First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Grade:

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Indefinite Integrals

1. 12 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate the indefinite integral $\int (\ln x)^2\, dx$ for $x > 0$.

Answer: $x(\ln x)^2 - 2x(\ln x - 1) + C$

Solution: We do integration by parts with:

$$u(x) = (\ln x)^2 \Rightarrow u'(x) = 2\frac{1}{x}\ln x,$$
$$v'(x) = 1 \Rightarrow v(x) = x.$$

$$\int (\ln x)^2\, dx = x(\ln x)^2 - \int 2\ln x\, dx$$

and then again integration by parts on $\int \ln x\, dx$ with $u(x) = \ln x$ and $v'(x) = 1$, and finally get:

$$\int (\ln x)^2\, dx = x(\ln x)^2 - 2x(\ln x - 1) + C$$

(b) Calculate the indefinite integral $\int 3x\sqrt{3-3x}\, dx$ for $x < 1$.

Answer: $-\frac{2}{5}(3-3x)^{3/2}(\frac{2}{3}+x)+C$

Solution: We take $u(x) = 3 - 3x$, then we have $u'(x) = -3$ and we replace $3x$ by $3 - u(x)$, such that we write

$$I = \int 3x\sqrt{3-3x}\, dx = -\frac{1}{3}\int (-3)3x\sqrt{3-3x}\, dx = -\frac{1}{3}\int (3-u)u^{1/2}u'\, dx$$

and apply substitution rule as:

$$-\int (3-u)u^{1/2}u'\, dx = -\left(\int 3u^{1/2} - u^{3/2}\, du\right)_{u=3-3x}$$

Anti-differentiating the simple polynomial function $3u^{1/2} - u^{3/2}$ and eventually substituting $u(x) = 3 - 3x$, we finally get:

$$I = -\frac{2}{3}\left((3-3x)^{3/2} - \frac{1}{5}(3-3x)^{5/2}\right) + C = -\frac{2}{5}(3-3x)^{3/2}(\frac{2}{3}+x) + C$$

Note that this problem can also be solved by IBP (but more challenging) with:

$$u(x) = 3x \Rightarrow u'(x) = 3,$$
$$v'(x) = (3-3x)^{1/2} \Rightarrow v(x) = \frac{2}{3}\left(\frac{-1}{3}\right)(3-3x)^{3/2} = \frac{-2}{9}(3-3x)^{3/2}.$$

such that

$$\begin{aligned} I &= 3x\left(\frac{-2}{9}\right)(3-3x)^{3/2} - \int 3\left(\frac{-2}{9}\right)(3-3x)^{3/2}\,dx \\ &= \left(-\frac{2}{3}x\right)(3-3x)^{3/2} + \frac{2}{3}\int (3-3x)^{3/2}\,dx \end{aligned}$$

Given that the anti-derivative of $\int (3-3x)^{3/2}\,dx$ is $\frac{2}{5}\left(\frac{-1}{3}\right)(3-3x)^{5/2} + C = -\frac{2}{15}(3-3x)^{5/2} + C$, we get:

$$\begin{aligned} I = (3-3x)^{3/2}\left(-\frac{2}{3}x - \frac{4}{45}(3-3x)\right) + C &= (3-3x)^{3/2}\left(-\frac{4}{15} - \frac{2}{5}x\right) + C \\ &= -\frac{2}{5}(3-3x)^{3/2}\left(\frac{2}{3} + x\right) + C \end{aligned}$$

(c) (A Little Harder): Calculate the indefinite integral $\int \tan^3(6x)\sec^3(6x)\,dx$.

Answer: $\frac{1}{30}\sec^5(6x) - \frac{1}{18}\sec^3(6x) + C$

Solution: We use the substitution $u(x) = 6x$, $u'(x) = 6$ to rewrite the indefinite integral as:

$$I = \int \tan^3(6x)\sec^3(6x)\,dx = \frac{1}{6}\int 6\tan^3(6x)\sec^3(6x)\,dx$$
$$= \frac{1}{6}\left(\int \tan^3 u \sec^3 u\,du\right)_{u=6x}$$

Then it is classical trigonometric integral, we hold $\tan u \sec u$, replace $\tan^2 u$ by $\sec^2 u - 1$, and do another substitution $v(u) = \sec u$, $v'(u) = \tan u \sec u$ to get:

$$I = \frac{1}{6}\int (v^2-1)v^2 v'\,du = \frac{1}{6}\left(\int (v^2-1)v^2\,dv\right)_{v=\sec u} = \frac{1}{6}\left[\frac{1}{5}v^5 - \frac{1}{3}v^3\right]_{v=\sec u} + C$$

Finally we substitute $v = \sec u$ and $u = 6x$, which boils down to substituting $v = \sec(6x)$ to establish that:

$$I = \frac{1}{30}\sec^5(6x) - \frac{1}{18}\sec^3(6x) + C$$

Definite Integrals

2. 8 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate $\int_0^\pi 3\sin^3 x\,dx$.

Answer: 4

Solution: This is a trigonometric integral that is calculated as:

$$I = \int_0^\pi 3\sin^3 x\,dx = 3\int_0^\pi \sin x\,\sin^2 x\,dx = 3\int_0^\pi \sin x(1-\cos^2 x)\,dx$$

which gives:

$$I = 3\left[-\cos x + \frac{\cos^3 x}{3}\right]_0^\pi = \left[\cos^3 x - 3\cos x\right]_0^\pi = (-1+3)-(1-3) = 4$$

(b) Calculate $\int_1^2 \frac{x-1}{\sqrt{2x+1-x^2}}\,dx$.

Answer: $\sqrt{2}-1$

Solution: We can rewrite $2x+1-x^2$ as $2-(x-1)^2$ and use a trigonometric substitution as

$$x-1 = \sqrt{2}\sin\theta \quad , \quad x'(\theta) = \frac{dx}{d\theta} = \sqrt{2}\cos\theta,$$
$$x = 1 \Rightarrow \theta = 0 \quad , \quad x = 2 \Rightarrow \theta = \pi/4$$

to get:

$$I = \int_1^2 \frac{x-1}{\sqrt{2x+1-x^2}}\,dx = \int_0^{\pi/4} \frac{\sqrt{2}\sin\theta}{\sqrt{2-2\sin^2\theta}}\sqrt{2}\cos\theta\,d\theta$$

Now we replace $\sqrt{1-\sin^2\theta}$ by $\sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$ as $\cos\theta$ is positive on $[0,\pi/4]$ and finally calculate:

$$I = \sqrt{2}\int_0^{\pi/4} \sin\theta\,d\theta = \sqrt{2}[-\cos\theta]_0^{\pi/4} = \sqrt{2}-1$$

Note that this problem can also be solved by standard substitution: $u(x) = 2x+1-x^2$, $u'(x) = 2-2x = -2(x-1)$, $u(1) = 2$, $u(2) = 1$ as

$$\int_1^2 \frac{x-1}{\sqrt{2x+1-x^2}}\,dx = -\frac{1}{2}\int_1^2 \frac{-2(x-1)}{\sqrt{2x+1-x^2}}\,dx = -\frac{1}{2}\int_1^2 u'u^{-1/2}\,dx$$

and then

$$-\frac{1}{2}\int_1^2 u'u^{-1/2}\,dx = -\frac{1}{2}\int_2^1 u^{-1/2}\,du = [-u^{1/2}]_2^1 = -1 + 2^{1/2}$$

Riemann Sum and FTC

3. 12 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Which definite integral corresponds to $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2i\cos(\frac{i^2}{n^2}+1)}{n^2}$?

(A) $2\int_0^1 x\cos(x^2+1)dx$

(B) $\int_0^2 x\cos(x^2+1)dx$

(C) $\int_0^1 x\cos(x^2+1)dx$

(D) $2\int_0^1 \sqrt{x}\cos(x+1)dx$

(E) $\int_0^2 \sqrt{x}\cos(x+1)dx$

Answer: A

Solution: Pick $x_i = \frac{i}{n}$, so $x_0 = 0$, $x_n = 1$ and $\Delta x = \frac{1}{n}$. Then we can rewrite the summation as:

$$2\sum_{i=1}^{n} x_i \cos(x_i^2+1)\Delta x$$

which corresponds to the Right Riemann Sum for option (A).

(b) Define $F(x)$ and $g(x)$ by $F(x) = \int_x^2 \ln t\, dt$ and $g(x) = x^2\, F(x)$ for $x > 1$. Calculate $g'(1)$.

Answer: $4\ln 2 - 2$

Solution: We use the product rule to get: $g'(x) = 2xF(x) + x^2\, F'(x)$.
By FTC I, we get $F'(x) = -\ln x$, such that:

$$g'(x) = 2x\int_x^2 \ln t\, dt - x^2\, \ln x$$

Since $\ln 1 = 0$, we get:

$$g'(1) = 2\cdot 1\cdot \int_1^2 \ln t\, dt = 2\cdot \int_1^2 \ln t\, dt$$

By IBP, we calculate:

$$\int_1^2 \ln t\, dt = [t\ln t]_1^2 - \int_1^2 1\, dt = 2\ln 2 - 1$$

and get $g'(1) = 4\ln 2 - 2$.

(c) Let $F(x) = \int_{x^2}^{x^3} 6e^{t^2}\,dt$. Find the equation of the tangent line to the graph of $y = F(x)$ at $x = 1$. Tip: recall that the tangent line to the graph of $y = F(x)$ at $x = x_0$ is given by the equation $y = F(x_0) + F'(x_0)(x - x_0)$.

Answer: $y = 6e(x - 1)$

Solution: We first write $F(x)$ for any real number c as:

$$F(x) = -\int_c^{x^2} 6e^{t^2}\,dt + \int_c^{x^3} 6e^{t^2}\,dt$$

Then use FTC I and the chain rule to get:

$$F'(x) = -6e^{x^4}2x + 6e^{x^6}3x^2$$

Then we calculate $F(1)$ and $F'(1)$, we get $F(1) = \int_1^1 6e^{t^2}\,dt = 0$ and $F'(1) = 6e$, and finally the equation of the tangent $y - F(1) = F'(1)(x - 1)$ becomes

$$y = 6e(x - 1)$$

Areas and volumes

Please write your answers in the boxes. **Do not use absolute values in your expressions, always work out: (i) the outer function and the inner function for volumes or (ii) which function lies above the other function for areas.**

4. 4 marks Write a definite integral, with specified limits of integration, for the volume obtained by revolving the bounded region between $y = x + 5$ and $y = 6\sqrt{x}$ about the vertical line $x = -1$. **Do not evaluate the integral.**

Answer: $\pi \int_6^{30} (y-4)^2 - (y^2/36+1)^2 \, dy$

Solution: Intersection points are given by $x + 5 = 6\sqrt{x}$.

Solving for x, we determine the 2 intersection points

$$I_1 = (1, 6) \quad , \quad I_2 = (25, 30).$$

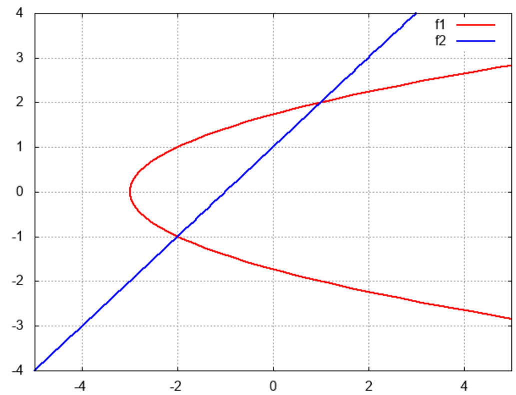
We integrate in y, hence we write x as a function of y for the 2 curves and apply a shift of $+1$, we finally establish:

$$\pi \int_6^{30} (y-4)^2 - (y^2/36+1)^2 \, dy.$$

5. (a) [2 marks] Sketch by hand the finite area enclosed by $y^2 = x + 3$ and $y = 1 + x$

Answer:

Solution: The area is the region enclosed between the red and blue curves:



(b) [4 marks] Write a definite integral with specific limits of integration that determines this finite area.

Answer: $\int_{-1}^{2} \left(-y^2 + y + 2\right)\, dy$

Solution: We first find the intersection between the two curves, given by the solution of:

$$y^2 - 3 = y - 1 \Leftrightarrow (y - 2)(y + 1) = 0.$$

We then label the curve $x_R = y^2 - 3$ and $x_B = y - 1$ and notice that $x_B \geq x_R$ for $-1 \leq y \leq 2$. The area is therefore given by the following definite integral:

$$A = \int_{-1}^{2} \left(y - 1 - y^2 + 3\right)\, dy = \int_{-1}^{2} \left(-y^2 + y + 2\right)\, dy$$

(c) 2 marks Evaluate the integral to compute the area enclosed.

Answer: $\frac{9}{2}$

Solution:

$$A = \left[-\frac{y^3}{3} + \frac{y^2}{2} + 2y \right]_{-1}^{2} = \frac{9}{2}$$