \pi_S^{-1}\phi_S(H)$ of $G_S(\mathbf{A}_f)$ equals $\widehat{R}_S^\times$ for some Eichler order $R_S \subset B_S$ whose level $\mathcal{N}_S$ is the "prime-to-S" part of $\mathcal{N}$: $\mathcal{N}_S = \prod_{Q\notin S} Q^{n_Q}$. For $g \in G_S(\mathbf{A}_f)$ and $x = G_S(\mathbf{Q})gH_S$ in

$$\mathcal{X}_H(S) \simeq \mathcal{X}_S/H_S = \overline{G_S(\mathbf{Q})}\backslash G_S(\mathbf{A}_f)/H_S = G_S(\mathbf{Q})\backslash G_S(\mathbf{A}_f)/H_S$$

put $\mathcal{O}(g) = g\widehat{R}_S g^{-1} \cap B_S$. This is an $\mathcal{O}_F$-order in B_S whose $B_S^\times$-conjugacy class depends only upon x. The isomorphism class of the group $\mathcal{O}(g)^\times/\mathcal{O}_F^\times$ also depends only upon x and since B_S is totally definite, this group is finite [**?**, p. 139]. The *weight* $\omega(x)$ of x is the order of this group: $\omega(x) = [\mathcal{O}(g)^\times : \mathcal{O}_F^\times]$. The *weight* of an element $s = (x_{S,\sigma})$ in $\mathcal{X}_H(\mathfrak{S}, \mathfrak{R})$ is then given by $\omega(s) = \prod_{S,\sigma} \omega(x_{S,\sigma})$.

Finally, we put

$$\Omega = \frac{1}{\Omega(\mathcal{G})} \cdot \left(\prod_{S \in \mathfrak{S}} \frac{\Omega(F)}{\Omega(B_S) \cdot \Omega(\mathcal{N}_S)} \right)^{|\mathfrak{R}|}$$

where

- $\Omega(F) = 2^{2[F:\mathbf{Q}]-1} [\mathcal{O}_F^\times : \mathcal{O}_F^{>0}]^{-1} \left|\zeta_F(-1)\right|^{-1}$,
- $\Omega(B_S) = \prod_{Q \in \mathrm{Ram}_f(B_S)} (\|Q\| - 1)$,
- $\Omega(\mathcal{N}_S) = \|\mathcal{N}_S\| \cdot \prod_{Q|\mathcal{N}_S} (\|Q\|^{-1} + 1)$ and
- $\Omega(\mathcal{G})$ is the order of the image of $\mathcal{G}$ in the Galois group $\mathrm{Gal}(F_1^+/F)$ of the narrow Hilbert class field F_1^+ of F.

Corollary 2.11 *For all $\epsilon > 0$, there exists a finite set $\overline{\mathcal{C}}(\epsilon) \subset \overline{\mathcal{H}}$ such that for all $s \in \mathcal{X}_H(\mathfrak{S}, \mathfrak{R})$ and $x \in \overline{\mathcal{H}} \setminus \overline{\mathcal{C}}(\epsilon)$,*

$$\left| \mathrm{Prob}\left\{ \mathrm{RED}(\mathcal{G} \cdot x) = s \right\} - \frac{\Omega}{\omega(s)} \right| \leq \epsilon$$

if s belongs to $C^{-1}(\mathcal{G} \cdot \bar{x})$ and $\mathrm{Prob}\left\{ \mathrm{RED}(\mathcal{G} \cdot x) = s \right\} = 0$ *otherwise.*

The remainder of this first part of the paper is devoted to the proofs of Proposition 2.5, Theorem 2.9, Corollary 2.10, Corollary 2.11.

2.3 Proof of the main theorems: first reductions

Notations.

For a continuous function $f : \mathcal{X}(\mathfrak{S}, \mathfrak{R}) \to \mathbf{C}$ and $x \in \mathrm{CM}$, we put

$$A(f, x) = \int_{\mathcal{G}} f \circ \mathrm{RED}(g \cdot x) dg \quad \text{and} \quad B(f, x) = B(f, \bar{x}) = \int_{\mathcal{G}} I(f, g \cdot \bar{x}) dg$$

where $\bar{x} = C \circ \textsc{Red}(x)$, with $I(f,z) = \int_{C^{-1}(z)} f d\mu_z$ for $z \in \mathcal{Z}(\mathfrak{S}, \mathfrak{R})$.

Then the theorem says that for all $\epsilon > 0$, there exists a compact subset $\mathcal{C}(\epsilon) \subset \mathrm{CM}$ such that,

$$\forall x \in \mathcal{H},\, x \notin \mathcal{C}(\epsilon) : \quad |A(f,x) - B(f,x)| \leq \epsilon.$$

We claim that the functions $x \mapsto A(f,x)$ and $x \mapsto B(f,x)$ are well-defined. This is clear for $A(f,x)$, as $g \mapsto f \circ \textsc{Red}(g \cdot x)$ is continuous on $\mathcal{G}$. For $B(f,x)$, we claim that $g \mapsto I(f, g\cdot\bar{x})$ is also continuous. Since $g \mapsto g\cdot\bar{x}$ is continuous, it is sufficient to show that $z \mapsto I(f,z)$ is continuous on $\mathcal{Z}(\mathfrak{S}, \mathfrak{R})$. Note that for $u \in G(\mathfrak{S}, \mathfrak{R})$,

$$I(f, z\cdot u) - I(f,z) = \int_{C^{-1}(z\cdot u)} f d\mu_{z\cdot u} - \int_{C^{-1}(z)} f d\mu_z = \int_{C^{-1}(z)} (f(\star u) - f)\, d\mu_z.$$

Since f is continuous and $\mathcal{X}(\mathfrak{S}, \mathfrak{R})$ is compact, f is uniformly continuous. It follows that $I(f, z\cdot u) - I(f,z)$ is small when u is small and $z \mapsto I(f,z)$ is indeed continuous.

To prove the theorem, we may assume that f is locally constant. Indeed, there exists a locally constant function $f' : \mathcal{X}(\mathfrak{S}, \mathfrak{R}) \to \mathbf{C}$ such that $\|f - f'\| \leq \epsilon/3$. If the theorem were known for f', we could find a compact subset $\mathcal{C}(\epsilon) \subset \mathrm{CM}$ such that $|A(f',x) - B(f',x)| \leq \epsilon/3$ for all $x \in \mathcal{H}$ with $x \notin \mathcal{C}(\epsilon)$, thus obtaining

$$\begin{aligned} &|A(f,x) - B(f,x)| \\ &\quad \leq \ |A(f,x) - A(f',x)| + |A(f',x) - B(f',x)| + |B(f',x) - B(f,x)| \\ &\quad \leq \ \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

A decomposition of $\mathcal{G}\cdot\mathcal{H}\cdot H$.

From now on, we shall thus assume that f is locally constant. Let H be a compact open subgroup of $G(\mathbf{A}_f)$ such that f factors through $\mathcal{X}(\mathfrak{S}, \mathfrak{R})/H(\mathfrak{S}, \mathfrak{R})$, where $H(\mathfrak{S}, \mathfrak{R}) = \prod_{S,\sigma} H(S)$ with $H(S) = \phi_S(H)$. Then

- $x \mapsto A(x) = \int_{\mathcal{G}} f \circ \textsc{Red}(g\cdot x) dg$ factors through $\mathcal{G}\backslash\mathrm{CM}/H$,
- $z \mapsto I(z) = \int_{C^{-1}(z)} f d\mu_z$ factors through $\mathcal{Z}(\mathfrak{S}, \mathfrak{R})/H(\mathfrak{S}, \mathfrak{R})$, hence
- $x \mapsto B(x) = B(\bar{x}) = \int_{\mathcal{G}} I(g\cdot\bar{x}) dg$ factors through $\mathcal{G}\backslash\mathrm{CM}/H$

(where $\bar{x} = C \circ \textsc{Red}(x) \in \mathcal{Z}(\mathfrak{S}, \mathfrak{R})$ as usual).

For a nonzero nilpotent element $N \in B_P$, the formula $u(t) = 1 + tN$ defines a group isomorphism $u : F_P \to U = u(F_P) \subset B_P^\times$. We say that $U = \{u(t)\}$ is a one parameter unipotent subgroup of $B_P^\times$.

Proposition 2.12 *There exists:* (1) *a* finite *set* $\mathcal{I}$, (2) *for each* $i \in \mathcal{I}$, *a point* $x_i \in \mathcal{H}$ *and a one parameter unipotent subgroup* $U_i = \{u_i(t)\}$ *of* $B_P^\times$, *and* (3) *a compact open subgroup* κ *of* $F_P^\times$ *such that*

1. $\mathcal{G} \cdot \mathcal{H} \cdot H = \bigcup_{i \in \mathcal{I}} \bigcup_{n \geq 0} \mathcal{G} \cdot x_i \cdot u_i(\kappa_n) \cdot H$, *and*
2. $\forall i \in \mathcal{I}$ *and* $\forall n \geq 0$, $\mathcal{G} \cdot x_i \cdot u_i(\kappa_n) \cdot H = \mathcal{G} \cdot x_i u_{i,n} \cdot H$,

where $\kappa_n = \varpi_P^{-n} \kappa \subset F_P^\times$ *and* $u_{i,n} = u_i(\varpi_P^{-n}) \in u_i(\kappa_n)$.

Proof. Section 2.6.

Unipotent orbits: reduction of Theorem 2.9

This decomposition allows us to switch from Galois (=toric) orbits to unipotent orbits of CM points. To deal with the latter, we have the following proposition. We fix a CM point $x \in \mathcal{H}$ and a one parameter unipotent subgroup $U = \{u(t)\}$ in $B_P^\times$. We also choose a Haar measure $\lambda = dt$ on F_P. Then Theorem 2.9 follows from Proposition 2.12 and

Proposition 2.13 *Under the assumptions of Theorem 2.9, for almost all* $g \in \mathrm{Gal}_K^{ab}$,

$$\lim_{n \to \infty} \frac{1}{\lambda(\kappa_n)} \int_{\kappa_n} f \circ \textsc{Red}(g \cdot x \cdot u(t)) dt = \int_{C^{-1}(g \cdot \bar{x})} f d\mu_{g \cdot \bar{x}}.$$

Proof. Section 2.5.

To deduce Theorem 2.9, we may argue as follows. By taking the integral over $g \in \mathcal{G}$ and using (a) Lebesgue's dominated convergence theorem to exchange $\int_{\mathcal{G}}$ and $\lim_n$, and (b) Fubini's theorem to exchange $\int_{\mathcal{G}}$ and $\int_{\kappa_n}$, we obtain:

$$\lim_{n \to \infty} \frac{1}{\lambda(\kappa_n)} \int_{\kappa_n} A(x \cdot u(t)) dt = B(x).$$

This holds for all x and u.

Then for $x = x_i$ and $u = u_i$, we also know from part (2) of Proposition 2.12 that $t \mapsto A(x_i \cdot u_i(t))$ is constant on κ_n, equal to $A(x_i u_{i,n})$. In particular,

$$\forall i \in \mathcal{I}: \quad \lim_{n\to\infty} A(x_i u_{i,n}) = B(x_i).$$

Fix $\epsilon > 0$ and choose $N \geq 0$ such that

$$\forall n > N, \forall i \in \mathcal{I}: \quad |A(x_i u_{i,n}) - B(x_i)| \leq \epsilon.$$

Put $\mathcal{C}(\epsilon) = \bigcup_{i\in\mathcal{I}} \bigcup_{n=0}^{N} \mathcal{G} \cdot x_i u_i(\kappa_n) \cdot H$, a compact subset of CM.

For any $x \in \mathcal{H}$, there exists $i \in \mathcal{I}$ and $n \geq 0$ such that x belongs to $\mathcal{G} \cdot x_i u_{i,n} H$, so that $A(x) = A(x_i u_{i,n})$ and $B(x) = B(x_i u_{i,n}) = B(x_i)$. If $x \notin \mathcal{C}(\epsilon)$, $n > N$ and $|A(x) - B(x)| \leq \epsilon$, QED.

Reduction of Corollaries 2.10 and 2.11.

Let H be a compact open subgroup of $G(\mathbf{A}_f)$ and let $f : \mathcal{X}(\mathfrak{S}, \mathfrak{R}) \to \{0, 1\}$ be the characteristic function of some $s \in \mathcal{X}_H(\mathfrak{S}, \mathfrak{R})$, say $s = \widetilde{s} \cdot H(\mathfrak{S}, \mathfrak{R})$ with $\widetilde{s} \in \mathcal{X}(\mathfrak{S}, \mathfrak{R})$. The function $z \mapsto I(f, z) = \int_{C^{-1}(z)} f d\mu_z$ factors through $\mathcal{Z}_H(\mathfrak{S}, \mathfrak{R})$ and equals 0 outside $C(s) = C(\widetilde{s}) \cdot H(\mathfrak{S}, \mathfrak{R})$. Let $I(s)$ be its value on $C(s)$.

For $\widetilde{x} \in$ CM, we easily obtain:

- $A(f, \widetilde{x}) = \text{Prob}\{\text{Red}(\mathcal{G} \cdot x) = s\}$ where x is the image of $\widetilde{x}$ in CM_H,
- $B(f, \widetilde{x}) = 0$ if $\bar{x} = C \circ \text{Red}(x)$ does not belong to $\mathcal{G} \cdot C(s)$, and
- $B(f, \widetilde{x}) = I(s)/\Omega(\mathcal{G}, H)$ otherwise, where $\Omega(\mathcal{G}, H)$ is the common size of all $\mathcal{G}$-orbits in $\mathcal{Z}(\mathfrak{S}, \mathfrak{R})/H(\mathfrak{S}, \mathfrak{R}) \simeq \prod_{S,\sigma} \mathcal{Z}/\text{nr}(H)$, which is also the size of the $\mathcal{G}$-orbits in $\mathcal{Z}/\text{nr}(H)$.

If $\text{nr}(H)$ is the maximal compact subgroup $\widehat{\mathcal{O}}_F^\times$ of $Z(\mathbf{A}_f)$ (which occurs when $H = \widehat{R}^\times$ for some Eichler order $R \subset B$), $\mathcal{Z}/\text{nr}(H) \simeq \text{Gal}(F_1^+/F)$ and $\Omega(\mathcal{G}, H)$ is the order of the image of $\mathcal{G}$ in $\text{Gal}(F_1^+/F)$: $\Omega(\mathcal{G}, H) = \Omega(\mathcal{G})$.

The main theorem asserts that for all $\epsilon > 0$, there exists a compact subset $\mathcal{C}(\epsilon)$ of CM such that for all $x \in \overline{\mathcal{H}} \setminus \overline{\mathcal{C}}(\epsilon)$ (where $\overline{\mathcal{H}}$ and $\overline{\mathcal{C}}(\epsilon)$ are the images of $\mathcal{H}$ and $\mathcal{C}(\epsilon)$ in CM_H),

$$|\text{Prob}\{\text{Red}(\mathcal{G} \cdot x) = s\} - I(s)/\Omega(\mathcal{G}, H)| \leq \epsilon$$

if $s \in C^{-1}(\mathcal{G} \cdot \bar{x})$ and $\mathrm{Prob}\,\{\mathrm{Red}(\mathcal{G} \cdot x) = s\} = 0$ otherwise. Note that $\overline{\mathcal{C}}(\epsilon)$ is finite, being compact and discrete. To prove the corollaries, it remains to (1) show that $I(s)$ is nonzero and (2) compute $I(s)$ exactly when H arises from an Eichler order in B.

Write $s = (x_{S,\sigma})$ with $x_{S,\sigma} = \widetilde{x}_{S,\sigma} H_S$ in $\mathcal{X}(S)/H(S) \simeq \mathcal{X}_S/H_S$ ($S \in \mathfrak{S}$, $\sigma \in \mathfrak{R}$ and $\widetilde{x}_{S,\sigma} \in \mathcal{X}_S$). Then $I(s) = \prod_{S,\sigma} I(s)_{S,\sigma}$ with

$$I(s)_{S,\sigma} = \int_{(c_S \circ q_S)^{-1}(z_{S,\sigma})} f_{S,\sigma} d\mu_{z_{S,\sigma}}$$

where $z_{S,\sigma} = c_S \circ q_S(\widetilde{x}_{S,\sigma}) \in \mathcal{Z}$ and $f_{S,\sigma} : \mathcal{X}_S \to \{0,1\}$ is the characteristic function of $x_{S,\sigma}$.

Proposition 2.14 (1) *For all $S \in \mathfrak{S}$ and $\sigma \in \mathfrak{R}$, $I(s)_{S,\sigma} > 0$.*

(2) *If $H = \widehat{R}^\times$ for some Eichler order $R \subset B$ of level $\mathcal{N}$,*

$$I(s)_{S,\sigma} = \frac{1}{\omega(x_{S,\sigma})} \cdot \frac{\Omega(F)}{\Omega(B_S) \cdot \Omega(\mathcal{N}_S)}$$

with $\omega(\star)$, $\Omega(F)$, $\Omega(B_S)$ and $\Omega(\mathcal{N}_S)$ as in section 2.2.4.

Proof. See section 2.4.2, especially Proposition 2.18.

In particular, $I(s) > 0$ and if $H = \widehat{R}^\times$ with R as above,

$$I(s) = \frac{1}{\omega(s)} \cdot \left(\prod_{S \in \mathfrak{S}} \frac{\Omega(F)}{\Omega(B_S) \cdot \Omega(\mathcal{N}_S)} \right)^{|\mathfrak{R}|}.$$

Thus we obtain Corollaries 2.10 and 2.11.

2.4 Further reductions

The arguments of the last section have reduced our task to proving Propositions 2.5, 2.12, 2.13, and 2.14. In this section, we make some further steps in this direction. Section 2.4.1 gives the proof of Proposition 2.5. Section 2.4.2 gives the proof of Proposition 2.14. Finally, Section 2.4.3 is a step towards Ratner's theorem and the proof of Proposition 2.14.

Throughout this section, S is a finite set of finite places of F subject to the condition **S1** to **S3** of section 2.1.1.

2.4.1 Existence of a measure and proof of Proposition 2.5

These are compact spaces with a continuous right action of $G_S^1(\mathbf{A}_f) = \ker(\mathrm{nr}_S)$ and $G^1(S) = \ker(\mathrm{nr}'_S)$ respectively.

We shall repeatedly apply the following principle:

Lemma 2.15 *[**?**, Lemme 1.2 page 105] Suppose that L and C are topological groups with L locally compact and C compact. If Λ is a discrete and cocompact subgroup of $L \times C$, the projection of Λ to L is a discrete and cocompact subgroup of L.*

By [**?**, Théorème 1.4 page 61], $G_S^1(\mathbf{Q})$ diagonally embedded in $G_S^1(\mathbf{A}_f) \times G_S^1(\mathbf{A}_\infty)$ is a discrete and cocompact subgroup. Since $G_S^1(\mathbf{A}_\infty)$ is compact, $G_S^1(\mathbf{Q})$ is also discrete and cocompact in $G_S^1(\mathbf{A}_f)$. Since the sequence

$$1 \to \ker(\pi_S) \to G_S^1(\mathbf{A}_f) \to G^1(S) \to 1$$

is split exact with $\ker(\pi_S)$ compact, $G^1(S, \mathbf{Q}) = \pi_S(G_S^1(\mathbf{Q}))$ is again a discrete and cocompact subgroup of $G^1(S)$.

Lemma 2.16 *The fibers of $c_S \circ q_S$ are the $G_S^1(\mathbf{A}_f)$-orbits in $\mathcal{X}_S$. For $g \in G_S(\mathbf{A}_f)$ and $x = \overline{G_S(\mathbf{Q})}g$ in $\mathcal{X}_S$, the stabilizer of x in $G_S^1(\mathbf{A}_f)$ is a discrete and cocompact subgroup of $G_S^1(\mathbf{A}_f)$ given by $\mathrm{Stab}_{G_S^1(\mathbf{A}_f)}(x) = g^{-1}G_S^1(\mathbf{Q})g$.*

Proof. Fix $x = \overline{G_S(\mathbf{Q})}g$ in $\mathcal{X}_S$ and put $z = c_S \circ q_S(x) = \overline{Z(\mathbf{Q})^+}\mathrm{nr}_S(g) \in \mathcal{Z}$. The fiber of $c_S \circ q_S$ above z is the image of $L = \mathrm{nr}_S^{-1}\left(\overline{Z(\mathbf{Q})^+}\mathrm{nr}_S(g)\right)$ in $\mathcal{X}_S$ and the stabilizer of x in $G_S^1(\mathbf{A}_f)$ equals $M = G_S^1(\mathbf{A}_f) \cap g^{-1}\overline{G_S(\mathbf{Q})}g$. We have to show that $L = \overline{G_S(\mathbf{Q})}gG_S^1(\mathbf{A}_f)$ and $M = g^{-1}G_S^1(\mathbf{Q})g$.

We break this up into a series of steps.

Step 1: *$\overline{G_S(\mathbf{Q})}gG_S^1(\mathbf{A}_f)$ is closed in $G_S(\mathbf{A}_f)$.* This is equivalent to saying that the $G_S^1(\mathbf{A}_f)$-orbit of x is closed in $\mathcal{X}_S$. Since M contains $g^{-1}G_S^1(\mathbf{Q})g$ which is cocompact in $G_S^1(\mathbf{A}_f)$, M itself is cocompact in $G_S^1(\mathbf{A}_f)$. It follows that $x \cdot G_S^1(\mathbf{A}_f)$ is compact, hence closed in $\mathcal{X}_S$.

Step 2: $L = \overline{G_S(\mathbf{Q})}gG_S^1(\mathbf{A}_f)$. Since $\mathrm{nr}_S : G_S(\mathbf{A}_f) \to Z(\mathbf{A}_F)$ is open, L is the closure of $\mathrm{nr}_S^{-1}\left(Z(\mathbf{Q})^+\mathrm{nr}_S(g)\right)$ in $G_S(\mathbf{A}_f)$. The norm theorem [**?**, Théorème 4.1 p. 80] implies that $\mathrm{nr}_S^{-1}\left(Z(\mathbf{Q})^+\mathrm{nr}_S(g)\right) = G_S(\mathbf{Q})gG_S^1(\mathbf{A}_f)$ and then $L = \overline{G_S(\mathbf{Q})}gG_S^1(\mathbf{A}_f)$ by **(1)**.

Step 3: $\overline{G_S(\mathbf{Q})} = \overline{Z(\mathbf{Q})} G_S(\mathbf{Q})$ **.** This is easy. See for instance the proof of Corollary 3.10.

Step 4: $M = g^{-1} G_S^1(\mathbf{Q}) g$**.** Suppose that γ belongs to $M = G_S^1(\mathbf{A}_f) \cap g^{-1}\overline{G_S(\mathbf{Q})} g$. By **(3)**, $\gamma = g^{-1} \lambda g_{\mathbf{Q}} g$ for some $\lambda \in \overline{Z(\mathbf{Q})}$ and $g_{\mathbf{Q}} \in G_S(\mathbf{Q})$ with $\mathrm{nr}_S(\gamma) = 1$. Then $\alpha = \lambda^2 = \mathrm{nr}(g_{\mathbf{Q}}^{-1})$ belongs to $\overline{Z(\mathbf{Q})}^2 \cap Z(\mathbf{Q})^+ \subset Z(\mathbf{A}_f)^2 \cap Z(\mathbf{Q})^+$. Since α belongs to $Z(\mathbf{Q})^+ \subset F^\times$, we may form the abelian extension $F(\sqrt{\alpha})$ of F. Since α also belongs to $Z(\mathbf{A}_f)^2$, this extension splits everywhere and is therefore trivial: $\alpha = \lambda_0^2$ for some $\lambda_0 \in F^\times$. Then λ/λ_0 is an element of order 2 in $\overline{Z(\mathbf{Q})} \cap \widehat{\mathcal{O}}_F^\times$. Since $\overline{Z(\mathbf{Q})} \cap \widehat{\mathcal{O}}_F^\times = \overline{\mathcal{O}_F^\times}$ is isomorphic to the profinite completion of $O_F^\times$ (a finite type $\mathbf{Z}$-module), λ/λ_0 actually belongs to $\{\pm 1\}$, the torsion subgroup of $\mathcal{O}_F^\times$. We have shown that λ belongs to $Z(\mathbf{Q})$, hence $\gamma = g^{-1}\lambda g_{\mathbf{Q}} g$ belongs to $g^{-1} G_S(\mathbf{Q}) g \cap G_S^1(\mathbf{A}_f) = g^{-1} G_S^1(\mathbf{Q}) g$.

Since (1) q_S identifies $\mathcal{X}_S/\ker(\pi_S)$ with $\mathcal{X}(S)$ and (2) $G_S^1(\mathbf{A}_f) \simeq G^1(S) \times \ker(\pi_S)$ with $\ker(\pi_S)$ compact, we obtain:

Lemma 2.17 *The fibers of c_S are the $G^1(S)$-orbits in $\mathcal{X}(S)$. For $g \in G(S)$ and $x = \overline{G(S, \mathbf{Q})} g$ in $\mathcal{X}(S)$, the stabilizer of x in $G^1(S)$ is a discrete and cocompact subgroup of $G^1(S)$ given by $\mathrm{Stab}_{G^1(S)}(x) = g^{-1} G^1(S, \mathbf{Q}) g$.*

For $z \in \mathcal{Z}$ and $x \in (c_S \circ q_S)^{-1}(z)$, the map $g \mapsto x \cdot g$ induces a $G_S^1(\mathbf{A}_f)$-equivariant homeomorphism between $\mathrm{Stab}(x)\backslash G_S^1(\mathbf{A}_f)$ and $(c_S \circ q_S)^{-1}(z)$. Similarly, any $x \in c_S^{-1}(z)$ defines a $G^1(S)$-equivariant homeomorphism between $\mathrm{Stab}(x)\backslash G^1(S)$ and $c_S^{-1}(z)$. Proposition 2.5 easily follows.

2.4.2 A computation.

Any Haar measure μ^1 on $G_S^1(\mathbf{A}_f)$ induces a collection of $G_S^1(\mathbf{A}_f)$-invariant Borel measures μ_z^1 on the fibers $(c_S \circ q_S)^{-1}(z)$ of $c_S \circ q_S : \mathcal{X}_S \to \mathcal{Z}$. These measures are characterized by the fact that for any compact open subgroup H_S^1 of $G_S^1(\mathbf{A}_f)$ and any $x \in (c_S \circ q_S)^{-1}(z)$,

$$\mu_z^1(x \cdot H_S^1) = \frac{\mu^1(H_S^1)}{\left|\mathrm{Stab}_{H_S^1}(x)\right|}$$

($\mathrm{Stab}_{H_S^1}(x) = \mathrm{Stab}_{G_S^1(\mathbf{A}_f)}(x) \cap H_S^1$ is indeed finite since $\mathrm{Stab}_{G_S^1(\mathbf{A}_f)}(x)$ is discrete while H_S^1 is compact). One easily checks that $\mu_{z\cdot g}^1(\star g)$ equals μ_z^1 on

$(c_S \circ q_S)^{-1}(z)$ for any $g \in G_S(\mathbf{A}_f)$. It follows that these measures assign the same volume λ to each fiber of $c_S \circ q_S$, and $\mu_z^1 = \lambda \mu_z$ on $(c_S \circ q_S)^{-1}(z)$.

We shall now simultaneously determine λ (or find out which normalization of μ^1 yields $\lambda = 1$) and compute a formula for

$$\varphi_z(x) = \mu_z\left(xH_S \cap (c_S \circ q_S)^{-1}(z)\right) \qquad (x \in \mathcal{X}_S, z \in \mathcal{Z})$$

where H_S is a compact open subgroup of $G_S(\mathbf{A}_f)$. The map $z \mapsto \varphi_z(x)$ factors through $\mathcal{Z}/\mathrm{nr}_S(H_S)$ and equals 0 outside $c_S \circ q_S(xH_S) = c_S \circ q_S(x) \cdot \mathrm{nr}_S(H_S)$.

Let $z_1, \cdots, z_n$ be a set of representatives for $\mathcal{Z}/\mathrm{nr}_S(H_S)$ and for $1 \le i \le n$, let $x_{i,1}, \cdots, x_{i,n_i}$ be a set of representatives in $(c_S \circ q_S)^{-1}(z_i)$ of

$$(c_S \circ q_S)^{-1}(z_i \mathrm{nr}_S(H_S))/H_S = (c_S \circ q_S)^{-1}(z_i) \cdot H_S/H_S.$$

The $x_{i,j}$'s then form a set of representatives for $\mathcal{X}_S/H_S$ and

$$\textstyle\sum_{i,j} \varphi_{z_i}(x_{i,j}) = \sum_i \mu_{z_i}\left(\cup_{j=1}^{n_i} x_{i,j} H_S \cap (c_S \circ q_S)^{-1}(z_i)\right) = \sum_i 1 = n \qquad (1)$$

since $(x_{i,j}H_S)_{j=1}^{n_i}$ covers $(c_S \circ q_S)^{-1}(z_i)$.

To compute $\varphi_z(x)$, we may assume that $z = c_S \circ q_S(x)$. Choose $g \in G_S(\mathbf{A}_f)$ such that $x = \overline{G_S(\mathbf{Q})}g$ and put $H_S^1 = H_S \cap G_S^1(\mathbf{A}_f)$. By Lemma 2.16, the map $b \mapsto x \cdot b$ yields a bijection

$$g^{-1}G_S^1(\mathbf{Q})g \backslash (g^{-1}\overline{G_S(\mathbf{Q})}g \cdot H_S) \cap G_S^1(\mathbf{A}_f)/H_S^1 \xrightarrow{\simeq} xH_S \cap (c_S \circ q_S)^{-1}(z)/H_S^1. \qquad (2)$$

Note that $g^{-1}\overline{G_S(\mathbf{Q})}g \cdot H_S = g^{-1}G_S(\mathbf{Q})g \cdot H_S$. Let $(a_k b_k)_{k=1}^m$ be a set of representatives for the left hand side of (2), with a_k in $g^{-1}G_S(\mathbf{Q})g$, b_k in H_S and $\mathrm{nr}_S(a_k b_k) = 1$. Since $x \cdot a_k = x$ and b_k normalizes H_S^1,

$$\varphi_z(x) = \sum_{k=1}^m \mu_z\left(x \cdot a_k b_k H_S^1\right) = \frac{m}{|H_S^1 \cap g^{-1}G_S^1(\mathbf{Q})g|} \times \frac{\mu^1(H_S^1)}{\lambda} \cdot .$$

On the other hand, the map $a_k b_k \mapsto \mathrm{nr}_S(a_k) = \mathrm{nr}_S(b_k)^{-1}$ yields a bijection between the left hand side of (2) and

$$\mathrm{nr}_S\left(H_S \cap g^{-1}G_S(\mathbf{Q})g\right) \backslash \mathrm{nr}_S\left(H_S\right) \cap \mathrm{nr}_S(G_S(\mathbf{Q})).$$

Since $\mathrm{nr}_S(G_S(\mathbf{Q})) = Z(\mathbf{Q})^+$, we obtain

$$\varphi_z(x) = \frac{|q(g, H_S)|}{|k(g, H_S)|} \times \frac{\mu^1(H_S^1)}{\lambda} \qquad (3)$$

where $k(g,H_S)$ and $q(g,H_S)$ are respectively the kernel and cokernel of

$$gH_Sg^{-1}\cap G_S(\mathbf{Q}) \xrightarrow{\mathrm{nr}_S} \mathrm{nr}_S(H_S)\cap Z(\mathbf{Q})^+.$$

When $H_S = \widehat{R}_S^\times$ for some *Eichler* order R_S in B_S, the following simplifications occur:

- $\mathrm{nr}_S(H_S) = \widehat{\mathcal{O}}_F^\times$, so that $n = |\mathcal{Z}/\mathrm{nr}_S(H_S)| = \left|\widehat{F}^\times/F^{>0}\widehat{\mathcal{O}}_F^\times\right| = h_F^+$ is the order of the narrow class group of F. Note that $h_F^+ = h_F\cdot[\mathcal{O}_F^{>0} : (\mathcal{O}_F^\times)^2]$, where h_F is the class number of F and $(\mathcal{O}_F^\times)^2 = \{x^2 \mid x\in\mathcal{O}_F^\times\}$.
- The map $g\mapsto L(g) = g\cdot\widehat{R}_S\cap B_S$ yields a bijection between $\mathcal{X}_S/H_S = G_S(\mathbf{Q})\backslash G_S(\mathbf{A}_f)/H_S$ and the set of isomorphism classes of nonzero right R-ideals in B_S. Moreover, the left order $\mathcal{O}(g)$ of $L(g)$ equals $g\widehat{R}_Sg^{-1}\cap B_S$, so that $\mathcal{O}(g)^\times = gH_Sg^{-1}\cap G_S(\mathbf{Q})$.
- The following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \to & \mathcal{O}_F^\times & \to & \mathcal{O}(g)^\times & \to & \mathcal{O}(g)^\times/\mathcal{O}_F^\times & \to & 1 \\ & & {\scriptstyle 2}\downarrow & & {\scriptstyle \mathrm{nr}_S}\downarrow & & \downarrow & & \\ 1 & \to & \mathcal{O}_F^{>0} & \to & \mathcal{O}_F^{>0} & \to & 1 & & \end{array}$$

  yields an exact sequence

$$1\to\{\pm1\}\to k(g,H_S)\to\mathcal{O}(g)^\times/\mathcal{O}_F^\times\to\mathcal{O}_F^{>0}/(\mathcal{O}_F^\times)^2\to q(g,H_S)\to 1.$$

  In particular,

$$\frac{|q(g,H_S)|}{|k(g,H_S)|} = \frac{[\mathcal{O}_F^{>0}:(\mathcal{O}_F^\times)^2]}{2\cdot[\mathcal{O}(g)^\times:\mathcal{O}_F^\times]}.$$

Combining this, (1), (3) and [**?**, Corollaire 2.2 page 142], we obtain:

$$\frac{\mu^1(H_S^1)}{\lambda} = \frac{2^{[F:\mathbf{Q}]}\left|\zeta_F(-1)\right|^{-1}}{\|\mathcal{N}_S\|\cdot\prod_{Q\in\mathrm{Ram}_f(B_S)}(\|Q\|-1)\cdot\prod_{Q|\mathcal{N}_S}(\|Q\|^{-1}+1)}$$

where $\mathcal{N}_S$ is the level of R_S. This tells us how to normalize μ^1 in order to have $\lambda = 1$. We have proven:

Proposition 2.18 *Let H_S be a compact open subgroup of $G_S(\mathbf{A}_f)$. For $x \in \mathcal{X}_S$ and $z \in \mathcal{Z}$,*

$$\mu_z\left(xH_S \cap (c_S \circ q_S)^{-1}(z)\right) = \begin{cases} \frac{|q(g,H_S)|}{|k(g,H_S)|}\mu^1(H_S^1) & \text{if } z \in c_S \circ q_S(xH_S) \\ 0 & \text{otherwise} \end{cases}$$

where $x = \overline{G_S(\mathbf{Q})}g$, $k(g,H_S)$ and $q(g,H_S)$ are as above, $H_S^1 = H_S \cap G_S^1(\mathbf{A}_f)$ and μ^1 is the unique Haar measure on $G_S^1(\mathbf{A}_f)$ such that

$$\mu^1(H_S^1) = \frac{2^{[F:\mathbf{Q}]}\left|\zeta_F(-1)\right|^{-1}}{\prod_{Q \in \mathrm{Ram}_f(B_S)}(\|Q\|-1)}$$

when $H_S = \widehat{R}_S^\times$ for some maximal *order $R_S \subset B_S$. Moreover, if $H_S = \widehat{R}_S^\times$ for some Eichler order $R_S \subset B_S$ of level $\mathcal{N}_S$,*

$$\frac{|q(g,H_S)|}{|k(g,H_S)|}\mu^1(H_S^1) = \frac{1}{[\mathcal{O}(g)^\times : \mathcal{O}_F^\times]} \times \frac{\Omega(F)}{\Omega(B_S)\cdot\Omega(\mathcal{N}_S)}$$

with $\Omega(F)$, $\Omega(B_S)$ and $\Omega(\mathcal{N}_S)$ as in section 2.2.4.

2.4.3 P-adic uniformization.

Suppose moreover that P does not belong to S (this is the case for all $S \in \mathfrak{S}$). Since B splits at P, so does B_S.

Let H be a compact open subgroup of $G_S^1(\mathbf{A}_f)^P = \{x \in G_S^1(\mathbf{A}_f) \mid x_P = 1\}$. For $z \in \mathcal{Z}$, the right action of $G_S^1(\mathbf{A}_f)$ on $c_S \circ q_S^{-1}(z)$ induces a right action of $B_{S,P}^1 = \{b \in B_{S,P}^\times \mid \mathrm{nr}_S(b) = 1\}$ on $c_S \circ q_S^{-1}(z)/H$.

Lemma 2.19 *This action is transitive and the stabilizer of $x \in c_S \circ q_S^{-1}(z)/H$ is a discrete and cocompact subgroup $\Gamma_S(x)$ of $B_{S,P}^1$. For $x = \overline{G_S(\mathbf{Q})}gH$ (with $g \in G_S(\mathbf{A}_f)$), $\Gamma_S(x) = g_P^{-1}\Gamma_S g_P$ where $g_P \in B_{S,P}^\times$ is the P-component of g and $\Gamma_S = \Gamma_S(gHg^{-1})$ is the projection to $B_{S,P}^1$ of $G_S^1(\mathbf{Q}) \cap \left\{gHg^{-1}\cdot B_{S,P}^1\right\} \subset G_S^1(\mathbf{A}_f)$. The commensurator of Γ_S in $B_{S,P}^\times$ equals $F_P^\times B_S^\times$.*

Proof. The stabilizer of $\widetilde{x} = \overline{G_S(\mathbf{Q})}g$ in $G_S^1(\mathbf{A}_f)$ equals $\mathrm{Stab}(\widetilde{x}) = g^{-1}G_S^1(\mathbf{Q})g$ (by Lemma 2.16). The strong approximation theorem [**?**, Théorème 4.3 p.81] implies that $\mathrm{Stab}(\widetilde{x})B_{S,P}^1 H = G_S^1(\mathbf{A}_f)$. Using Lemma 2.16 again, we obtain

$$(c_S \circ q_S)^{-1}(z) = \widetilde{x}\cdot G_S^1(\mathbf{A}_f) = \widetilde{x}\cdot B_{S,P}^1 H = \widetilde{x}\cdot HB_{S,P}^1 = x \cdot B_{S,P}^1.$$

In particular, $B^1_{S,P}$ acts transitively on $(c_S \circ q_S)^{-1}(z)/H$. An easy computation shows that $\Gamma_S(x) = g_P^{-1}\Gamma_S g_P$ with Γ_S as above.

Put $U = gHg^{-1}\cdot B^1_{S,P}$. The continuous map $U \cap G^1_S(\mathbf{Q})\backslash U \hookrightarrow G^1_S(\mathbf{Q})\backslash G^1_S(\mathbf{A}_f)$ is (1) open since U is open in $G^1_S(\mathbf{A}_f)$ and (2) surjective by the strong approximation theorem. In particular, $U \cap G^1_S(\mathbf{Q})$ is a discrete and cocompact subgroup of U. Since $U = gHg^{-1} \times B^1_{S,P}$ (with gHg^{-1} compact), the projection Γ_S of $U \cap G^1_S(\mathbf{Q})$ to $B^1_{S,P}$ is indeed discrete and cocompact in $B^1_{S,P}$.

Finally, since the compact open subgroups of $G^1_S(\mathbf{A}_f)^P$ are all commensurable, neither the commensurability class of Γ_S nor its commensurator in $B^\times_{S,P}$ depends upon g or H. When $g = 1$ and $H = \widehat{R}^\times \cap G^1_S(\mathbf{A}_f)^P$ for some Eichler order $R \subset B_S$, Γ_S is the image in $B^1_{S,P}$ of the subgroup $\{x \in R[1/P]^\times \mid \mathrm{nr}_S(x) = 1\}$ of $B^\times_S$. The commensurator of Γ_S in $B^\times_{S,P}$ then equals $F^\times_P B^\times_S$ by [**?**, Corollaire 1.5, p. 106].

Similarly, let H be a compact open subgroup of $G^1(S)^P = \{x \in G^1(S) \mid x_P = 1\}$. Then $B^1_{S,P}$ acts on $c_S^{-1}(z)/H$ and we have the following lemma.

Lemma 2.20 *This action is transitive and the stabilizer of $x \in c_S^{-1}(z)/H$ is a discrete and cocompact subgroup $\Gamma_S(x)$ of $B^1_{S,P}$. For $x = \overline{G(S,\mathbf{Q})}gH$ with g in $G(S)$, $\Gamma_S(x) = g_P^{-1}\Gamma_S g_P$ where $\Gamma_S = \Gamma_S(gHg^{-1})$ is the projection to $B^1_{S,P}$ of $G^1(S,\mathbf{Q}) \cap \left\{gHg^{-1} \cdot B^1_{S,P}\right\} \subset G^1(S)$. The commensurator of Γ_S in $B^\times_{S,P}$ equals $F^\times_P B^\times_S$.*

Proof. The proof is similar, using Lemma 2.17 instead of 2.16. Alternatively, we may deduce the results for c_S from those for $c_S \circ q_S$ as follows. Put $H' = \pi_S^{-1}(H)$. Then H' is a compact open subgroup of $G^1_S(\mathbf{A}_f)$ and q_S induces a $B^1_{S,P}$-equivariant homeomorphism between $(c_S \circ q_S)^{-1}(z)/H'$ and $c_S^{-1}(z)/H$.

In particular, the map $b \mapsto x \cdot b$ induces a $B^1_{S,P}$-equivariant homeomorphism

$$\Gamma_S(x)\backslash B^1_{S,P} \xrightarrow{\simeq} c_S^{-1}(z)/H.$$

Since $\Gamma_S(x)$ is discrete and cocompact in $B^1_{S,P} \simeq \mathrm{SL}_2(F_P)$, there exists a unique $B^1_{S,P}$-invariant Borel probability measure on the left hand side. It corresponds on the right hand side to the image of the measure μ_z through the (proper) map $c_S^{-1}(z) \to c_S^{-1}(z)/H$: the latter is indeed yet another $B^1_{S,P}$-invariant Borel probability measure.

2.5 Reduction of Proposition 2.13 to Ratner's theorem

Let us fix a point $x \in \mathrm{CM}$, a one parameter unipotent subgroup $U = \{u(t)\}$ in $B_P^\times$, a compact open subgroup κ in $F_P^\times$ and a Haar measure $\lambda = dt$ on F_P. For $n \geq 0$, we put $\kappa_n = \varpi_P^{-n}\kappa$ so that $\lambda(\kappa_n) \to \infty$ as $n \to \infty$. For $\gamma \in \mathrm{Gal}_K^{\mathrm{ab}}$ and $t \in F_P$,

$$C \circ \textsc{Red}(\gamma \cdot x \cdot u(t)) = \gamma \cdot \bar{x} \quad \text{with } \bar{x} = C \circ \textsc{Red}(x) \in \mathcal{Z}(\mathfrak{S}, \mathfrak{R})$$

where $\mathfrak{S}, \mathfrak{R}$, C and $\textsc{Red}$ are as in section 2.2.1. Our aim is to prove the following two propositions, which together obviously imply Proposition 2.13.

Proposition 2.21 *Suppose that* $\textsc{Red}(x \cdot U)$ *is dense in* $C^{-1}(\bar{x})$*. Then for any continuous function* $f : C^{-1}(\bar{x}) \to \mathbf{C}$,

$$\lim_{n\to\infty} \frac{1}{\lambda(\kappa_n)} \int_{\kappa_n} f \circ \textsc{Red}(x \cdot u(t))dt = \int_{C^{-1}(\bar{x})} f d\mu_{\bar{x}}.$$

Proposition 2.22 *Under the assumptions of Theorem 2.9,* $\textsc{Red}(\gamma \cdot x \cdot U)$ *is dense in* $C^{-1}(\gamma \cdot \bar{x})$ *for almost all* $\gamma \in \mathrm{Gal}_K^{ab}$.

2.5.1 Reduction of Proposition 2.21.

We may assume that f is locally constant (by the same argument that we already used in section 2.3). In this case, there exists a compact open subgroup H of $G^1(\mathbf{A}_f)$ such that f factors through $C^{-1}(\bar{x})/H(\mathfrak{S}, \mathfrak{R})$. For our purposes, it will be sufficient to assume that f is right $H(\mathfrak{S}, \mathfrak{R})$-invariant when H is a compact open subgroup of $G^1(\mathbf{A}_f)^P = \{x \in G^1(\mathbf{A}_f) \mid x_P = 1\}$. Here, $H(\mathfrak{S}, \mathfrak{R}) = \prod_{S,\sigma} H(S)$ with $H(S) = \phi_S(H)$ as usual.

For such an H, the right action of $G^1(\mathfrak{S}, \mathfrak{R})$ on $C^{-1}(\bar{x})$ induces a right action of $\prod_{S,\sigma} B^1_{S,P}$ on $C^{-1}(\bar{x})/H(\mathfrak{S}, \mathfrak{R})$ which together with the isomorphism

$$\prod_{S,\sigma} \phi_{S,P} : (B_P^1)^{\mathfrak{S}\times\mathfrak{R}} \xrightarrow{\simeq} \prod_{S,\sigma} B^1_{S,P}$$

yields a right action of $(B_P^1)^{\mathfrak{S}\times\mathfrak{R}}$ on $C^{-1}(\bar{x})/H(\mathfrak{S}, \mathfrak{R})$.

By Lemma 2.20, the map $(b_{S,\sigma}) \mapsto \textsc{Red}(x) \cdot (\phi_{S,P}(b_{S,\sigma}))$ yields a $(B_P^1)^{\mathfrak{S}\times\mathfrak{R}}$-equivariant homeomorphism

$$\Gamma(x, H)\backslash(B_P^1)^{\mathfrak{S}\times\mathfrak{R}} \xrightarrow{\simeq} C^{-1}(\bar{x})/H(\mathfrak{S}, \mathfrak{R}) \qquad (4)$$

where $\Gamma(x, H)$ is the stabilizer of $\textsc{Red}(x) \cdot H(\mathfrak{S}, \mathfrak{R})$ in $(B_P^1)^{\mathfrak{S}\times\mathfrak{R}}$. Note that $\Gamma(x, H)$ equals $\prod_{S,\sigma} \Gamma_{S,\sigma}(x, H)$ where for each $S \in \mathfrak{S}$ and $\sigma \in \mathfrak{R}$,

$$\Gamma_{S,\sigma}(x, H) = \phi_{S,P}^{-1} \left\{ \mathrm{Stab}_{B^1_{S,P}} \left(\mathrm{RED}_S(\sigma \cdot x) \cdot H(S) \right) \right\}$$

is a discrete and cocompact subgroup of $B_P^1 \simeq \mathrm{SL}_2(F_P)$.

Under this equivariant homeomorphism,

- the image of $t \mapsto \textsc{Red}(x \cdot u(t))$ in $C^{-1}(\bar{x})/H(\mathfrak{S}, \mathfrak{R})$ corresponds to the image of $t \mapsto \Delta \circ u(t)$ in $\Gamma(x, H)\backslash(B_P^1)^{\mathfrak{S}\times\mathfrak{R}}$, where $\Delta : B_P^1 \to (B_P^1)^{\mathfrak{S}\times\mathfrak{R}}$ is the diagonal map;

- the image of $\mu_{\bar{x}}$ on $C^{-1}(\bar{x})/H(\mathfrak{S}, \mathfrak{R})$ corresponds to the (unique) $(B_P^1)^{\mathfrak{S}\times\mathfrak{R}}$-invariant Borel probability measure on $\Gamma(x, H)\backslash(B_P^1)^{\mathfrak{S}\times\mathfrak{R}}$.

Writing $\mu_{\Gamma(x,H)}$ for the latter measure, the above discussion shows that Proposition 2.21 is a consequence of the following purely P-adic statement, itself a special case of a theorem of Ratner, Margulis, and Tomanov.

Proposition 2.23 *Suppose that $\Gamma(x, H) \cdot \Delta(U)$ is dense in $(B_P^1)^{\mathfrak{S}\times\mathfrak{R}}$. Then for any continuous function $f : \Gamma(x, H)\backslash(B_P^1)^{\mathfrak{S}\times\mathfrak{R}} \to \mathbf{C}$,*

$$\lim_{n\to\infty} \frac{1}{\lambda(\kappa_n)} \int_{\kappa_n} f(\Delta \circ u(t))dt = \int_{\Gamma(x,H)\backslash(B_P^1)^{\mathfrak{S}\times\mathfrak{R}}} f d\mu_{\Gamma(x,H)}.$$

Proof. See section 2.7.

2.5.2 Reduction of Proposition 2.22

We keep the above notations and choose:

- an element $g \in G(\mathbf{A}_f)$ such that $x = \overline{T(\mathbf{Q})}g$ in CM;

- for each $\sigma \in \mathfrak{R}$, an element $\lambda_\sigma \in T(\mathbf{A}_f)$ such that $\sigma = \mathrm{rec}_K(\lambda_\sigma) \in \mathrm{Gal}_K^{\mathrm{ab}}$.

For $S \in \mathfrak{S}$ and $\sigma \in \mathfrak{R}$, we thus obtain (using Lemma 2.20):

- $\mathrm{RED}_S(\sigma \cdot x) = \overline{G(S, \mathbf{Q})}\phi_S(\lambda_\sigma g)$ and

- $\Gamma_{S,\sigma}(x, H) = g_P^{-1}\lambda_{\sigma,P}^{-1}\Gamma^0_{S,\sigma}(x, H)\lambda_{\sigma,P}g_P$

where $\lambda_{\sigma,P}$ and g_P are the P-components of λ_σ and g while $\Gamma^0_{S,\sigma}(x,H)$ is the inverse image (through $\phi_{S,P} : B^1_P \to B^1_{S,P}$) of the projection to $B^1_{S,P}$ of

$$G^1(S,\mathbf{Q}) \cap \left\{\phi_S\left(\lambda_\sigma g H g^{-1}\lambda_\sigma^{-1}\right) \cdot B^1_{S,P}\right\} \subset G^1(S).$$

For a subgroup Γ of B^1_P, we denote by $[\Gamma]$ the commensurability class of Γ in B^1_P, namely the set of all subgroups of B^1_P which are commensurable with Γ. The group $B^\times_P$ acts on the right on the set of all commensurability classes (by $[\Gamma]\cdot b = [b^{-1}\Gamma b]$) and the stabilizer of $[\Gamma]$ for this action is nothing but the commensurator of Γ in $B^\times_P$.

Since the compact open subgroups of $G^1(\mathbf{A}_f)^P$ are all commensurable, the commensurability class $[\Gamma^0_S]$ of $\Gamma^0_{S,\sigma}(x,H)$ does not depend upon H, x or σ (but it does depend on S). Similarly, the commensurability class $[\Gamma_{S,\sigma}(x)]$ of $\Gamma_{S,\sigma}(x,H)$ does not depend upon H and $[\Gamma_{S,\sigma}(x)] = [\Gamma^0_S]\cdot\lambda_{\sigma,P}g_P$. Changing x to $\gamma\cdot x$ ($\gamma \in \mathrm{Gal}^{\mathrm{ab}}_K$) changes g to λg, where λ is an element of $T(\mathbf{A}_f)$ such that $\gamma = \mathrm{rec}_K(\lambda)$. In particular,

$$[\Gamma_{S,\sigma}(\gamma\cdot x)] = [\Gamma^0_S]\cdot\lambda_{\sigma,P}\lambda_P g_P = [\Gamma^0_S]\cdot\lambda_P\lambda_{\sigma,P}g_P$$

where λ_P is the P-component of λ. On the other hand, the stabilizer of $[\Gamma^0_S]$ in $B^\times_P$ equals $F^\times_P\phi_S^{-1}(B^\times_S)$ by Lemma 2.20. Since $K^\times_P \cap F^\times_P\phi_S^{-1}(B^\times_S) = F^\times_P K^\times$,

$$[\Gamma_{S,\sigma}(\gamma\cdot x)] = [\Gamma_{S,\sigma}(\gamma'\cdot x)] \iff \gamma \equiv \gamma' \mod \mathrm{Gal}^{P-\mathrm{rat}}_K.$$

With these notations, we have

Proposition 2.24 *Under the assumptions of Theorem 2.9, for (S,σ) and (S',σ') in $\mathfrak{S}\times\mathfrak{R}$ with $(S,\sigma) \neq (S',\sigma')$, the set*

$$\mathfrak{B}((S,\sigma),(S',\sigma')) = \left\{\gamma \in \mathrm{Gal}^{ab}_K;\ [\Gamma_{S,\sigma}(\gamma\cdot x)]\cdot U = [\Gamma_{S',\sigma'}(\gamma\cdot x)]\cdot U\right\}$$

is the disjoint union of countably many cosets of Gal^{P-rat}_K in Gal^{ab}_K.

Proof. Fix $(S,\sigma) \neq (S',\sigma')$ in $\mathfrak{S}\times\mathfrak{R}$. We have to show that (under the assumptions of Theorem 2.9) the image of

$$\mathfrak{B}' = \left\{\lambda_P \in K^\times_P;\ [\Gamma^0_S]\cdot\lambda_{\sigma,P}\lambda_P g_P\cdot U = [\Gamma^0_{S'}]\cdot\lambda_{\sigma',P}\lambda_P g_P\cdot U\right\}$$

in $K^\times_P/F^\times_P K^\times$ is at most countable. For that purpose we may as well consider the image of $\mathfrak{B}'$ in $K^\times_P/F^\times_P$.

We first consider the case where $S \neq S'$. In this case, we claim that $\mathfrak{B}'$ is empty. In fact: For $S \neq S'$, we claim that

$$[\Gamma_S^0] \cdot B_P^\times \neq [\Gamma_{S'}^0] \cdot B_P^\times.$$

To see this, suppose that $[\Gamma_{S'}^0] = [\Gamma_S^0] \cdot b$ for some $b \in B_P^\times$. Then $b^{-1} F_P^\times \phi_{S,P}^{-1}(B_S^\times) b = F_P^\times \phi_{S',P}^{-1}(B_{S'}^\times)$, so that $F_P^\times B_{S'}^\times = F_P^\times \phi(B_S^\times)$ in $B_{S',P}^\times$, where $\phi : B_{S,P} \to B_{S',P}$ is the isomorphism of F_P-algebras which sends α to $\phi_{S',P}(b^{-1}\phi_{S,P}^{-1}(\alpha)b)$. Since $F_P B_S = F_P^\times B_S^\times \cup \{0\}$ and similarly for $B_{S'}$, $F_P B_{S'} = F_P \phi(B_S)$.

We contend that ϕ maps B_S to $B_{S'}$. Indeed, suppose that α belongs to B_S and choose $\eta \in F$ such that $\mathrm{Tr}(\alpha+\eta) = \mathrm{Tr}(\alpha) + 2\eta \neq 0$. Since $\alpha+\eta$ belongs to B_S, there exists $\mu \in F_P$ and $\beta \in B_{S'}$ such that $\phi(\alpha+\eta) = \mu\beta$. Taking traces on both sides we obtain $\mu = \frac{\mathrm{Tr}(\alpha+\eta)}{\mathrm{Tr}(\beta)} \in F$, so that $\phi(\alpha+\eta) = \phi(\alpha)+\eta$ belongs to $B_{S'}$, and so does $\phi(\alpha)$.

By symmetry, $\phi^{-1}(B_{S'}) \subset B_S$ and ϕ yields an isomorphism of F-algebras between B_S and $B_{S'}$. This is a contradiction, since B_S and $B_{S'}$ are non-isomorphic quaternion algebras over F when $S \neq S'$. This proves the proposition when $S \neq S'$.

Next we consider the case where $S = S'$ but $\sigma \neq \sigma'$. In this case, an element λ_P in $K_P^\times$ belongs to $\mathfrak{B}'$ if and only if there exists $t \in F_P$ such that

$$b(t) = \lambda_P w(t) \lambda_P^{-1} (\lambda_{\sigma,P} \lambda_{\sigma',P}^{-1}) \in F_P^\times B_S^\times, \tag{5}$$

for $w(t) = \lambda_{\sigma,P} \phi_S(g_P u(t) g_P^{-1}) \lambda_{\sigma,P}^{-1}$. We contend that this condition can only be satisfied for countably many λ_P modulo $F_P^\times$.

Suppose first that $K_P^\times$ normalizes the unipotent subgroup W of elements of the form $w(t)$, for all $t \in F_P$. In this situation, $K_P^\times$ is a split torus, and we claim that (5) never holds for any λ_P and t. To see this, observe that if $k \in K_p$ is arbitrary, then, in view of the representation of elements of $K_P^\times$ and W by triangular matrices, the commutator $[k, b(t)]$ is unipotent. (This also follows from standard facts about Borel subgroups.) Since $b(t) \in F_P^\times B_S^\times$, we can apply this to elements of $K_P \cap B_S = K$, to conclude that either $B_S^\times$ contains nontrivial unipotent elements, or that $[k, b(t)]$ is trivial for all k. The former is impossible, since B_S is a definite quaternion algebra, so we conclude that $b(t)$ commutes with $K^\times$ which implies that $b(t) \in K_P$. It follows that $b(t) \in F_P^\times B_S^\times \cap K_P = F_P^\times K^\times$. But now looking at the form of $b(t)$ shows that $w(t) = 1$ and $(\lambda_{\sigma,P} \lambda_{\sigma',P}^{-1}) \in F_P^\times K^\times$, which contradicts the fact that $\sigma \not\equiv \sigma'$ mod $\mathrm{Gal}_K^{P-\mathrm{rat}}$.

It remains to dispose of the situation where K_P fails to normalize W. In this case, we may argue as follows. Since $w(t)$ is unipotent, the left-hand-side of (5) has norm independent of λ_P. On the other hand, the set $F_P^\times B_S^\times$ contains only countably many elements of given norm. It follows that there are only countably many possibilities for the left-hand-side of (5). Thus consider a given element α in $F_P^\times B_S^\times$. We want to count the number of cosets $\lambda_P F_P^\times \in K_P^\times / F_P^\times$ such that there there exists $t \in F_P$ with

$$\lambda_P w(t) \lambda_P^{-1} = \alpha(\lambda_{\sigma',P} \lambda_{\sigma,P}^{-1}). \tag{6}$$

Note that since $\lambda_{\sigma',P}\lambda_{\sigma,P}^{-1}$ is not an element of $\mathrm{Gal}_K^{P-\mathrm{rat}}$ by assumption, any such t is neccesarily nontrivial. But since the normalizer of W in $K_P^\times$ is precisely $F_P^\times$, we see that if λ_P and λ'_P belong to different $F_P^\times$-cosets, then the conjugates of W by λ_P and λ'_P have trivial intersection. It follows that for each α, there is at most a unique coset in $\lambda_P F_P^\times \in K_P^\times / F_P$ such that (6) holds for some t. Since there are only countably many possibilities for α, our contention follows. □

We may now prove Proposition 2.22. Put

$$\mathfrak{B} = \bigcup_{(S,\sigma)\neq(S',\sigma')} \mathfrak{B}((S,\sigma),(S',\sigma'))$$

so that $\mathfrak{B}$ is again the disjoint union of countably many cosets of $\mathrm{Gal}_K^{P-\mathrm{rat}}$ in $\mathrm{Gal}_K^{\mathrm{ab}}$. Since any such coset is *negligible*, so is $\mathfrak{B}$. We claim that $\textsc{Red}(\gamma\cdot x\cdot U)$ is dense in $C^{-1}(\gamma\cdot\bar{x})$ if γ belongs to $\mathrm{Gal}_K^{\mathrm{ab}} \setminus \mathfrak{B}$. In fact:

Lemma 2.25 *For $\gamma \in \mathrm{Gal}_K^{ab}$, the following conditions are equivalent:*

1. $\textsc{Red}(\gamma\cdot x\cdot U)$ *is dense in* $C^{-1}(\gamma\cdot\bar{x})$.
2. $\Gamma(\gamma\cdot x, H)\cdot\Delta(U)$ *is dense in* $(B_P^1)^{\mathfrak{S}\times\mathfrak{R}}$ *($\forall H$ compact open in $G^1(\mathbf{A}_f)^P$).*
3. $\Gamma(\gamma\cdot x, H)\cdot\Delta(U)$ *is dense in* $(B_P^1)^{\mathfrak{S}\times\mathfrak{R}}$ *($\exists H$ compact open in $G^1(\mathbf{A}_f)^P$).*
4. γ *does not belong to* $\mathfrak{B}$.

Proof. Lemma 2.17 implies that a subset Z of $C^{-1}(\bar{\gamma}\cdot x)$ is everywhere dense if and only if for every compact open subgroup H of $G^1(\mathbf{A}_f)^P$, the image of Z in the quotient $C^{-1}(\bar{\gamma}\cdot x)/H(\mathfrak{S},\mathfrak{R})$ is everywhere dense. Applying this to $Z = \textsc{Red}(\gamma\cdot x\cdot U)$ yields (1) ⇔ (2). But now (2) implies (3) and (4) is equivalent to (3) for any H by Proposition 2.35 below. □

In summary, the arguments of this section show that Proposition 2.13 follows from Proposition 2.23, together with Proposition 2.35 below.

2.6 Proof of Proposition 2.12.

Let V be a simple left B_P-module, so that $V \simeq F_P^2$ as an F_P-module and $V \simeq K_P$ as a K_P-module. We fix a K_P-basis e of V and an $\mathcal{O}_{F_P}$-basis $(1, \omega)$ of $\mathcal{O}_{K_P}$. Then $(e, \omega e)$ is an F_P-basis of V, which we use to identify $B_P \simeq \mathrm{End}_{F_P}(V)$ with $M_2(F_P)$. Under this identification, the element $x = \alpha + \beta\omega$ of K_P corresponds to the matrix $\left(\begin{smallmatrix} \alpha & -\beta\theta \\ \beta & \alpha+\beta\tau \end{smallmatrix}\right)$ with $\theta = \mathrm{nr}(\omega)$ and $\tau = \mathrm{Tr}(\omega)$.

Let $\mathcal{L}$ be the set of all $\mathcal{O}_{F_P}$-lattices in V. To each L in $\mathcal{L}$, we may attach an integer $n(L)$ as follows. The set $\mathcal{O}(L) = \{\lambda \in K_P; \lambda L \subset L\}$ is an $\mathcal{O}_{F_P}$-order in K and therefore equals $\mathcal{O}_n = \mathcal{O}_{F_P} + P^n \mathcal{O}_{K_P}$ for a unique integer n: we take $n(L) = n$. From a matrix point of view, $n(L)$ is the smallest integer $n \geq 0$ such that $\varpi_P^n \left(\begin{smallmatrix} 0 & -\theta \\ 1 & \tau \end{smallmatrix}\right) L \subset L$.

Lemma 2.26 *The map $L \mapsto n(L)$ induces a bijection $K_P^\times \backslash \mathcal{L} \to \mathbf{N}$.*

Proof. For $\lambda \in K_P^\times$ and $L \in \mathcal{L}$, $\mathcal{O}(\lambda L) = \mathcal{O}(L)$, so that $n(\lambda \cdot L) = n(L)$: our map is well-defined. Conversely, suppose that $n(L) = n(L') = n$ for $L, L' \in \mathcal{L}$. Since both L and L' are free, rank one $\mathcal{O}_n$-submodules of $V = K_P \cdot e$, there exists $\lambda \in K_P^\times$ such that $\lambda \cdot L = L'$: our map is injective. It is also surjective, since $n(\mathcal{O}_n \cdot e) = n$ for all $n \in \mathbf{N}$.

Put $L_0 = \mathcal{O}_0 \cdot e$, $R = \mathrm{End}(L_0) = M_2(\mathcal{O}_{F_P})$, $\delta = \left(\begin{smallmatrix} \varpi_P & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $u(t) = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ for $t \in F_P$. Then:

Lemma 2.27 *For $n \geq 0$ and $t \in \mathcal{O}_{F_P}^\times$,*

$$n\Big(u(\varpi_P^{-n} t) \cdot L_0\Big) = 2n \quad \textit{and} \quad n\Big(u(\varpi_P^{-n} t) \cdot \delta L_0\Big) = 2n + 1.$$

Proof. Left to the reader.

Let us consider a P-isogeny class $\mathcal{H} \subset \mathrm{CM}$, a compact open subgroup $\mathcal{G} \subset \mathrm{Gal}_K^{\mathrm{ab}}$ and a compact open subgroup $H \subset G(\mathbf{A}_f)$. We choose an element $x_0 \in \mathcal{H}$ such that $x_0 = \overline{T(\mathbf{Q})} \cdot g_0$ for some $g_0 \in G(\mathbf{A}_f)$ whose P-component equals 1. Let $K_P^{\mathcal{G}}$ be the kernel of

$$K_P^\times \hookrightarrow T(\mathbf{A}_f) \xrightarrow{\mathrm{rec}_K} \mathrm{Gal}_K^{\mathrm{ab}} \to \mathrm{Gal}_K^{\mathrm{ab}}/\mathcal{G}.$$

This is an open subgroup of finite index in $K_P^\times$. Let N be a positive integer such that

- $1 + P^N \mathcal{O}_{K_P}^\times \subset K_P^{\mathcal{G}}$ and
- H contains the image of $R(N)^\times = 1 + P^N R \subset B_P^\times$ in $G(\mathbf{A}_f)$.

We denote by $\mathcal{I}_1 \subset K_P^\times$ (resp. $\mathcal{I}_2 \subset R^\times$) a chosen set of representatives for $K_P^\times / K_P^{\mathcal{G}}$ (resp. $R^\times / R(N)^\times$) and put $\mathcal{I} = \mathcal{I}_1 \times \{0, 1\} \times \mathcal{I}_2$. For $i = (\lambda, \epsilon, r) \in \mathcal{I}$, we put

$$x_i = x_0 \cdot \lambda \delta^\epsilon r \in \mathcal{H} \qquad \text{and} \qquad u_i(t) = (\delta^\epsilon r)^{-1} u(t) (\delta^\epsilon r) \ (t \in F_P).$$

We finally put $\kappa = 1 + P^{N+1} \mathcal{O}_{F_P}$, a compact open subgroup of $F_P^\times$.

Proposition 2.28 *With notations as above,*

1. $\mathcal{G} \cdot \mathcal{H} \cdot H = \cup_{i \in \mathcal{I}} \cup_{n \geq 0} \mathcal{G} \cdot x_i u_i(\kappa_n) \cdot H$ *and*
2. $\forall i \in \mathcal{I}$ *and* $\forall n \geq 0$, $\mathcal{G} \cdot x_i u_i(\kappa_n) \cdot H = \mathcal{G} \cdot x_i u_{i,n} \cdot H$

where $\kappa_n = \varpi_P^{-n} \kappa$ *and* $u_{i,n} = u(\varpi_P^{-n}) \subset u_i(\kappa_n)$.

Proof. Note that $x_i \cdot u_i(t) = x_0 \cdot \lambda u(t) \delta^\epsilon r$.

(1) We have to show that any element x of $\mathcal{H}$ belongs to $\mathcal{G} \cdot x_i u_i(\kappa_n) \cdot H$ for some $i \in \mathcal{I}$ and $n \geq 0$. Write $x = x_0 \cdot b$ with $b \in B_P^\times$.

Consider the lattice $b \cdot L_0 \subset V$ and write $n(b \cdot L_0) = 2n + \epsilon$ with $\epsilon \in \{0, 1\}$. By Lemma 2.26 and 2.27, there exists $\lambda_0 \in K_P^\times$ such that $b \cdot L_0 = \lambda_0 \cdot u(\varpi_P^{-n}) \delta^\epsilon \cdot L_0$, hence $b = \lambda_0 \cdot u(\varpi_P^{-n}) \delta^\epsilon \cdot r_0$ for some $r_0 \in R^\times$. By definition of $\mathcal{I}_1$ and $\mathcal{I}_2$, there exists $\lambda \in \mathcal{I}_1$, $k \in K_P^{\mathcal{G}}$, $r \in \mathcal{I}_2$ and $h \in R(N)^\times$ such that $\lambda_0 = k \cdot \lambda$ and $r_0 = rh$. Put $i = (\lambda, \epsilon, r) \in \mathcal{I}$, $t = \varpi_P^{-n} \in \kappa_n$ and $\sigma = \mathrm{rec}_K(k) \in \mathcal{G}$. Since $x_0 \cdot k = \sigma \cdot x_0$, we obtain

$$x = x_0 \cdot b = \sigma \cdot x_0 \cdot \lambda u(t) \delta^\epsilon r h = \sigma \cdot (x_i u_i(t)) \cdot h \in \mathcal{G} \cdot x_i \cdot u_i(\kappa_n) \cdot H.$$

(2) We have to show that for $i = (\lambda, \epsilon, r) \in \mathcal{I}$, $n \geq 0$ and $a \in \kappa$,

$$x_i \cdot u_i(\varpi_P^{-n} a) \in \mathcal{G} \cdot x_i u_{i,n} \cdot H.$$

Put $y_a = 1 - a^{-1}$, $\lambda_{n,a} = 1 + \varpi_P^n y_a \omega \in K_P^\times$ and $\sigma_{n,a} = \mathrm{rec}_K(\lambda_{n,a})$. Since a belongs to $\kappa = 1 + P^{N+1} O_{F_P}^\times$, y_a belongs to P^{N+1}, $\lambda_{n,a}$ belongs to $K_P^{\mathcal{G}}$ and $\sigma_{n,a}$ belongs to $\mathcal{G}$. As a matrix,

$$\lambda_{n,a} = \begin{pmatrix} 1 & -\theta \varpi_P^n y_a \\ \varpi_P^n y_a & 1 + \tau \varpi_P^n y_a \end{pmatrix} \in K_P^\times \subset \mathrm{GL}_2(F_P).$$

In particular,

$$\delta^{-\epsilon} u(-\varpi_P^{-n}) \cdot \lambda_{n,a} \cdot u(\varpi_P^{-n} a) \delta^\epsilon = \begin{pmatrix} 1 - y_a & -(\theta \varpi_P^n + \tau) \varpi_P^{-\epsilon} y_a \\ \varpi_P^{n+\epsilon} y_a & 1 + (a + \tau \varpi_P^n) y_a \end{pmatrix} \equiv 1 \mod P^N.$$

In other words: there exists $r' \in R(N)^\times$ such that $\lambda_{n,a} u(\varpi_P^{-n} a) \delta^\epsilon = u(\varpi_P^{-n}) \delta^\epsilon r'$. We thus obtain:

$$\begin{aligned} \sigma_{n,a} \cdot x_i \cdot u_i(\varpi^{-n} a) &= x_0 \cdot \lambda \lambda_{n,a} u(\varpi_P^{-n} a) \delta^\epsilon r \\ &= x_0 \cdot \lambda u(\varpi_P^{-n}) \delta^\epsilon r' r \\ &= x_i \cdot u_{i,n} \cdot h \end{aligned}$$

with $h = r^{-1} r' r \in R(N)^\times \subset H$. QED.

2.7 An application of Ratner's Theorem

In this section, we study the distribution of certain unipotent flows on $X = \Gamma \backslash G^r$, where $G = \mathrm{SL}_2(F_P)$, r is a positive integer and $\Gamma = \Gamma_1 \times \cdots \times \Gamma_r$ is a product of cocompact lattices in G. Our key tool is the following special case of a theorem of Margulis and Tomanov [**?**, Theorem 11.2] (see also Ratner's Theorem 3 in [**?**]). We fix a Haar measure λ on F_P.

Theorem 2.29 (Uniform Distribution) *Let $V = \{v(t)\}$ be a one-parameter unipotent subgroup of G^r.*

1. *For every $x \in X$, there exists*
   - *a closed subgroup $L \supset V$ of G^r such that $\overline{x \cdot V} = x \cdot L$, and*
   - *an L-invariant Borel probability measure μ on X supported on $\overline{x \cdot V}$.*

2. *With x and μ as above, for every continuous function f on X and every compact set κ of F_P with positive measure, we have*

$$\lim_{|s|\to\infty} \frac{1}{\lambda(s\cdot\kappa)} \int_{s\cdot\kappa} f(x\cdot v(t))d\lambda(t) = \int_X f(y)d\mu(y).$$

Here λ denotes a choice of Haar measure on F_P.

The measure μ is uniquely determined by x and V. On the other hand, we may replace the closed subgroup $L \supset V$ of G^r by $\Sigma = \{g \in G^r \mid \mu \text{ is } g\text{-invariant}\}$. Indeed, Σ is a closed subgroup of G^r which contains L and therefore also V. Since μ is Σ-invariant, so is its support $\overline{x\cdot V} = x\cdot L$. In particular, $\overline{x\cdot V} = x\cdot\Sigma$.

Suppose now that $V = \Delta(U)$, where $\Delta : G \to G^r$ is the diagonal map and $U = \{u(t)\}$ is a (non-trivial) one-parameter unipotent subgroup of G. In this case, a result of M. Ratner shows that Σ contains some "twisted" diagonal:

Lemma 2.30 *There exists an element $c \in U^r$ such that $c\Delta(G)c^{-1} \subset \Sigma$.*

Proof. This is Corollary 4 of Theorem 6 in [**?**] when $F_P = \mathbf{Q}_p$ (note that the centralizer of $\Delta(U)$ in G^r equals $\{\pm U\}^r$). The case of general F_P seems to be well-known to the experts, see for instance the notes of N. Shah [**?**].

This leaves only finitely many possible values for $\Omega = c^{-1}\Sigma c$. Indeed:

Lemma 2.31 *For any subgroup Ω of G^r such that $\Delta(G) \subset \Omega$, there exists a partition $\{I_\alpha\}$ of $\{1,\cdots,r\}$ such that*

$$\prod_\alpha \Delta^{I_\alpha}(G) \subset \Omega \subset \{\pm1\}^r \cdot \prod_\alpha \Delta^{I_\alpha}(G)$$

where $\Delta^{I_\alpha}(G)$ is the diagonal subgroup of $\{(g_i) \in G^r; \forall i \notin I_\alpha,\ g_i = 1\}$.

Proof. This is a slight generalization of Proposition 3.10 of [**?**]. According to the latter, there exists a partition $\{I_\alpha\}$ of $\{1,\cdots,r\}$ such that

$$\{\pm1\}^r \cdot \Omega = \{\pm1\}^r \cdot \textstyle\prod_\alpha \Delta^{I_\alpha}(G).$$

Taking the derived group on both sides gives $\prod_\alpha \Delta^{I_\alpha}(G) = [\Omega:\Omega] \subset \Omega$.

The equivalence relation $\sim$ on $\{1, \cdots, r\}$ which is defined by the above partition can easily be retrieved from $\overline{x \cdot \Delta(U)} = x \cdot \Sigma$ by the following rule: for $1 \leq i, j \leq r$, $i \sim j$ if and only if the projection

$$\overline{x \cdot \Delta(U)} \subset X \to \Gamma_i \backslash G \times \Gamma_j \backslash G$$

is *not* surjective. On the other hand, this equivalence relation can also be used to characterize those Ω-orbits which are closed subsets of X:

Lemma 2.32 *For $g = (g_i) \in G^r$, the Ω-orbit of $y = \Gamma \cdot g$ is closed in X if and only for all $1 \leq i, j \leq r$ with $i \sim j$, $g_i^{-1}\Gamma_i g_i$ and $g_j^{-1}\Gamma_j g_j$ are commensurable in G.*

Proof. The map $\omega \mapsto y \cdot \omega$ induces a continuous bijection $\theta : g^{-1}\Gamma g \cap \Omega \backslash \Omega \to y \cdot \Omega$. We first claim that $y \cdot \Omega$ is closed in X if and only if $g^{-1}\Gamma g \cap \Omega \backslash \Omega$ is compact. The if part is trivial: if $g^{-1}\Gamma g \cap \Omega \backslash \Omega$ is compact, so is $\theta(g^{-1}\Gamma g \cap \Omega \backslash \Omega) = y \cdot \Omega$. To prove the converse, it is sufficient to show that θ is an homeomorphism when $y \cdot \Omega$ is closed (hence compact). Now if $y \cdot \Omega$ is a closed subset of X, $\Gamma \cdot g \cdot \Omega$ is a closed subset of G^r and $g^{-1}\Gamma g \cdot \Omega$ is a Baire space. Since Γ is countable (being discrete in a σ-compact space), it follows that Ω is open in $g^{-1}\Gamma g \cdot \Omega$ and θ is indeed an homeomorphism.

Put $\Omega' = \prod_\alpha \Delta^{I_\alpha}(G)$. Since $\Omega' \subset \Omega \subset \{\pm 1\}^r \cdot \Omega'$, $g^{-1}\Gamma g \cap \Omega$ is cocompact in Ω if and only if $g^{-1}\Gamma g \cap \Omega'$ is cocompact in Ω'. Note that $g^{-1}\Gamma g \cap \Omega' \backslash \Omega' \simeq \prod_\alpha \Gamma_\alpha \backslash G$ where $\Gamma_\alpha = \cap_{i \in I_\alpha} g_i^{-1}\Gamma_i g_i$, and $\Gamma_\alpha \backslash G$ is compact if and only if $g_i^{-1}\Gamma_i g_i$ is commensurable with $g_j^{-1}\Gamma_j g_j$ for all $i, j \in I_\alpha$. This finishes the proof of the lemma.

We thus obtain a second characterization of the equivalence relation $\sim$.

Definition 2.33 We say that two subgroups Γ and Γ' of G are *U-commensurable* if there exists $u \in U$ such that Γ and $u^{-1}\Gamma u$ are commensurable.

Corollary 2.34 *Write $x = \Gamma \cdot g$ with $g = (g_i) \in G^r$. For $1 \leq i, j \leq r$, $i \sim j$ if and only if $g_i^{-1}\Gamma_i g_i$ and $g_j^{-1}\Gamma_j g_j$ are U-commensurable in G.*

Proof. Write $c = (c_i) \in U^r$ and put $y = x \cdot c = \Gamma \cdot gc$. Then $y \cdot \Omega = x \cdot \Sigma = \overline{x \cdot V}$ is a closed subset of X. The lemma implies that $(g_i c_i)^{-1}\Gamma_i(g_i c_i)$ and $(g_j c_j)^{-1}\Gamma_j(g_j c_j)$ are commensurable in G when $i \sim j$. Conversely, suppose that $g_i^{-1}\Gamma_i g_i$ and $\alpha^{-1}g_j^{-1}\Gamma_j g_j\alpha$ are commensurable for some $\alpha \in U$. Put $\Gamma' = \Gamma_i \times \Gamma_j$, $X' = \Gamma' \backslash G^2$, $c' = (1, \alpha)$ and $\Delta'(g) = (g, g)$ for $g \in G$. Let $p : X \to X'$ be the obvious projection. The lemma implies that $p(x) \cdot c'\Delta'(G)c'^{-1}$ is closed in X', so that

$$p(\overline{x \cdot \Delta(U)}) \subset \overline{p(x) \cdot \Delta'(U)} \subset p(x) \cdot c'\Delta'(G)c'^{-1}.$$

In particular, $p(\overline{x \cdot \Delta(U)}) \neq X'$ and $i \sim j$.

Proposition 2.35 *The following conditions are equivalent:*

1. $\overline{x \cdot \Delta(U)} = X$.
2. *For all* $1 \leq i \neq j \leq r$, $g_i^{-1}\Gamma_i g_i$ *and* $g_j^{-1}\Gamma_j g_j$ *are* not U*-commensurable.*

The measure μ *of Theorem 2.29 is then the (unique)* G^r*-invariant Borel probability measure on* X*.*

Proof. Both conditions are equivalent to the assertion that the partition $\{I_\alpha\}$ of $\{1, \cdots, r\}$ is trivial. In that case, $\Omega = G^r = \Sigma$ and μ is G^r-invariant.

3 The case of Shimura curves

3.1 Shimura Curves.

3.1.1 Definitions

We start by defining the Shimura curves. Let $\{\tau_1, \cdots, \tau_d\} = \mathrm{Hom}_{\mathbf{Q}}(F, \mathbf{R})$ be the set of real embeddings of F. We shall always view F as a subfield of $\mathbf{R}$ or $\mathbf{C}$ through τ_1. Let S be a set of finite primes such that $|S| + d$ is odd, and let B denote the quaternion algebra over F which ramifies precisely at $S \cup \{\tau_2, \cdots, \tau_d\}$ (a finite set of even order). Let G be the reductive group over $\mathbf{Q}$ whose set of points on a commutative $\mathbf{Q}$-algebra A is given by $G(A) = (B \otimes A)^\times$.

In particular, $G_{\mathbf{R}} \simeq G_1 \times \cdots \times G_d$ where $B_{\tau_i} = B \otimes_{F,\tau_i} \mathbf{R}$ and G_i is the algebraic group over $\mathbf{R}$ whose set of points on a commutative $\mathbf{R}$-algebra A is given by $G_i(A) = (B_{\tau_i} \otimes_{\mathbf{R}} A)^\times$. Fix $\epsilon = \pm 1$ and let X be the $G(\mathbf{R})$-conjugacy class of the morphism from $\mathbb{S} \overset{\text{def}}{=} \mathrm{Res}_{\mathbf{C}/\mathbf{R}}(\mathbb{G}_{m,\mathbf{C}})$ to $G_{\mathbf{R}}$ which maps

$z = x + iy \in \mathbb{S}(\mathbf{R}) = \mathbf{C}^\times$ to

$$\left[\begin{pmatrix} x & y \\ -y & x \end{pmatrix}^\epsilon, 1, \cdots, 1 \right] \in G_1(\mathbf{R}) \times \cdots \times G_d(\mathbf{R}) \simeq G(\mathbf{R}).$$

We have used an isomorphism of $\mathbf{R}$-algebras $B_{\tau_1} \simeq M_2(\mathbf{R})$ to identify G_1 and $\mathrm{GL}_{2,\mathbf{R}}$; the resulting conjugacy class X does not depend upon this choice (but it does depend on ϵ, cf. section 3.3.1 below).

For every compact open subgroup H of $G(\mathbf{A}_f)$ (where $\mathbf{A}_f = \mathbf{Q} \otimes \widehat{\mathbf{Z}}$ is the ring of finite adeles of $\mathbf{Q}$), the quotient of $G(\mathbf{A}_f)/H \times X$ by the diagonal left action of $G(\mathbf{Q})$ is a Riemann surface

$$M_H^{\mathrm{an}} \stackrel{\mathrm{def}}{=} G(\mathbf{Q}) \backslash \left(G(\mathbf{A}_f)/H \times X \right)$$

which is compact unless $d = 1$ ($F = \mathbf{Q}$) and $S = \emptyset$ ($G = \mathrm{GL}_2$). The *Shimura curve* M_H is Shimura's canonical model for M_H^{an}. It is a smooth curve over F (the reflex field) whose underlying Riemann surface $M_H(\mathbf{C})$ equals M_H^{an}.

3.1.2 CM points.

Among the models of M_H^{an}, the Shimura curve M_H is characterized by specifying the action of Galois (the "reciprocity law") on certain *special points.* A morphism $h : \mathbb{S} \to G_{\mathbf{R}}$ in X is special if it factors through the real locus of some $\mathbf{Q}$-rational subtorus of G and a point x in M_H^{an} is special if $x = [g, h]$ with h special (and g in $G(\mathbf{A}_f)$).

Now let K be an imaginary quadratic extension of F such that there exists some embedding $K \to B$. Put $T = \mathrm{Res}_{K/\mathbf{Q}}(\mathbb{G}_{m,K})$. Any embedding $K \hookrightarrow B$ yields an embedding $T \hookrightarrow G$. In the sequel, we shall fix an embedding of K in B, and study those special points in X or M_H^{an} for which $h : \mathbb{S} \to G_{\mathbf{R}}$ factors through the morphism $T_{\mathbf{R}} \hookrightarrow G_{\mathbf{R}}$ which is induced by the fixed F-embedding $K \hookrightarrow B$. We shall refer to such points as *CM points.* We denote by CM_H the set of CM points in $M_H^{\mathrm{an}} = M_H(\mathbf{C})$. It is clear that this set is nonempty. Furthermore, Shimura's theory implies that any CM point is algebraic, defined over the maximal abelian extension K^{ab} of K (see section 3.2.4 below).

3.1.3 Integral models and supersingular points.

Let v be a finite place of F where B is split and put $\mathcal{S} = \mathrm{Spec}\, \mathcal{O}(v)$ where $\mathcal{O}(v)$ is the local ring of F at v. We denote by F_v and $\mathcal{O}_v$ the completion of

F at v and its ring of integers. For simplicity, we shall only consider level structures $H \subset G(\mathbf{A}_f)$ which decompose as $H = H^v H_v$ where H^v (resp. H_v) is a compact open subgroup of

$$G(\mathbf{A}_f)^v = \{g \in G(\mathbf{A}_f) \mid g_v = 1\} \quad (\text{resp. } B_v^\times \subset G(\mathbf{A}_f)).$$

In the non-compact (classical) case where $F = \mathbf{Q}$ and $G = \mathrm{GL}_2$, it is well-known that M_H is a coarse moduli space which classifies elliptic curves (with level structures) over extensions of $\mathbf{Q}$. Extending the moduli problem to elliptic curves over $\mathcal{S}$-schemes, we obtain a regular model $\mathbf{M}_H/\mathcal{S}$ of M_H. A geometric point in the special fiber of $\mathbf{M}_H$ is *supersingular* if it corresponds to (the class of) a supersingular elliptic curve.

In the general (compact) case, the Shimura curve M_H may not be a moduli space. However, provided that H^v is *sufficiently small* (a condition depending upon H_v), Carayol describes in [**?**] a proper and regular model $\mathbf{M}_H/\mathcal{S}$ of M_H, which is smooth when H_v is a maximal compact open subgroup of B_v^*. When H^v fails to be sufficiently small in the sense of [**?**], we let $\mathbf{M}_H/\mathcal{S}$ be the quotient of $\mathbf{M}_{H'}$ by the $\mathcal{S}$-linear right action of H/H', where $H' = H'^v H_v$ for a sufficiently small compact, open and normal subgroup H'^v of H^v. Then $\mathbf{M}_H/\mathcal{S}$ is again a proper and regular model of M_H which is smooth when H_v is maximal (cf. [**?**, p. 508]), and it does not depend upon the choice of H'^v.

These models form a projective system $\{\mathbf{M}_H\}_H$ of proper $\mathcal{S}$-schemes with finite flat transition maps, whose limit $\mathbf{M} = \varprojlim \mathbf{M}_H$ has a right action of $G(\mathbf{A}_f)$ and carries an $\mathcal{O}_v$-divisible module $\mathbf{E}$ of height 2 (cf. [**?**, Appendice] for the definition and basic properties of $\mathcal{O}_v$-divisible modules). A geometric point x in the special fiber of $\mathbf{M}$ is said to be *ordinary* if $\mathbf{E} \mid x$ is isomorphic to the product of the $\mathcal{O}_v$-divisible constant module $F_v/\mathcal{O}_v$ with Σ_1, the unique $\mathcal{O}_v$-formal module of height 1. Otherwise, x is *supersingular* and $\mathbf{E} \mid x$ is isomorphic to Σ_2, the unique $\mathcal{O}_v$-formal module of height 2. A supersingular point in the special fiber of $\mathbf{M}_H$ is one which lifts to a supersingular point in $\mathbf{M}$.

In the classical case, the supersingular points also have such a description, with $\mathbf{E}$ equal to the relevant Barsotti-Tate group in the universal elliptic curve on $\mathbf{M} = \varprojlim \mathbf{M}_H$.

3.1.4 Reduction maps.

Let us choose a place $\bar{v}$ of K^{ab} above v, with ring of integers $\mathcal{O}(\bar{v}) \subset K^{\mathrm{ab}}$ and residue field $\mathbf{F}(\bar{v})$, an algebraic closure of the residue field $\mathbf{F}(v)$ of v.

Consider the specialization maps:

$$M_H(K^{\rm ab}) = \mathbf{M}_H(K^{\rm ab}) \leftarrow \mathbf{M}_H(\mathcal{O}(\bar{v})) \rightarrow \mathbf{M}_H(\mathbf{F}(\bar{v}))$$

In the compact case, $\mathbf{M}_H$ is proper over $\mathcal{S}$ and the first of these two maps is a bijection by the valuative criterion of properness. In the classical non-compact case, the first map is still injective ($\mathbf{M}_H$ is separated over $\mathcal{S}$); by [**?**, Theorem 6], its image contains CM_H. In both cases, we obtain a reduction map

$$\mathrm{RED}_v : \mathrm{CM}_H \rightarrow \mathbf{M}_H(\mathbf{F}(\bar{v})).$$

Let $\mathbf{M}_H^{ss}(v)$ be the set of supersingular points in $\mathbf{M}_H(\mathbf{F}(\bar{v}))$.

Lemma 3.1 *If v does not split in K,* $\mathrm{RED}_v(\mathrm{CM}_H) \subset \mathbf{M}_H^{ss}(v)$.

Proof. (Sketch) Let $\mathbf{E}^0$ be the $\mathcal{O}_v$-divisible module $\mathbf{E}$ "up to isogeny". There is an F_v-linear right action of $G(\mathbf{A}_f)$ on $\mathbf{E}^0$ covering the right action of $G(\mathbf{A}_f)$ on $\mathbf{M}$ (see [**?**, 7.5] for the compact case). For any point x on $\mathbf{M}$, we thus obtain an F_v-linear right action of $\mathrm{Stab}_{G(\mathbf{A}_f)}(x)$ on $\mathbf{E}^0 \mid x$. If x is a CM point, say $x = [g, h]$ for some $g \in G(\mathbf{A}_f)$ and $h : \mathbb{S} \rightarrow T_{\mathbf{R}} \hookrightarrow G_{\mathbf{R}}$ in X, $g^{-1}T(\mathbf{Q})g \subset \mathrm{Stab}_{G(\mathbf{A}_f)}(x)$ and the induced F_v-linear action of $T(\mathbf{Q}) = K^*$ on $\mathbf{E}^0 \mid x$ (or its inverse, depending upon ϵ) arises from an F_v-linear right K_v-module structure on $\mathbf{E}^0 \mid x$. The connected part of the special fiber $\mathbf{E}^0 \mid \mathrm{RED}_v(x)$ therefore inherits a K_v-module structure. Since $\mathrm{End}_{F_v}(\Sigma_1^0) \simeq F_v$, this connected part can not be isomorphic to Σ_1^0 unless v splits in K.

3.1.5 Connected components.

We now want to define yet another type of "reduction map". Recall from Shimura's theory that the natural map from $M_H^{\rm an}$ to its set of connected components $\pi_0(M_H^{\rm an})$ corresponds to an F-morphism $c : M_H \rightarrow \mathcal{M}_H$ between the Shimura curve M_H and a zero-dimensional Shimura variety $\mathcal{M}_H$ over F whose finitely many points are algebraic over the maximal abelian extension $F^{\rm ab} \subset K^{\rm ab}$ of F. Since $\mathbf{M}_H$ is regular (hence normal), this morphism extends over $\mathcal{S}$ to a morphism $c : \mathbf{M}_H \rightarrow \boldsymbol{\mathcal{M}}_H$ between $\mathbf{M}_H$ and the normalization $\boldsymbol{\mathcal{M}}_H$ of $\mathcal{S}$ in $\mathcal{M}_H$, a finite and regular $\mathcal{S}$-scheme. With $\mathcal{Z}_H \stackrel{\rm def}{=} \pi_0(M_H^{\rm an}) =$

$\mathcal{M}_H(K^{\mathrm{ab}})$ and $\mathcal{Z}_H(v) \stackrel{\mathrm{def}}{=} \boldsymbol{\mathcal{M}}_H(\mathbf{F}(\overline{v}))$, the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{CM}_H & \stackrel{\mathrm{RED}_v}{\longrightarrow} & \mathbf{M}_H^{\mathrm{ss}}(v) \\ c\downarrow & & \downarrow c \\ \mathcal{Z}_H & \stackrel{\mathrm{RED}_v}{\longrightarrow} & \mathcal{Z}_H(v) \end{array}$$

Put $\mathcal{X}_H(v) = \mathbf{M}_H^{ss}(v) \times_{\mathcal{Z}_H(v)} \mathcal{Z}_H$. If v does not split in K, we thus obtain a *reduction map* and a *connected component map*

$$\begin{array}{ccccc} \mathrm{CM}_H & \stackrel{\mathrm{RED}_v}{\longrightarrow} & \mathcal{X}_H(v) & \stackrel{c_v}{\longrightarrow} & \mathcal{Z}_H \\ x & \longmapsto & (\mathrm{RED}_v(x), c(x)) & \longmapsto & c(x) \end{array} \qquad (7)$$

The composite map $c = c_v \circ \mathrm{RED}_v : \mathrm{CM}_H \to \mathcal{Z}_H$ does not depend upon v and commutes with the action of $\mathrm{Gal}_K^{\mathrm{ab}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(K^{\mathrm{ab}}/K)$ on both sides.

Remark 3.2 In the compact case, $\mathbf{M}_H \stackrel{c}{\longrightarrow} \boldsymbol{\mathcal{M}}_H \to \mathcal{S}$ is the Stein factorization

$$\mathbf{M}_H \to \mathrm{Spec}(\Gamma(\mathbf{M}_H, \mathcal{O}_{\mathbf{M}_H})) \to \mathcal{S}$$

of the proper morphism $\mathbf{M}_H \to \mathcal{S}$. Indeed, since $\boldsymbol{\mathcal{M}}_H$ is affine, c factors through an S-morphism $\alpha : \mathrm{Spec}(\Gamma(\mathbf{M}_H, \mathcal{O}_{\mathbf{M}_H})) \to \boldsymbol{\mathcal{M}}_H$. Over the generic point of $\mathcal{S}$, $\alpha_{/F} : \mathrm{Spec}(\Gamma(M_H, \mathcal{O}_{M_H})) \to \mathcal{M}_H$ is a morphism between finite tale F-schemes which induces a bijection on complex points: it is therefore an isomorphism. Since $\mathbf{M}_H$ is a regular scheme which is proper and flat over $\mathcal{S}$, $\mathrm{Spec}(\Gamma(\mathbf{M}_H, \mathcal{O}_{\mathbf{M}_H}))$ is a normal scheme which is finite and flat over $\mathcal{S}$. It follows that α is an isomorphism.

3.1.6 Simultaneous reduction maps.

Let $\mathfrak{S}$ be a finite set of finite places of F which are non-split in K and away from S: for each $v \in \mathfrak{S}$, K_v is a field and $B_v \simeq M_2(F_v)$. Let also $\mathfrak{R}$ be a finite set of Galois elements in $\mathrm{Gal}_K^{\mathrm{ab}}$. We put

$$\mathcal{X}_H(\mathfrak{S}, \mathfrak{R}) \stackrel{\mathrm{def}}{=} \prod_{v \in \mathfrak{S}, \sigma \in \mathfrak{R}} \mathcal{X}_H(v) \quad \text{and} \quad \mathcal{Z}_H(\mathfrak{S}, \mathfrak{R}) \stackrel{\mathrm{def}}{=} \prod_{v \in \mathfrak{S}, \sigma \in \mathfrak{R}} \mathcal{Z}_H(v)$$

and define *a simultaneous reduction map* and *a connected component map*

$$\mathrm{R{\scriptstyle ED}} : \mathrm{CM}_H \to \mathcal{X}_H(\mathfrak{S}, \mathfrak{R}) \quad \text{and} \quad C : \mathcal{X}_H(\mathfrak{S}, \mathfrak{R}) \to \mathcal{Z}_H(\mathfrak{S}, \mathfrak{R})$$

by $\mathrm{R{\scriptstyle ED}}(x) = (\mathrm{RED}_v(\sigma x))_{v \in \mathfrak{S}, \sigma \in \mathfrak{R}}$ and $C(x_{v,\sigma}) = (c_v(x_{v,\sigma}))_{v \in \mathfrak{S}, \sigma \in \mathfrak{R}}$. For $x \in \mathrm{CM}_H$ and $\tau \in \mathrm{Gal}_K^{\mathrm{ab}}$,

$$C \circ \mathrm{R{\scriptstyle ED}}(\tau x) = (\tau \sigma c(x))_{v \in \mathfrak{S}, \sigma \in \mathfrak{R}}.$$

3.1.7 Main theorem.

Let P be a maximal ideal of $\mathcal{O}_F$ and suppose that $P \notin S \cup \mathfrak{S}$. In particular, $B_P \simeq M_2(F_P)$. We make no assumptions on P relative to K: P may either split, ramify or be inert in K. Then the following definitions reprise the ones already made in the first section of the paper: we have chosen to repeat them here for the convenience of the reader.

Definition 3.3 We say that two points x and $y \in M_H^{\mathrm{an}}$ are *P-isogeneous* if $x = [g, h]$ and $y = [g', h]$ for some $h \in X$ and $g, g' \in G(\mathbf{A}_f)$ such that $g_w = g'_w$ for every finite place $w \neq P$ of F.

Note that a point which is P-isogeneous to a CM point is again a CM point.

Definition 3.4 An element $\sigma \in \mathrm{Gal}_K^{\mathrm{ab}}$ is *P-rational* if $\sigma = \mathrm{rec}_K(\lambda)$ for some $\lambda \in \widehat{K}^\times$ whose P-component λ_P belongs to the subgroup $K^\times \cdot F_P^\times$ of $K_P^\times$. We denote by $\mathrm{Gal}_K^{P-\mathrm{rat}} \subset \mathrm{Gal}_K^{\mathrm{ab}}$ the subgroup of all P-rational elements.

In the above definition, $\mathrm{rec}_K : \widehat{K}^\times \twoheadrightarrow \mathrm{Gal}_K^{\mathrm{ab}}$ is Artin's reciprocity map. We normalize the latter by specifying that it sends local uniformizers to *geometric* Frobeniuses.

Theorem 3.5 *Suppose that the finite subset $\mathfrak{R}$ of Gal_K^{ab} consists of elements which are pairwise distinct modulo Gal_K^{P-rat}. Let $\mathcal{H} \subset \mathrm{CM}_H$ be a P-isogeny class of CM points and let $\mathcal{G}$ be a compact open subgroup of Gal_K^{ab}. Then for all but finitely many points $x \in \mathcal{H}$,*

$$\textsc{Red}(\mathcal{G} \cdot x) = C^{-1}(C \circ \textsc{Red}(\mathcal{G} \cdot x)).$$

Remark 3.6 When the level structure H arises from an Eichler order in B, our proof of this surjectivity statement yields a little bit more: for any $y \in \mathcal{X}_H(\mathfrak{S}, \mathfrak{R})$, we can compute the asymptotic behavior of the probability that $\textsc{Red}(g \cdot x) = y$ for some $g \in \mathcal{G}$, as x goes to infinity inside $\mathcal{H}$ (see Corollary 2.11).

3.2 Uniformization.

Write CM, $\mathbf{M}^{ss}(v)$, $\mathcal{Z}$, $\mathcal{Z}(v)$ and $\mathcal{X}(v) = \mathbf{M}^{ss}(v) \times_{\mathcal{Z}(v)} \mathcal{Z}$ for the projective limits of $\{\mathrm{CM}_H\}$, $\{\mathbf{M}_H^{ss}(v)\}$, $\{\mathcal{Z}_H\}$, $\{\mathcal{Z}_H(v)\}$ and $\{\mathcal{X}_H(v)\}$. These sets now have a right action of $G(\mathbf{A}_f)$. For $X \in \{\mathrm{CM}, \mathbf{M}^{ss}(v), \mathcal{Z}, \mathcal{Z}(v)\}$, the natural map $X/H \to X_H$ is a bijection while $\mathcal{X}(v)/H \to \mathcal{X}_H(v)$ is surjective. The projective limit of (7) yields $G(\mathbf{A}_f)$-equivariant maps

$$\mathrm{CM} \xrightarrow{\mathrm{RED}_v} \mathcal{X}(v) \xrightarrow{c_v} \mathcal{Z}$$

which we shall now compute.

3.2.1 CM points.

From [**?**, Proposition 2.1.10] or [**?**, Theorem 5.27],

$$M^{\mathrm{an}} \stackrel{\mathrm{def}}{=} \varprojlim M_H^{\mathrm{an}} = G(\mathbf{Q})\backslash \left(G(\mathbf{A}_f)/\overline{Z(\mathbf{Q})} \times X\right)$$

where $Z = \mathrm{Res}_{F/\mathbf{Q}}(\mathbb{G}_{m,F})$ is the center of G and $\overline{Z(\mathbf{Q})}$ is the closure of $Z(\mathbf{Q})$ in $Z(\mathbf{A}_f)$. Inside M^{an}, $\mathrm{CM} \stackrel{\mathrm{def}}{=} \varprojlim \mathrm{CM}_H$ corresponds to those elements which can be represented by (g, h) with $g \in G(\mathbf{A}_f)$ and h a CM point in X. Let us construct such an h and show that any other CM point belongs to the same $G(\mathbf{Q})$-orbit.

Since K_v is a field for all $v \in S \cup \{\tau_2, \cdots, \tau_d\}$ (the set of places of F where B ramifies), there exists an F-embedding $\iota : K \hookrightarrow B$. Moreover, any other F-embedding $K \hookrightarrow B$ is conjugated to ι by an element of $B^\times = G(\mathbf{Q})$. We use ι to identify T as a $\mathbf{Q}$-rational subtorus of G and also chose an extension $\tau_1 : K \hookrightarrow \mathbf{C}$ of our distinguished embedding $\tau_1 : F \hookrightarrow \mathbf{R}$. In the sequel, we shall always view K as a subfield of $\mathbf{C}$ through τ_1.

Put $T_i = \mathrm{Res}_{K_{\tau_i}/\mathbf{R}}(\mathbb{G}_{m,K_{\tau_i}})$ (with $K_{\tau_i} = K \otimes_{F,\tau_i} \mathbf{R}$), so that $T_{\mathbf{R}} \simeq T_1 \times \cdots \times T_d$ and this decomposition is compatible with the decomposition $G_{\mathbf{R}} \simeq G_1 \times \cdots \times G_d$ of section 3.1.1. Moreover, $\tau_1 : K \hookrightarrow \mathbf{C}$ induces an isomorphism between K_{τ_1} and $\mathbf{C}$ which allows us to identify T_1 and $\mathbb{S}$. There are exactly two morphisms s and $\bar{s} : \mathbb{S} \to T_{\mathbf{R}}$ whose composite with $\iota_{\mathbf{R}} : T_{\mathbf{R}} \hookrightarrow G_{\mathbf{R}}$ belongs to X. They are characterized by

$$s(z) = (z^\epsilon, 1, \cdots, 1) \quad \text{and} \quad \bar{s}(z) = (\bar{z}^\epsilon, 1, \cdots, 1) \quad \text{for } z \in \mathbf{C}^\times = \mathbb{S}(\mathbf{R}).$$

Finally, there exists an element $b \in B^\times = G(\mathbf{Q})$ such that $b\iota(\lambda)b^{-1} = \iota(\bar{\lambda})$ for all $\lambda \in K$ (where $\lambda \mapsto \bar{\lambda}$ is the non-trivial F-automorphism of K). But then $b(\iota_{\mathbf{R}} \circ \bar{s})b^{-1} = \iota_{\mathbf{R}} \circ s$, so that $h = \iota_{\mathbf{R}} \circ s$ and $\bar{h} = \iota_{\mathbf{R}} \circ \bar{s}$ belong to the same $G(\mathbf{Q})$-orbit in X. Since the centralizer of h in $G(\mathbf{Q})$ equals $T(\mathbf{Q})$, we obtain:

Lemma 3.7 *The map $g \mapsto [1,h] \cdot g = [g,h]$ induces a bijection*

$$\overline{T(\mathbf{Q})} \backslash G(\mathbf{A}_f) \xrightarrow{\simeq} \mathrm{CM}$$

where $\overline{T(\mathbf{Q})}$ is the closure of $T(\mathbf{Q})$ in $T(\mathbf{A}_f)$.

Proof. The above discussion gives a bijection $T(\mathbf{Q})\backslash G(\mathbf{A}_f)/\overline{Z(\mathbf{Q})} \simeq \mathrm{CM}$. We claim that $T(\mathbf{Q})\overline{Z(\mathbf{Q})} = \overline{T(\mathbf{Q})}$. Indeed, $\overline{Z(\mathbf{Q})}$ is the product of $Z(\mathbf{Q}) = F^\times$ with the closure of $\mathcal{O}_F^\times$ in $\widehat{\mathcal{O}}_F^\times \subset Z(\mathbf{A}_f) = \widehat{F}^\times$ (this holds more generally for any number field). Therefore, $T(\mathbf{Q})\overline{Z(\mathbf{Q})} = K^\times\overline{\mathcal{O}_F^\times}$ and

$$T(\mathbf{Q})\overline{Z(\mathbf{Q})} \cap \widehat{\mathcal{O}}_K^\times = \mathcal{O}_K^\times\overline{\mathcal{O}_F^\times} = \cup_{\alpha \in O_K^\times/O_F^\times} \alpha\overline{\mathcal{O}_F^\times}.$$

Since $[\mathcal{O}_K^\times : \mathcal{O}_F^\times]$ is finite, $T(\mathbf{Q})\overline{Z(\mathbf{Q})}$ is a locally closed, hence closed subgroup of $T(\mathbf{A}_f)$. Our claim easily follows.

3.2.2 Connected components.

Let $G(\mathbf{R})^+$ and $Z(\mathbf{R})^+$ be the identity components of $G(\mathbf{R})$ and $Z(\mathbf{R})$ and put $G(\mathbf{Q})^+ = G(\mathbf{Q}) \cap G(\mathbf{R})^+$ and $Z(\mathbf{Q})^+ = Z(\mathbf{Q}) \cap Z(\mathbf{R})^+$. Thus, $G(\mathbf{R})^+$ is the set of elements in $G(\mathbf{R})$ whose projection to $G_1(\mathbf{R}) \simeq GL_2(\mathbf{R})$ has a positive determinant while $Z(\mathbf{Q})^+$ is the subgroup of totally positive elements in $Z(\mathbf{Q}) = F^\times$. Let $X^+ = G(\mathbf{R})^+ \cdot h$ be the connected component of h in X. Since $G(\mathbf{Q})$ is dense in $G(\mathbf{R})$, $G(\mathbf{Q}) \cdot X^+ = X$ and

$$M_H^{\mathrm{an}} \simeq G(\mathbf{Q})^+ \backslash \left(G(\mathbf{A}_f)/H \times X^+ \right).$$

It follows that $\mathcal{Z}_H = \pi_0(M_H^{\mathrm{an}}) \simeq G(\mathbf{Q})^+\backslash G(\mathbf{A}_f)/H$.

On the other hand, the reduced norm $\mathrm{nr} : B \to F$ induces a surjective morphism $\mathrm{nr} : G \to Z$ whose kernel $G^1 \subset G$ is the derived group of G. The norm theorem ($\mathrm{nr}(G(\mathbf{Q})^+) = Z(\mathbf{Q})^+$, [**?**, p. 80]) and the strong approximation theorem ($G^1(\mathbf{Q})$ is dense in $G^1(\mathbf{A}_f)$, [**?**, p. 81]) together imply that the reduced norm induces a bijection between $G(\mathbf{Q})^+\backslash G(\mathbf{A}_f)/H$ and $Z(\mathbf{Q})^+\backslash Z(\mathbf{A}_f)/\mathrm{nr}(H)$. With $\mathcal{Z} \stackrel{\mathrm{def}}{=} \varprojlim \mathcal{Z}_H$, we thus obtain:

Lemma 3.8 *The map $g \mapsto c([1,h]) \cdot g = c([g,h])$ factors through the reduced norm and yields a bijection*

$$\overline{Z(\mathbf{Q})^+} \backslash Z(\mathbf{A}_f) \xrightarrow{\simeq} \mathcal{Z}$$

where $\overline{Z(\mathbf{Q})^+}$ is the closure of $Z(\mathbf{Q})^+$ in $Z(\mathbf{A}_f)$.

3.2.3 Supersingular points.

Proposition 3.9 (1) *The right action of* $G(\mathbf{A}_f)$ *on* $\mathcal{X}(v) \stackrel{\text{def}}{=} \varprojlim \mathcal{X}_H(v)$ *is transitive and factors through the surjective group homomorphism*

$$(\mathbf{1}, \mathrm{nr}_v) : G(\mathbf{A}_f) = G(\mathbf{A}_f)^v \times B_v^\times \to G(\mathbf{A}_f)^v \times F_v^\times$$

where $G(\mathbf{A}_f)^v = \{g \in G(\mathbf{A}_f); g_v = 1\}$.

(2) *For any point* $x \in \mathcal{X}(v)$ *(such as* $x = \mathrm{RED}_v([1,h])$ *if* v *does not split in* K*), the stabilizer of* x *in* $G(\mathbf{A}_f)^v \times F_v^\times$ *may be computed as follows. Let* B' *be the quaternion algebra over* F *which is obtained from* B *by changing the invariants at* v *and* τ_1*:* B' *is totally definite and* $\mathrm{Ram}_f B' = \mathrm{Ram}_f B \cup \{v\}$*. Put* $G' = \mathrm{Res}_{F/\mathbf{Q}}(B'^\times)$*, a reductive group over* $\mathbf{Q}$ *with center* Z*. There exists an isomorphism* $\phi_x^v : G(\mathbf{A}_f)^v \xrightarrow{\simeq} G'(\mathbf{A}_f)^v$ *such that* $(\phi_x^v, \mathbf{1}) : G(\mathbf{A}_f)^v \times F_v^\times \xrightarrow{\simeq} G'(\mathbf{A}_f)^v \times F_v^\times$ *maps* $\mathrm{Stab}(x)$ *to the image of* $G'(\mathbf{Q})\overline{Z(\mathbf{Q})} \subset G'(\mathbf{A}_f)$ *through the (surjective) map*

$$(\mathbf{1}, \mathrm{nr}_v) : G'(\mathbf{A}_f) = G'(\mathbf{A}_f)^v \times B'^\times_v \to G'(\mathbf{A}_f)^v \times F_v^\times.$$

Proof. In the compact case, this is exactly how Carayol describes the action of $G(\mathbf{A}_f)$ on a set which he denotes by S, cf. Proposition 11.2 of [**?**]. The fact that Carayol's set S equals our $\mathcal{X}(v)$ follows from the discussion of [**?**, Section 10.1]. The non-compact case is similar.

Define

$$G(v) \stackrel{\text{def}}{=} G'(\mathbf{A}_f)^v \times F_v^\times,$$

$$\phi_x(v) \stackrel{\text{def}}{=} (\phi_x^v, \mathrm{nr}_v) : G(\mathbf{A}_f) \twoheadrightarrow G(v),$$

and let $G(\mathbf{Q}, v)$ be the image of $G'(\mathbf{Q})$ in $G(v)$:

$$G(\mathbf{Q}, v) = (\mathbf{1}, \mathrm{nr}_v)(G'(\mathbf{Q})).$$

Corollary 3.10 *The map* $g \mapsto x \cdot g$ *factors through* $\phi_x(v)$ *and induces a bijection*

$$\overline{G(\mathbf{Q}, v)} \backslash G(v) \xrightarrow{\simeq} \mathcal{X}(v)$$

where $\overline{G(\mathbf{Q}, v)}$ *is the closure of* $G(\mathbf{Q}, v)$ *in* $G(v)$.

Proof. We have to show that $\overline{G(\mathbf{Q},v)} = (\mathbf{1}, \mathrm{nr}_v)(G'(\mathbf{Q})\overline{Z(\mathbf{Q})})$. We first claim that $G'(\mathbf{Q})\overline{Z(\mathbf{Q})}$ is locally closed (hence closed) in $G'(\mathbf{A}_f)$. Indeed, let R be a maximal $\mathcal{O}_F$-order in B'. As $\overline{Z(\mathbf{Q})} = F^\times \overline{\mathcal{O}_F^\times}$,

$$\left(G'(\mathbf{Q})\overline{Z(\mathbf{Q})}\right) \cap \widehat{R}^\times = \left(B'^\times \overline{\mathcal{O}_F^\times}\right) \cap \widehat{R}^\times = R^\times \overline{\mathcal{O}_F^\times} = \cup_{\alpha \in R^\times / O_F^\times} \alpha \overline{\mathcal{O}_F^\times}$$

is closed because $[R^\times : \mathcal{O}_F^\times]$ is finite (use [**?**, p. 139]). The map $\mathrm{nr}_v : B'^\times_v \to F_v^\times$ is open and surjective with a *compact* kernel: it is therefore a closed map, and so is $(\mathbf{1}, \mathrm{nr}_v) : G'(\mathbf{A}_f) \to G(v)$. In particular, $\overline{(\mathbf{1}, \mathrm{nr}_v)(G'(\mathbf{Q})\overline{Z(\mathbf{Q})})}$ is closed in $G(v)$, so that $\overline{G(\mathbf{Q},v)} \subset (\mathbf{1}, \mathrm{nr}_v)(G'(\mathbf{Q})\overline{Z(\mathbf{Q})})$ and the other inclusion is trivial.

3.2.4 Reciprocity laws.

We now want to describe the reciprocity laws for CM points and connected components, following [**?**] instead of [**?**] (see the remark at the end of [**?**, §12]). In particular: (1) reciprocity maps send uniformizers to geometric Frobenius; (2) Galois actions on geometric points are *left* actions.

Let $\mu : \mathbb{G}_{m,\mathbf{C}} \to T_{\mathbf{C}}$ be the cocharacter which is defined by $\mu(z) = s \circ r(z)$, where $r : \mathbb{G}_{m,\mathbf{C}} \to \mathbb{S}_{\mathbf{C}} \simeq \mathbb{G}_{m,\mathbf{C}} \times \mathbb{G}_{m,\mathbf{C}}$ maps z to $(z, 1)$ (and $\mathbb{S}_{\mathbf{C}} \simeq \mathbb{G}_{m,\mathbf{C}} \times \mathbb{G}_{m,\mathbf{C}}$ is induced by $z \otimes_{\mathbf{R}} a \mapsto (za, \bar{z}a)$ for $z \in \mathbf{C}$ and a in some $\mathbf{C}$-algebra A). The isomorphism

$$T_{\mathbf{C}} \xrightarrow{\lambda \otimes a \mapsto (\tau(\lambda)a)_\tau} \mathbb{G}_{m,\mathbf{C}}^{\mathrm{Hom}(K,\mathbf{C})}$$

yields a bijection between the set of cocharacters of T and $\mathbf{Z}^{\mathrm{Hom}(K,\mathbf{C})}$, with $\sigma \in \mathrm{Aut}(\mathbf{C})$ acting on the latter set by $(n_\tau)_\tau \cdot \sigma = (n_{\sigma\tau})_\tau$. The cocharacter μ corresponds to $n_\tau = \epsilon$ if $\tau = \tau_1$ and $n_\tau = 0$ otherwise. In particular, the field of definition of μ equals $\tau_1(K) \simeq K$ and the morphism

$$T = \mathrm{Res}_{K/\mathbf{Q}}(\mathbb{G}_{m,K}) \xrightarrow{\mathrm{Res}_{K/\mathbf{Q}}(\mu)} \mathrm{Res}_{K/\mathbf{Q}}(T_K) \xrightarrow{\mathrm{Norm}_{K/\mathbf{Q}}} T$$

sends z to z^ϵ. We thus obtain:

Lemma 3.11 *The CM points are algebraic, defined over the maximal abelian extension K^{ab} of K. For $\sigma = \mathrm{rec}_K(\lambda)$ with $\lambda \in T(\mathbf{A}_f) = \widehat{K}^\times$, the action of σ on* CM $\simeq \overline{T(\mathbf{Q})} \backslash G(\mathbf{A}_f)$ *is given by multiplication on the left by λ^ϵ.*

Similarly:

Lemma 3.12 *The connected components are defined over the maximal abelian extension F^{ab} of F. For $\sigma = \mathrm{rec}_F(\lambda)$ with $\lambda \in Z(\mathbf{A}_f) = \widehat{F}^\times$, the action of σ on $\mathcal{Z} \simeq \overline{Z(\mathbf{Q})}^+ \backslash Z(\mathbf{A}_f)$ is given by multiplication by λ^ϵ.*

In particular, the pro-tale F-scheme $\mathcal{M} \overset{\text{def}}{=} \varprojlim \mathcal{M}_H$ together with its right action of $G(\mathbf{A}_f)$ is (non-canonically) isomorphic to $\mathrm{Spec}(F^{\mathrm{ab}})$ on which $G(\mathbf{A}_f)$ acts through $g \mapsto \mathrm{Spec}(\sigma)$ with $\sigma = \mathrm{rec}_F(\mathrm{nr}(g)^\epsilon)$, while $\boldsymbol{\mathcal{M}} \overset{\text{def}}{=} \varprojlim \boldsymbol{\mathcal{M}}_H$ is (non-canonically) isomorphic to the spectrum of the ring of v-integers in F^{ab}. It follows that the reduction map $\mathcal{Z} = \mathcal{M}(F^{\mathrm{ab}}) \to \mathcal{Z}(v) = \boldsymbol{\mathcal{M}}(\mathbf{F}(\overline{v}))$ identifies $\mathcal{Z}(v)$ with $\mathcal{Z}/\mathcal{O}_v^\times$ (viewing $\mathcal{O}_v^\times$ as a subgroup of $Z(\mathbf{A}_f) = Z(\mathbf{A}_f)^v \times F_v^\times$). Since $\mathcal{O}_v^\times$ also acts trivially on $\mathbf{M}^{ss}(v) = \varprojlim \mathbf{M}_H^{ss}(v)$ (cf. [**?**, section 11.2]), the projection $\mathcal{X}(v) \to \mathbf{M}^{ss}(v)$ also identifies $\mathbf{M}^{ss}(v)$ with $\mathcal{X}(v)/\mathcal{O}_v^\times$ (viewing now $\mathcal{O}_v^\times$ as a subgroup of $G(v) = G'(\mathbf{A}_f)^v \times F_v^\times$).

Corollary 3.13 *If* $\mathrm{nr}(H_v) = \mathcal{O}_v^\times$ (1) $\boldsymbol{\mathcal{M}}_H$ *is a finite tale $\mathcal{S}$-scheme,* (2) $\mathcal{Z}_H \simeq \mathcal{Z}_H(v)$ *and* (3) $\mathcal{X}(v)/H \simeq \mathcal{X}_H(v) \simeq \mathbf{M}_H^{ss}(v)$.

Proof. In general, $\boldsymbol{\mathcal{M}}_H$ is isomorphic to the spectrum of the ring of v-integers in the abelian extension F_H of F which is cut out by $\mathrm{rec}_K(\mathrm{nr}(H))$. If $\mathcal{O}_v^\times \subset \mathrm{nr}(H)$, F_H is unramified at v and $\boldsymbol{\mathcal{M}}_H$ is therefore a finite tale $\mathcal{S}$-scheme. This proves (1) and (2), and (2) implies that $\mathcal{X}_H(v) = \mathbf{M}_H^{ss}(v) \times_{\mathcal{Z}_H(v)} \mathcal{Z}_H \simeq \mathbf{M}_H^{ss}(v)$. Finally, since $\mathcal{X}(v)/\mathcal{O}_v^\times \simeq \mathbf{M}^{ss}(v)$, $\mathcal{X}(v)/H \simeq \mathbf{M}^{ss}(v)/H \simeq \mathbf{M}_H^{ss}(v)$.

Remark 3.14 The assumption $\mathrm{nr}(H_v) = \mathcal{O}_v^\times$ holds true when $H = \widehat{R}^\times$ for some Eichler order $R \subset B$.

3.2.5 Conclusion.

Putting lemmas 3.7, 3.8 and Corollary 3.10 together, we obtain a commutative diagram

$$\begin{array}{ccccc} \overline{T(\mathbf{Q})}\backslash G(\mathbf{A}_f) & \xrightarrow{(1)} & \overline{G(\mathbf{Q},v)}\backslash G(v) & \xrightarrow{(2)} & \overline{Z(\mathbf{Q})^+}\backslash Z(\mathbf{A}_f) \\ \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\ \mathrm{CM} & \xrightarrow{\mathrm{RED}_v} & \mathcal{X}(v) & \xrightarrow{c_v} & \mathcal{Z} \end{array}$$

where (1) is induced by $\phi_x(v) : G(\mathbf{A}_f) \to G(v)$ (with $x = \mathrm{RED}_v([1,h])$) while (2) is induced by the morphism

$$\begin{array}{rcl} G'(\mathbf{A}_f)^v \times F_v^\times = G(v) & \longrightarrow & Z(\mathbf{A}_f) = Z(\mathbf{A}_f)^v \times F_v^\times \\ (g^v, \lambda_v) & \longmapsto & (\mathrm{nr}(g^v), \lambda_v) \end{array}$$

For a compact open subgroup H of $G(\mathbf{A}_f)$, put $H(v) = \phi_x(v)(H) \subset G(v)$. We thus obtain a diagram

$$\begin{array}{ccccc} T(\mathbf{Q})\backslash G(\mathbf{A}_f)/H & \longrightarrow & G(\mathbf{Q},v)\backslash G(v)/H(v) & \longrightarrow & Z(\mathbf{Q})^+\backslash Z(\mathbf{A}_f)/\mathrm{nr}(H) \\ \simeq\downarrow & & \downarrow & & \simeq\downarrow \\ \mathrm{CM}_H & \xrightarrow{\mathrm{RED}_v} & \mathcal{X}_H(v) & \xrightarrow{c_v} & \mathcal{Z}_H \end{array}$$

in which the middle vertical arrow is surjective (and a bijection when $H = \widehat{R}^\times$ for some Eichler order $R \subset B$). Theorem 3.5 is therefore a consequence of a special case (S) of Theorem 2.9, corresponding to the situation where $\mathfrak{S}$ (in the notations of Theorem 2.9) equals $\{\{v\}, v \in \mathfrak{S}\}$ (in the notations of Theorem 3.5).

3.3 Complements

3.3.1 On the parameter $\epsilon = \pm 1$.

Let us fix an isomorphism of $\mathbf{R}$-algebras between B_{τ_1} and $M_2(\mathbf{R})$, thus obtaining an isomorphism of group schemes over $\mathbf{R}$ between G_1 and $\mathrm{GL}_2(\mathbf{R})$. Let X_ϵ be the $G(\mathbf{R})$-conjugacy class of the morphism $h_\epsilon : \mathbb{S} \to G_{\mathbf{R}}$ which sends $z = x + iy$ to

$$h_\epsilon(z) = \left[\begin{pmatrix} x & y \\ -y & x \end{pmatrix}^\epsilon, 1, \cdots, 1 \right] \in G_1(\mathbf{R}) \times \cdots \times G_d(\mathbf{R}) \simeq G(\mathbf{R})$$

and let $\{M_H(\epsilon)\}$ be the corresponding collection of Shimura curves. We thus have a compatible system of isomorphisms $\psi_H(\epsilon) : M_H(\epsilon) \times_F \mathbf{C} \to M_H^{\mathrm{alg}}(\epsilon)$, where $M_H^{\mathrm{alg}}(\epsilon)$ is the algebraic curve over $\mathbf{C}$ whose underlying Riemann surface equals

$$M_H^{\mathrm{an}}(\epsilon) = G(\mathbf{Q})\backslash \left(G(\mathbf{A}_f)/H \times X_\epsilon \right).$$

The topology, the differentiable structure and the real analytic structure of X_ϵ are induced from those of $G(\mathbf{R})$ through the map $g \mapsto g h_\epsilon g^{-1}$. For $h \in X_\epsilon$ and $z \in \mathbf{C}^\times = \mathbb{S}(\mathbf{R})$, the map $x \mapsto h(z) x h(z)^{-1}$ fixes h and therefore induces an $\mathbf{R}$-linear map $T_h(\mathrm{ad}h(z))$ on the tangent space T_hX_ϵ of X_ϵ at h. The almost complex structure on X_ϵ is characterized by the fact that $T_h(\mathrm{ad}h(z))$ acts by multiplication by $z/\bar{z}$ on T_hX_ϵ for all $h \in X_\epsilon$ and $z \in \mathbf{C}^\times$. This almost complex structure is known to be integrable.

Remark 3.15 Most authors replace X_ϵ by $\mathbf{C} - \mathbf{R}$ with $G(\mathbf{R})$ acting through the projection on the first component $G_1(\mathbf{R}) \simeq \mathrm{GL}_2(\mathbf{R})$, by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \lambda = \frac{a\lambda+b}{c\lambda+d}$ $(\lambda \in \mathbf{C}-\mathbf{R})$. This corresponds to $\epsilon = 1$. Indeed, the map $gh_1g^{-1} \mapsto g \cdot i$ yields a diffeomorphism between X_1 and $\mathbf{C}-\mathbf{R}$ and for $z \in \mathbf{C}^\times$, the derivative of $\lambda \mapsto gh_1(z)g^{-1} \cdot \lambda$ at $\lambda = g \cdot i$ equals $z/\bar{z}$. On the other hand, our main reference [**?**] on Shimura curves very explicitly uses $\epsilon = -1$. While it seems clear that Carayol's constructions could easily be transfered to the $\epsilon = 1$ case, we will show below that the choice of ϵ is, in fact, irrelevant.

From the above discussion, we know that the $G(\mathbf{R})$-equivariant map $\Phi : X_\epsilon \to X_{-\epsilon}$ which sends h to h^{-1} is an antiholomorphic diffeomorphism. For any compact open subgroup H of $G(\mathbf{A}_f)$, Φ therefore induces an antiholomorphic diffeomorphism between $M_H^{\mathrm{an}}(\epsilon)$ and $M_H^{\mathrm{an}}(-\epsilon)$ and an antilinear isomorphism between $M_H^{\mathrm{alg}}(\epsilon)$ and $M_H^{\mathrm{alg}}(-\epsilon)$, namely an isomorphism of *schemes* $\Phi : M_H^{alg}(\epsilon) \to M_H^{alg}(-\epsilon)$ such that the diagram

$$\begin{array}{ccc} M_H^{\mathrm{alg}}(\epsilon) & \xrightarrow{\Phi} & M_H^{\mathrm{alg}}(-\epsilon) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{C}) & \xrightarrow{\mathrm{Spec}(\tau)} & \mathrm{Spec}(\mathbf{C}) \end{array}$$

is commutative (τ=complex conjugation).

For any scheme X over $\mathrm{Spec}(\mathbf{C})$, we denote by $\tau X \to \mathrm{Spec}(\mathbf{C})$ the pullback of $X \to \mathrm{Spec}(\mathbf{C})$ through $\mathrm{Spec}(\tau) : \mathrm{Spec}(\mathbf{C}) \to \mathrm{Spec}(\mathbf{C})$. The above diagram thus yields an isomorphism of complex curves between $M_H^{\mathrm{alg}}(\epsilon)$ and $\tau M_H^{\mathrm{alg}}(-\epsilon)$ which together with $\psi_H(\epsilon)$ and $\psi_H(-\epsilon)$ induces an isomorphism

$$\Phi' : M_H(\epsilon) \times_F \mathbf{C} \to M_H(-\epsilon) \times_F \mathbf{C} \simeq \tau(M_H(-\epsilon) \times_F \mathbf{C})$$

(recall that F is embedded in $\mathbf{C}$ through $\tau_1 : F \hookrightarrow \mathbf{R}$). In other words, $M_H(-\epsilon)$ is a twist of $M_H(\epsilon)$. We shall now determine this twist.

For $\sigma \in \mathrm{Aut}(\mathbf{C}/F)$, let $\rho(\sigma)$ be the F-automorphism of $M_H(\epsilon)$ such that $\rho(\sigma) \cdot [g, h] = [g\lambda, h]$ for $g \in G(\mathbf{A}_f)$ and $h \in X_\epsilon$, where λ is any element of $Z(\mathbf{A}_f)$ (the center of $G(\mathbf{A}_f)$) such that $\mathrm{rec}_F(\lambda) = \sigma$ in $\mathrm{Gal}_F^{\mathrm{ab}}$.

One easily checks that $\sigma \mapsto \rho(\sigma)$ is a well-defined group homomorphism $\rho : \mathrm{Aut}(\mathbf{C}/F) \to \mathrm{Aut}_F(M_H(\epsilon))$ which factors through $\mathrm{Gal}(F'_H/F)$ where F'_H is the abelian extension of F corresponding to the subgroup $F^\times \cdot (Z(\mathbf{A}_f) \cap H)$ of $Z(\mathbf{A}_f) = \widehat{F}^\times$.

Lemma 3.16 *Φ' realizes $M_H(-\epsilon)$ as the twist of $M_H(\epsilon)$ by $\rho^{-\epsilon}$.*

Proof. On the level of complex points, Φ' is the composite of Φ with the action of complex conjugation. The latter is described by a conjecture of Langlands [**?**], proven in [**?**]. We obtain: for $x = [g, h] \in M_H^{\mathrm{an}}(\epsilon)$ (with $g \in G(\mathbf{A}_f)$ and $h \in X_\epsilon$), $\Phi'(x) = [g, \bar{h}^{-1}] \in M_H^{\mathrm{an}}(-\epsilon)$ where $\bar{h} : \mathbb{S} \to G_{\mathbf{R}}$ maps z to $h(\bar{z})$. Note that $h \mapsto \bar{h}$ is indeed an involution of X_ϵ since

$$\bar{h}_\epsilon(x+iy) = \left[\begin{pmatrix} x & -y \\ y & x \end{pmatrix}^\epsilon, 1, \cdots, 1 \right] = \omega h_\epsilon(x+iy)\omega^{-1}$$

where $\omega = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 1, \cdots, 1 \right] \in G(\mathbf{R})$.

If h is a special point of X_ϵ, $\bar{h}^{-1}$ is a special point of $X_{-\epsilon}$. More precisely, suppose that $h : \mathbb{S} \to G_{\mathbf{R}}$ factors through $T'_{\mathbf{R}}$ for some *maximal* $\mathbf{Q}$-rational subtorus $T' \subset G$. Let $\mu_h : \mathbb{G}_{m,\mathbf{C}} \to T'_{\mathbf{C}}$ be the induced cocharacter ($\mu_h(z) = h(z, 1)$), let $E_h \subset \mathbf{C}$ be the field of definition of μ_h (so that $F \subset E_h$) and put $\mathrm{rec}_h = \mathrm{Norm}_{E_h/\mathbf{Q}} \circ \mathrm{Res}_{E_h/\mathbf{Q}}(\mu_h)$:

$$\mathrm{rec}_h : \mathrm{Res}_{E_h/\mathbf{Q}}(\mathbb{G}_{m,E_h}) \to \mathrm{Res}_{E_h/\mathbf{Q}}(T'_{E_h}) \to T'.$$

Let also $\mu_0 : \mathbb{G}_{m,\mathbf{C}} \to Z_{\mathbf{C}} \subset T'_{\mathbf{C}}$ be the cocharacter defined by

$$\mu_0(z) = \left[\begin{pmatrix} z & \\ & z \end{pmatrix}, 1, \cdots, 1 \right] \in Z(\mathbf{R}) \subset G(\mathbf{R}) \simeq G_1(\mathbf{R}) \times \cdots \times G_d(\mathbf{R}).$$

Then μ_0 is defined over F and

$$Z = \mathrm{Res}_{F/\mathbf{Q}}(\mathbb{G}_{m,F}) \overset{\mathrm{Res}_{F/\mathbf{Q}}(\mu_0)}{\longrightarrow} \mathrm{Res}_{F/\mathbf{Q}}(Z_F) \overset{\mathrm{Norm}_{F/\mathbf{Q}}}{\longrightarrow} Z$$

is the identity map. Since $\mu_h \cdot \mu_{\bar{h}} = \mu_0^\epsilon$, $\mu_{\bar{h}}$ is also defined over E_h and

$$\mathrm{rec}_h \cdot \mathrm{rec}_{\bar{h}} = \mathrm{Norm}_{E_h/F}^\epsilon : \mathrm{Res}_{E_h/\mathbf{Q}}(\mathbb{G}_{m,E_h}) \to \mathrm{Res}_{F/\mathbf{Q}}(\mathbb{G}_{m,F}) = Z \subset T'.$$

It follows that (1) for $g \in G(\mathbf{A}_f)$, both $x = [g, h]$ and $\Phi'(x) = [g, \bar{h}^{-1}]$ are defined over the maximal abelian extension $E_h^{\mathrm{ab}} \subset \mathbf{C}$ of E_h; (2) for $\lambda \in \widehat{E}_h^\times = \mathrm{Res}_{E_h/\mathbf{Q}}(\mathbf{A}_f)$ and $\sigma = \mathrm{rec}_{E_h}(\lambda) \in \mathrm{Gal}_{E_h}^{\mathrm{ab}}$,

$$\begin{aligned}\Phi'(\rho(\sigma)^{-\epsilon}(\sigma \cdot x)) &= [\mathrm{rec}_h(\lambda) g \mathrm{Norm}_{E_h/F}^{-\epsilon}(\lambda), \bar{h}^{-1}] \\ &= [\mathrm{rec}_h(\lambda) \mathrm{Norm}_{E_h/F}^{-\epsilon}(\lambda) g, \bar{h}^{-1}] \\ &= [\mathrm{rec}_{\bar{h}^{-1}}(\lambda) g, \bar{h}^{-1}] \\ &= \sigma \cdot \Phi'(x).\end{aligned}$$

Our claim now easily follows from the uniqueness of canonical models.

As a scheme over F, the twist $M_H(\epsilon)'$ of $M_H(\epsilon)$ by $\rho^{-\epsilon}$ may be constructed as the quotient of $M_H(\epsilon) \times_{\mathrm{Spec}(F)} \mathrm{Spec}(F'_H)$ by the (right) action of $\mathrm{Gal}(F'_H/F)$ which maps σ to the F-automorphism $\alpha(\sigma) = (\rho(\sigma)^\epsilon, \mathrm{Spec}(\sigma))$ of $M_H(\epsilon) \times_F \mathrm{Spec}(F'_H)$.

Lemma 3.17 *Suppose that $H = \bar{H}$ where $g \mapsto \bar{g}$ is the anticommutative involution of $G(\mathbf{A}_f)$ which is induced by the canonical involution of B. Then $M_H(\epsilon)'$ is isomorphic to $M_H(\epsilon)$. In particular, $M_H(\epsilon) \simeq M_H(-\epsilon)$.*

Proof. We shall construct an involution θ of $M_H(\epsilon)$ with the property that for all $\sigma \in \Gamma_H \stackrel{\mathrm{def}}{=} \mathrm{Gal}(F'_H/F)$,

$$\theta \circ \alpha(\sigma) = (1 \times \mathrm{Spec}(\sigma)) \circ \theta \quad \text{on } M_H(\epsilon) \times_F \mathrm{Spec}(F'_H).$$

Such a θ induces an F-isomorphism between $M_H(\epsilon)'$ and $M_H(\epsilon)$.

Recall that $\mathcal{M}_H(\epsilon) = \mathrm{Spec}\left(\Gamma(\mathcal{O}_{M_H(\epsilon)}, M_H(\epsilon))\right)$ is non-canonically isomorphic to $\mathrm{Spec}(F_H)$ where F_H is the abelian extension of F cut out by $F_{>0}^\times \cdot \mathrm{nr}(H) \subset \widehat{F}^\times$. Since $(F^\times \cdot H \cap \widehat{F}^\times)^2 \subset F_{>0}^\times \cdot \mathrm{nr}(H)$, there is a well defined group homomorphism $\kappa : \Gamma_H \to \mathrm{Gal}(F_H/F)$ given by $\kappa(\mathrm{rec}_F(\lambda)) = \mathrm{rec}_F(\lambda^2)$ for $\lambda \in \widehat{F}^\times$. It follows from the discussion after Lemma 3.12 that for $\sigma \in \Gamma_H$ and $x \in M_H(\epsilon)(\mathbf{C})$,

$$c(\rho(\sigma)(x)) = \kappa(\sigma)^\epsilon \cdot c(x) \quad \text{in } \mathcal{M}_H(\epsilon)(\mathbf{C}) = \mathcal{M}_H(\epsilon)(F_H).$$

On the other hand, F'_H is a subfield of F_H whenever $F_{>0}^\times \cdot \mathrm{nr}(H) \subset F^\times \cdot H \cap \widehat{F}^\times$. This is indeed the case when $\bar{H} = H$. In particular, we may choose an F-morphism $\mathcal{M}_H(\epsilon) \to \mathrm{Spec}(F'_H)$, thus obtaining an F-morphism $c' : M_H(\epsilon) \to$

$\mathrm{Spec}(F'_H)$ such that

$$\forall \sigma \in \Gamma_H, \quad c' \circ \rho(\sigma) = \mathrm{Spec}(\sigma^{2\epsilon}) \circ c'.$$

Let A be an F-algebra and let $z = (x, y)$ be an A-valued point of the F-scheme $M_H(\epsilon) \times_F \mathrm{Spec}(F'_H)$. Then $c'(x)$ and y are A-valued points of $\mathrm{Spec}(F'_H)$. If $\mathrm{Spec}(A)$ is connected, there exists a unique element $\gamma \stackrel{\text{def}}{=} \gamma(z)$ in Γ_H such that $c'(x) = \mathrm{Spec}(\gamma^{-\epsilon}) \circ y$. This defines an F-morphism $z \mapsto \gamma(z)$ from $M_H(\epsilon) \times_F \mathrm{Spec}(F'_H)$ to the constant F-scheme Γ_H. For $z = (x, y)$ as above, we put $\theta(z) = (\rho(\gamma(z))(x), y)$. One easily checks that θ has the required properties.

When $H = \bar{H}$, we thus obtain an F-isomorphism between $M_H(\epsilon)$ and $M_H(-\epsilon)$. On the level of complex points, such an isomorphism is given by

$$[g, h] \in M_H^{\mathrm{an}}(\epsilon) \mapsto [\bar{g}^{-1}, \bar{h}^{-1}] \in M_H^{\mathrm{an}}(-\epsilon).$$

Note that the condition $H = \bar{H}$ defines a cofinal subset of the set of all compact open subgroups H of $G(\mathbf{A}_f)$. Also, $H = \bar{H}$ when $H = \widehat{R}^\times$ for some Eichler order R in B, in which case F_H and F'_H are respectively the Hilbert class field and the narrow Hilbert class field of F.

3.3.2 P-rational elements of $\mathrm{Gal}_K^{\mathrm{ab}}$.

It may seem rather surprising that the bizarre subgroup $\mathrm{Gal}_K^{P-\mathrm{rat}}$ of P-rational elements in $\mathrm{Gal}_K^{\mathrm{ab}}$ should play any role in the theory of CM points. For instance, $\mathrm{Gal}_K^{P-\mathrm{rat}}$ is *not* a closed subgroup of $\mathrm{Gal}_K^{\mathrm{ab}}$, although it contains the closed subgroup

$$\mathrm{Gal}(K^{\mathrm{ab}}/K[P^\infty]) = \mathrm{rec}_K \left\{ \lambda \in \widehat{O}_K^\times, \lambda_P \in O_{F_P}^\times \right\}.$$

The Galois group $\mathrm{Gal}(K[P^\infty]/K)$ is topologically isomorphic to $G_0 \times \mathbf{Z}_p^{[F_P:\mathbf{Q}_p]}$ where p is the residue characteristic of P and G_0 is a finite group, the torsion subgroup of $\mathrm{Gal}(K[P^\infty]/K)$. The subfield of $K[P^\infty]$ which is fixed by G_0 is the composite of all $\mathbf{Z}_p$-extensions of K which are unramified outside P and Galois and dihedral over F. The image of $\mathrm{Gal}_K^{P-\mathrm{rat}}$ in $\mathrm{Gal}(K[P^\infty]/K)$ is a *dense* but *countable* subgroup which is generated by the Frobeniuses of those primes of K which are not above P (the intersection of this subgroup with G_0 plays a key role in [**?**], where it is denoted by G_1). In particular, $\mathrm{Gal}_K^{P-\mathrm{rat}}$

is a *dense* but *negligible* (i.e. measurable with trivial measure) subgroup of $\mathrm{Gal}_K^{\mathrm{ab}}$. The map $\sigma = \mathrm{rec}_K(\lambda) \mapsto \lambda_P$ yields a bijection between $\mathrm{Gal}_K^{\mathrm{ab}}/\mathrm{Gal}_K^{P-\mathrm{rat}}$ and $K_P^\times/K^\times F_P^\times$.

However, the appearance of rational elements perhaps less surprising when one recalls that the present work originated in the study of elliptic curves over anticyclotomic towers of number fields, since the distinction between suitably defined rational and irrational elements of Galois groups occurs quite frequently in the context of Iwasawa theory. For instance, the celebrated theorems of Ferrero and Washington on the growth of class numbers in Z_p extensions of abelian fields rely crucially on the fact that nontrivial roots of unity are irrational. Another example of this occurs in recent work of Hida [**?**], [**?**] on anticyclotomic families of Hecke characters, where the key observation is the irrationality of certain Galois actions on Serre-Tate deformation spaces. Indeed, the irrationality arguments given by Ferrero and Washington were the original motivation for the introduction in [**?**] of rational and irrational elements to the study of CM points.

In this section, we shall provide some further evidence for the relevance P-rational elements by relating them to the André-Oort conjecture:

Proposition 3.18 *For $\sigma \in \mathrm{Gal}_K^{ab}$ and $x \in \mathrm{CM}_H$, put $\delta(x) = (x, \sigma x) \in M_H(\mathbf{C})^2$. The following conditions are equivalent.*

1. *σ is a P-rational element.*
2. *For any collection $\mathcal{E} \subset \mathrm{CM}_H$ of P-isogeneous CM points, the Zariski closure of $\delta(\mathcal{E})$ in $(M_H \times_F \mathbf{C})^2$ has dimension ≤ 1.*
3. *For some collection $\mathcal{E} \subset \mathrm{CM}_H$ of P-isogeneous CM points, the Zariski closure of $\delta(\mathcal{E})$ has dimension 1.*

For the proof of this proposition, we may and do assume that $H = \widehat{R}^\times$ for some maximal order $R \subset B$. For any CM point

$$x = [g] \in \mathrm{CM}_H = T(\mathbf{Q})\backslash G(\mathbf{A}_f)/H,$$

the stabilizer of x in $\mathrm{Gal}_K^{\mathrm{ab}}$ then equals $\mathrm{rec}_K(K^\times \mathcal{O}(x)^\times)$ where $\mathcal{O}(x) = K \cap gHg^{-1}$ is an $\mathcal{O}_F$-order in B. Moreover, there exists a unique integral ideal $\mathcal{C} \subset \mathcal{O}_F$ such that $\mathcal{O}(x) = \mathcal{O}_{K,\mathcal{C}} \stackrel{\mathrm{def}}{=} \mathcal{O}_F + \mathcal{C}\mathcal{O}_K$. We refer to $\mathcal{C} \stackrel{\mathrm{def}}{=} \mathcal{C}(x)$ as *the conductor* of x and denote by $\ell_P(x) \geq 0$ the exponent of P in $\mathcal{C}(x)$, so that

$\mathcal{C}(x) = \mathcal{C}_0(x) P^{\ell_P(x)}$ for some integral ideal $\mathcal{C}_0(x)$ which is relatively prime to P. By construction, $x \mapsto \mathcal{C}(x)$ is constant on $\mathrm{Gal}_K^{\mathrm{ab}}$-orbits while $x \mapsto \mathcal{C}_0(x)$ is constant on P-isogeny classes. It follows from [**?**, pp. 42-44] that the fibers of $\mathcal{C}$ are finite. In particular:

Lemma 3.19 *The function $x \mapsto \ell_P(x)$ has finite fibers on any P-isogeny class.*

This function is related to the usual distance d on the Bruhat-Tits tree

$$\mathcal{T} = F_P^\times \backslash B_P^\times / R_P^\times \simeq F_P^\times \backslash \mathrm{GL}_2(F_P)/\mathrm{GL}_2(\mathcal{O}_{F_P}).$$

Indeed, the group $K_P^\times$ acts on the left on $\mathcal{T}$ by isometries, and for $v = [b] \in \mathcal{T}$ (with $b \in B_P^\times$), the stabilizer of v in $K_P^\times$ equals $F_P^\times \mathcal{O}(v)^\times$where $\mathcal{O}(v) = K_P \cap b R_P b^{-1}$ is an $\mathcal{O}_{F_P}$-order in K_P. Just as above, there exists a unique integer $n \overset{\mathrm{def}}{=} n(v) \in \mathbf{N}$ such that $\mathcal{O}(v) = \mathcal{O}_n$ with $\mathcal{O}_n \overset{\mathrm{def}}{=} \mathcal{O}_{F_P} + P^n \mathcal{O}_{K_P}$ ($\mathcal{O}_n$ is the completion of $\mathcal{O}_{K,\mathcal{C}_0 P^n}$ at P for any integral ideal $\mathcal{C}_0 \subset \mathcal{O}_F$ which is relatively prime to P). It is clear that for a CM point $x = [g] \in \mathrm{CM}_H$, $\ell_P(x) = n(v)$ where $v = [g_P]$ ($g_P \in B_P^\times$ is the P-component of $g \in G(\mathbf{A}_f)$).

It is well-known that

- The map $v \mapsto n(v)$ yields a bijection between $K_P^\times \backslash \mathcal{T}$ and $\mathbf{N}$.
- The subset $\mathcal{T}_0 = \{v \in \mathcal{T};\, n(v) = 0\}$ of $\mathcal{T}$ consists of a vertex, two adjacent vertices or the set of vertices on a line in $\mathcal{T}$, depending upon whether P is inert, ramifies or splits in K.
- For any $v \in \mathcal{T}$, $n(v)$ is also the distance between v and $\mathcal{T}_0$.

In particular, suppose that $(v_n, v_{n-1}, \cdots, v_0)$ and $(w_m, w_{m-1}, \cdots, w_0)$ are geodesics in $\mathcal{T}$ from $v = v_n$ and $w = w_m$ to $\mathcal{T}_0$. Then $n(v_i) = i$ for $0 \le i \le n$ and $n(w_j) = j$ for $0 \le j \le m$. The geodesic γ between v and w may then be computed as follows:

- if $v_0 \neq w_0$, $\gamma = (v_n, v_{n-1}, \cdots, v_0, u_1, \cdots, u_{r-1}, w_0, w_1, \cdots, w_m)$ where $(v_0, u_1, \cdots, u_{r-1}, w_0)$ is the geodesic between v_0 and w_0 inside the connected subtree $\mathcal{T}_0$ of $\mathcal{T}$.
- if $v_0 = w_0$, $\gamma = (v_n, v_{n-1}, \cdots, v_c = w_c, w_{c+1}, \cdots, w_m)$ where c is the largest integer $\le n, m$ such that $v_c = w_c$.

In the special case where $w = \lambda v$ for some $\lambda \in K_P^\times$, $n = m = n(v)$ and $w_i = \lambda v_i$ for $0 \leq i \leq n$. If moreover $d(v, \lambda v) \leq 2n$, it thus must be that $v_0 = w_0$. With c as above, $d(v,w) = 2(n-c)$ and $v_c = w_c = \lambda v_c$, so that λ belongs to $F_P^\times \mathcal{O}_c^\times$. We have obtained:

Lemma 3.20 *Suppose that $d(v, \lambda v) \leq 2n(v)$ (with $v \in \mathcal{T}$ and $\lambda \in K_P^\times$). Then $d(v, \lambda v) = 2k$ for some $k \in \{0, \cdots, n(v)\}$ and λ belongs to $F_P^\times \mathcal{O}_{n(v)-k}^\times$.*

We may now sketch the proof of Proposition3.18. Of course, (2) implies (3).

(1) **implies** (2)**.**

We have to show that for any P-isogeny class $\mathcal{H} \subset CM_H$, $\delta(\mathcal{H})$ is contained in a one dimensional subscheme of $(M_H \times_F \mathbf{C})^2$ if $\sigma = \mathrm{rec}_K(\lambda)$ for some $\lambda \in T(\mathbf{A}_f) = \widehat{K}^\times$ whose P-component λ_P belongs to $F_P^\times$. Choose $g_0 \in G(\mathbf{A}_f)$ such that

$$\mathcal{H} = T(\mathbf{Q}) \backslash T(\mathbf{Q}) B_P^\times g_0 H/H \quad \text{inside} \quad \mathrm{CM}_H = T(\mathbf{Q}) \backslash G(\mathbf{A}_f)/H.$$

Using Lemma 3.11, we find that

$$\begin{aligned} \delta(\mathcal{H}) &= \left\{ ([bg_0], [\lambda^\epsilon b g_0])\,;\, b \in B_P^\times \right\} \\ &= \left\{ ([bg_0], [bg_0\gamma])\,;\, b \in B_P^\times \right\} \end{aligned}$$

where $\gamma = g_0^{-1} \lambda^\epsilon g_0$. Indeed, $bg_0\gamma = b\lambda^\epsilon g_0 = \lambda^\epsilon b g_0$ for any $b \in B_P^\times$ as λ_P^ϵ belongs to $F_P^\times$. In particular, $\delta(\mathcal{H})$ is contained in the 1-dimensional image of the (algebraic!) morphism $M_{H \cap \gamma H \gamma^{-1}} \to M_H^2$ which sends $[g,h]$ to $([g,h],[g\gamma,h])$ ($g \in G(\mathbf{A}_f)$, $h \in X$).

(3) **implies** (1)**.**

Write $\sigma = \mathrm{rec}_K(\lambda)$ with $\lambda \in \widehat{K}^\times$. Suppose that the Zariski closure of $\delta(\mathcal{E})$ in $(M_H \times_F \mathbf{C})^2$ has dimension 1 for some (infinite) collection $\mathcal{E}$ of P-isogeneous CM points. We have to show that the P-component $\lambda_P \in K_P^\times$ of λ belongs to $F_P^\times K^\times$.

By a proven case of the Andr-Oort conjecture [**?**, Theorem 1.2] there exists an infinite subset $\mathcal{E}' \subset \mathcal{E}$ and some element $\gamma \in G(\mathbf{A}_f)$ such that $\delta(\mathcal{E}')$ is contained in the image of a morphism $M_{H \cap \gamma H \gamma^{-1}} \to M_H^2$ as above. Fix $x = [g_0] \in \mathcal{E}'$ and let $\{g_1, \cdots, g_s\} \subset H$ be a set of representatives for

$H/H \cap \gamma H \gamma^{-1}$. For each $x' = [bg_0] \in \mathcal{E}'$ (with $b \in B_P^\times$), we know that $x = [bg_0 g_i]$ for any $i \in \{1, \cdots, s\}$ while

$$\sigma \cdot x \in \{[bg_0 g_1 \gamma], \cdots, [bg_0 g_s \gamma]\}.$$

Replacing g_0 by $g_0 g_i$ for a suitable $1 \leq i \leq s$ and using lemmas 3.11 and 3.19, we obtain: there exists a sequence $b_n \in B_P^\times$ such that (a) $\varphi(n) \stackrel{\text{def}}{=} \ell_P([b_n g_0])$ goes to infinity with n and (b) $[\lambda^\epsilon b_n g_0] = [b_n g_0 \gamma]$ for all $n \geq 0$. By (b), there exists $\lambda_n \in T(\mathbf{Q}) = K^\times$ and $h_n \in H$ such that for all $n \geq 0$,

$$\lambda_n \lambda^\epsilon b_n g_0 = b_n g_0 \gamma h_n \quad \text{in } G(\mathbf{A}_f).$$

Put $v_n = [b_n g_{0,P}] \in \mathcal{T}$ and $\mu_n = \lambda_n \lambda^\epsilon$. Since $\mu_{n,P} \cdot v_n = [b_n g_{0,P} \gamma_P]$, $d(v_n, \mu_{n,P} \cdot v_n) = d_0$ does not depend upon n. Pick $N \geq 0$ such that $\forall n \geq N$, $d_0 \leq 2\varphi(n)$. By Lemma 3.20, $d_0 = 2k$ and

$$\forall n \geq N: \quad \mu_{n,P} \in F_P^\times \mathcal{O}_{\varphi(n)-k}^\times. \tag{8}$$

On the other hand, $\mu_n \mu_N^{-1} = b_n g_0 \gamma h_n g_0^{-1} b_n^{-1} b_N g_0 h_N^{-1} \gamma^{-1} g_0^{-1} b_N^{-1}$. Away from P, this equation simplifies to $(\mu_n \mu_N^{-1})^P = (g_0 \gamma h_n h_N^{-1} \gamma^{-1} g_0^{-1})^P$, so that

$$(\mu_n \mu_N^{-1})_Q \in K_Q^\times \cap (g_0 \gamma)_Q R_Q^\times (g_0 \gamma)_Q^{-1} \subset \mathcal{O}_{K_Q}^\times \tag{9}$$

for all $Q \neq P$.

Let $U_F \subset U_K$ be the groups of all elements $z \in F^\times$ (resp. $z \in K^\times$) which are units away from P. Since K is a totally imaginary quadratic extension of F, $\text{rank}_{\mathbf{Z}} U_K = \text{rank}_{\mathbf{Z}} U_F$ if P does not split in K and $\text{rank}_{\mathbf{Z}} U_K = \text{rank}_{\mathbf{Z}} U_F + 1$ otherwise. Let U_K' be the subgroup of U_K defined by $U_K' = U_K \cap F_P^\times O_{K_P}^\times$. Then $U_F \subset U_K' \subset U_K$, and $[U_F : U_K']$ is finite. Let $\mathcal{R} \subset U_K'$ be a set of representatives for U_K'/U_F.

By (9), $\mu_n \mu_N^{-1} = \lambda_n \lambda_N^{-1}$ belongs to U_K for all $n \geq 0$. Then (8) shows that $\mu_n \mu_N^{-1}$ belongs to U_K' for all $n \geq N$. For such n's, we may thus write

$$\lambda_n = \lambda_N r(n) u(n) \quad \text{with } r(n) \in \mathcal{R} \text{ and } u(n) \in U_F.$$

Using (8) again, we find that $\lambda_N r(n) \lambda_P^\epsilon$ belongs to $F_P^\times \mathcal{O}_{\varphi(n)-k}^\times$ for all $n \geq N$. Choosing a subsequence on which $r(n) = r$ is constant, we finally obtain:

$$\lambda_N r \lambda_P^\epsilon \in F_P^\times = \cap_{n \geq 0} F_P^\times \mathcal{O}_n^\times.$$

Since $\lambda_N r$ belongs to $K^\times$, λ_P indeed belongs to $K^\times F_P^\times$.

Remark 3.21 More generally, it may be shown that for any infinite collection $\mathcal{E} \subset \mathrm{CM}_H$ of P-isogeneous CM points and any finite subset $\mathcal{R} = \{\sigma_1, \cdots, \sigma_r\}$ of $\mathrm{Gal}_K^{\mathrm{ab}}$, the Zariski closure of $\{(\sigma_1 x, \cdots, \sigma_r x); x \in \mathcal{E}\}$ in $V = (M_H \times_F \mathbf{C})^r$ contains a connected component of V if and only if the σ_i's are pairwise distinct modulo $\mathrm{Gal}_K^{P-\mathrm{rat}}$ (Hint: use section 7.3 of [**?**] and a variant of Proposition 2.1 of [**?**]).