First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Grade:

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Indefinite Integrals

1. 9 marks Each part is worth 3 marks. Please write your answers in the boxes.

   (a) Calculate the indefinite integral $I = \int x^2 e^{-3x^3}\, dx$.

   Answer: $I = -\frac{e^{-3x^3}}{9} + C$

   Solution: Let $u = 3x^3$, so $\frac{du}{dx} = 9x^2$. Then,

$$I = \frac{1}{9}\int e^{-u} du = -\frac{e^{-u}}{9} + C = -\frac{e^{-3x^3}}{9} + C.$$

   (b) Calculate the indefinite integral $I = \int \frac{3x-2}{x^2+6x+8}\, dx$ for $x > 0$.

   Answer: $7\ln|x+4| - 4\ln|x+2| + C$

   Solution: Factorise the denominator $x^2 + 6x + 8 = (x+2)(x+4)$. Write the integrand as partial fractions:

$$\frac{3x-2}{(x+2)(x+4)} = \frac{A}{x+2} + \frac{B}{x+4} = \frac{A(x+4) + B(x+2)}{(x+2)(x+4)}\,.$$

   Use $x = -2$, so $-8 = 2A$, that is, $A = -4$. Use $x = -4$, so $-14 = -2B$, that is, $B = 7$. Then,

$$I = -\int \frac{4}{x+2}dx + \int \frac{7}{x+4}dx = -4\ln|x+2| + 7\ln|x+4| + C\,.$$

(c) (A Little Harder): Calculate the indefinite integral $\int x^2 \sin x \, dx$.

Answer: $I = -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$

Solution: Use I.B.P. with $u = x^2$ and $v' = \sin(x)$ so $u' = 2x$ and $v = -\cos(x)$. Then,

$$I = -x^2\cos(x) + 2\int x\cos(x)\,dx\,.$$

Use I.B.P again to solve the integral on the R.H.S. with $w = x$ and $z' = \cos(x)$, so $w' = 1$ and $z = \sin(x)$. Then,

$$I = -x^2\cos(x) + 2\left(x\sin(x) - \int \sin(x)dx\right) = -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$$

Definite Integrals

2. 12 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate $I = \int_0^{\pi/8} \tan^5(2x)\sec^2(2x)\,dx$.

Answer: $I = \frac{1}{12}$

Solution: Let $u = \tan(2x)$, so $\frac{du}{dx} = 2\sec^2(2x)$, $u(0) = 0$ and $u(\frac{\pi}{8}) = 1$. Then,

$$I = \frac{1}{2}\int_0^1 u^5 du = \frac{1}{2}\left[\frac{u^6}{6}\right]_0^1 = \frac{1}{12}.$$

(b) Calculate $I = \int_1^e x^2 \ln x\,dx$.

Answer: $I = \frac{1+2e^3}{9}$

Solution: Use I.B.P. with $u = \ln(x)$ and $v' = x^2$, so $u' = \frac{1}{x}$ and $v = \frac{x^3}{3}$. Then,

$$\begin{aligned} I &= \left[\frac{x^3\ln(x)}{3}\right]_1^e - \frac{1}{3}\int_1^e x^3 \cdot \frac{1}{x}dx \\ &= \frac{e^3}{3} - \frac{1}{3}\left[\frac{x^3}{3}\right]_1^e \\ &= \frac{e^3}{3} - \frac{e^3}{9} + \frac{1}{9} = \frac{1+2e^3}{9}. \end{aligned}$$

(c) (A Little Harder): Calculate $I = \int_0^1 x^3\sqrt{1-x^2}\,dx$.

Answer: $I = \frac{2}{15}$

Solution: Let $x = \sin(\theta)$, so $\frac{dx}{d\theta} = \cos(\theta)$, $\theta = \arcsin(x)$, $\theta(0) = 0$ and $\theta(1) = \frac{\pi}{2}$. Then,

$$I = \int_0^{\frac{\pi}{2}} \sin^3(\theta)\sqrt{1-\sin^2(\theta)}\cos(\theta)d\theta = \int_0^{\frac{\pi}{2}} \sin^3(\theta)|\cos(\theta)|\cos(\theta)d\theta.$$

Since $\cos(\theta) \geq 0$ when $0 \leq \theta \leq \frac{\pi}{2}$, we have $|\cos(\theta)| = \cos(\theta)$. Then,

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \sin^3(\theta)\cos^2(\theta)d\theta \\ &= \int_0^{\frac{\pi}{2}} \sin^2(\theta)\cos^2(\theta)\sin(\theta)d\theta \\ &= \int_0^{\frac{\pi}{2}} (1-\cos^2(\theta))\cos^2(\theta)\sin(\theta)d\theta \end{aligned}$$

Now, let $u = \cos(\theta)$, so $\frac{du}{d\theta} = -\sin(\theta)$, $u(0) = 1$ and $u(\frac{\pi}{2}) = 0$. Then,

$$I = -\int_1^0 (1-u^2)u^2du = -\left[\frac{u^3}{3} - \frac{u^5}{5}\right]_1^0 = \frac{2}{15}.$$

Method II: Write the integral as

$$I \equiv \int_0^1 x^2\sqrt{1-x^2}\,(xdx)\,.$$

Set $u = 1-x^2$, so that $xdx = -du/2$. Since $x = 0$ and $x = 1$ maps to $u = 1$ and $u = 0$, we use $x^2 = 1-u$ and get

$$I = -\frac{1}{2}\int_1^0 (1-u)u^{1/2}\,du = \frac{1}{2}\int_0^1 \left(u^{1/2} - u^{3/2}\right)\,du = \frac{1}{2}\left(\frac{2}{3} - \frac{2}{5}\right) = \frac{2}{15}\,.$$

Method III:
Let $u = \sqrt{1-x^2}$, so that $\frac{du}{dx} = \frac{-x}{\sqrt{1-x^2}}$. We get $u(0) = 1$ and $u(1) = 0$. Then,

$$I = -\int_0^1 x^2(1-x^2)\left(\frac{-x}{\sqrt{1-x^2}}\right)dx = -\int_1^0 (1-u^2)u^2du = \frac{2}{15}.$$

Riemann Sum, FTC, and Volumes

3. [12 marks] Each part is worth 4 marks. Please write your answers in the boxes.

   (a) Calculate the infinite sum

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2i}{n^2\left(4+i^2/n^2\right)}$$

by first writing it as a definite integral. Then, **evaluate this integral**.

Answer: $\ln\left|\frac{5}{4}\right|$

Solution:

$$\frac{2i}{n^2\left(4+i^2/n^2\right)}=\frac{2\frac{i}{n}}{4+i^2/n^2}\cdot\frac{1}{n}.$$

So, $\Delta x=\frac{1}{n}$, $x_i=0+\frac{i}{n}$ and $x_i^*=x_i$. Then,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2i}{n^2\left(4+i^2/n^2\right)}=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2x_i^*}{4+(x_i^*)^2}\Delta x=\int_0^1\frac{2x}{4+x^2}dx.$$

Let $u=4+x^2$, so that $\frac{du}{dx}=2x$, and $u(0)=4$ and $u(1)=5$. Then,

$$\int_0^1\frac{2x}{4+x^2}dx=\int_4^5\frac{1}{u}du=\ln\left|\frac{5}{4}\right|$$

   (b) For $x>0$ define $F(x)=\int_1^x t^{1/2}\,dt$ and $g(x)=\sqrt{F(x^4)}$. Calculate $g'(2)$.

Answer: $\frac{2\cdot 2^3\sqrt{2^4}}{\sqrt{\frac{2}{3}((2^4)^{\frac{3}{2}}-1)}}$ or $\frac{64}{\sqrt{42}}$

Solution: First, $F(x)=\frac{2}{3}(x^{\frac{3}{2}}-1)$ and $F'(x)=\sqrt{x}$. Now differentiate g and use the chain rule twice to obtain,

$$g'(x)=\frac{4x^3F'(x^4)}{2\sqrt{F(x^4)}}$$
$$=\frac{2x^3\sqrt{x^4}}{\sqrt{\frac{2}{3}((x^4)^{\frac{3}{2}}-1)}}.$$

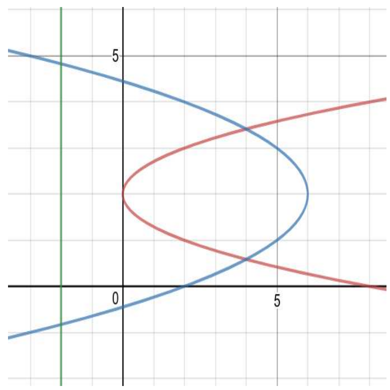
So,

$$g'(2)=\frac{2\cdot 2^3\sqrt{2^4}}{\sqrt{\frac{2}{3}((2^4)^{\frac{3}{2}}-1)}}=\frac{64}{\sqrt{\frac{2}{3}63}}=\frac{64}{\sqrt{42}}.$$

(c) Write a definite integral, with specified limits of integration, for the volume obtained by revolving the bounded region between $x = 2(y-2)^2$ and $x = 6-(y-2)^2$ about the vertical line $x = -2$. **Do not evaluate the integral**.

Answer: $\int_{2-\sqrt{2}}^{2+\sqrt{2}} \pi \left|(8-(y-2)^2)^2 - (2(y-2)^2+2)^2\right| dy$

Solution:



First, find the intersection points of the two curves $x_B = 2(y-2)^2$ (red curve) and $x_T = 6-(y-2)^2$ (blue curve) by setting

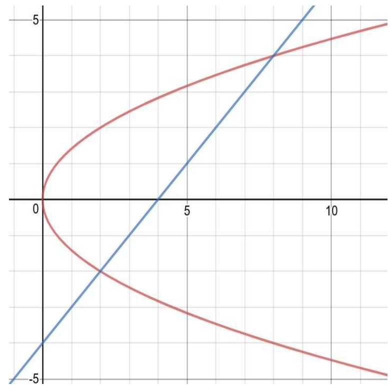
$$2(y-2)^2 = 6-(y-2)^2 .$$

This gives $y = 2+\sqrt{2}$ or $y = 2-\sqrt{2}$. Now, shift the functions so the rotation is around the y-axis. This gives $x = x_B + 2 = 2(y-2)^2 + 2$ and $x = x_T + 2 = 6-(y-2)^2 + 2$. Finally, integrate over y to get

$$V = \pi \int_{2-\sqrt{2}}^{2+\sqrt{2}} \left|\left(8-(y-2)^2\right)^2 - \left(2(y-2)^2+2\right)^2\right| dy .$$

4. (a) 2 marks Plot the finite area enclosed by $y^2 = 2x$ and $y = x - 4$.

Solution: The plot for $y^2 = 2x$ (red curve) and $y = x - 4$ (blue curve) is as shown, and the area is between these two curves.



(b) 4 marks Write a definite integral with specific limits of integration that determines this area. **Do not evaluate the integral**.

Solution: First, find the intersection points of the two curves from

$$\frac{y^2}{2} = y + 4\,.$$

This gives $y = 4$ or $y = -2$.

Now, since $x_T = y + 4$ and $x_B = y^2/2$ satisfies $x_T > x_B$ on $-2 < y < 4$, we integrate over y to find the area A as

$$A = \int_{-2}^{4} (x_T - x_B)\, dy = \int_{-2}^{4} \left(y + 4 - \frac{y^2}{2} \right) dy\,.$$

5. A solid has as its base the region in the xy-plane between $y = 1 - x^2/49$ and the x-axis. The cross-sections of the solid perpendicular to the x-axis are squares.

(a) 4 marks Write a definite integral that determines the volume of the solid.

Solution:

The cross section at x is given by a square of side $l(x) = 1 - \frac{x^2}{49}$

So the area of the cross section is $A(x) = \left(1 - \frac{x^2}{49}\right)^2$.

The volume is given by:

$$V = \int_{-7}^{7} \left(1 - \frac{x^2}{49}\right)^2 dx$$

(b) 2 marks **Evaluate the integral** to find the volume of the solid.

Solution: Let $u = \frac{x}{7}$, so that $\frac{du}{dx} = \frac{1}{7}$. Then, $u = -7$ when $x = -1$ and $u = 7$ when $x = 1$. This gives,

$$\begin{aligned} V &= \int_{-7}^{7} \left(1 - \left(\frac{x}{7}\right)^2\right)^2 dx = 7 \int_{-1}^{1} (1 - u^2)\, du \\ &= 14 \int_{0}^{1} (1 - 2u^2 + u^4)\, du = 14 \left[u - \frac{2u^3}{3} + \frac{u^5}{5}\right]_0^1 = \frac{112}{15}. \end{aligned}$$