First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Grade:

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Indefinite Integrals

1. 12 marks Each part is worth 4 marks. Please write your answers in the boxes.

   (a) Calculate the indefinite integral $\int x^2 e^{-x}\, dx$ for $x > 0$.

   Answer: $-e^{-x}(x^2 + 2x + 2) + C$

   Solution: We do integration by parts with:

   $$u(x) = x^2 \Rightarrow u'(x) = 2x,$$
   $$v'(x) = e^{-x} \Rightarrow v(x) = -e^{-x}.$$

   $$\int x^2 e^{-x}\, dx = -x^2 e^{-x} - \int 2x(-e^{-x})\, dx$$

   and then again integration by parts on $\int xe^{-x}\, dx$ with $u(x) = x$ and $v'(x) = e^{-x}$, and finally get:

   $$\int x^2 e^{-x}\, dx = -e^{-x}(x^2 + 2x + 2) + C$$

   (b) Calculate the indefinite integral $\int x\sqrt{3-x}\, dx$ for $x < 3$.

   Answer: $-\frac{2}{5}(3-x)^{3/2}(2+x) + C$

   Solution: We take $u(x) = 3 - x$, then we have $u'(x) = -1$ and we replace x by $3 - u(x)$, such that we write

   $$\int x\sqrt{3-x}\, dx = -\int (-1)x\sqrt{3-x}\, dx = -\int (3-u)u^{1/2}u'\, dx$$

   and apply substitution rule as:

   $$-\int (3-u)u^{1/2}u'\, dx = -\left(\int 3u^{1/2} - u^{3/2}\, du\right)_{u=3-x}$$

   Anti-differentiating the simple polynomial function $3u^{1/2} - u^{3/2}$ and eventually substituting $u(x) = 3 - x$, we finally get:

   $$\int x\sqrt{3-x}\, dx = -2\left((3-x)^{3/2} - \frac{1}{5}(3-x)^{5/2}\right) + C = -\frac{2}{5}(3-x)^{3/2}(2+x) + C$$

   Note that this problem can also be solved by IBP (but more challenging) with:

   $$u(x) = x \Rightarrow u'(x) = 1,$$
   $$v'(x) = (3-x)^{1/2} \Rightarrow v(x) = \frac{2}{3}(-1)(3-x)^{3/2} = -\frac{2}{3}(3-x)^{3/2}.$$

such that

$$I = x\left(-\frac{2}{3}\right)(3-x)^{3/2} - \int 1\left(\frac{-2}{3}\right)(3-x)^{3/2}\,dx$$
$$= \left(-\frac{2}{3}x\right)(3-x)^{3/2} + \frac{2}{3}\int (3-x)^{3/2}\,dx$$

Given that the anti-derivative of $\int (3-x)^{3/2}\,dx$ is $\frac{2}{5}(-1)(3-x)^{5/2} + C = -\frac{2}{5}(3-x)^{5/2} + C$, we get:

$$I = (3-x)^{3/2}\left(-\frac{2}{3}x - \frac{4}{15}(3-x)\right) + C = (3-x)^{3/2}\left(-\frac{12}{15} - \frac{2}{5}x\right) + C$$
$$= -\frac{2}{5}(3-x)^{3/2}(2+x) + C$$

(c) (A Little Harder): Calculate the indefinite integral $\int \tan^2(8x)\sec^4(8x)\,dx$.

Answer: $\frac{1}{24}\tan^3(8x) + \frac{1}{40}\tan^5(8x) + C$

Solution: We use the substitution $u(x) = 8x$, $u'(x) = 8$ to rewrite the indefinite integral as:

$$I = \int \tan^2(8x)\sec^4(8x)\,dx = \frac{1}{8}\int 8\tan^2(8x)\sec^4(8x)\,dx$$
$$= \frac{1}{8}\left(\int \tan^2 u \sec^4 u\,du\right)_{u=8x}$$

Then it is classical trigonometric integral, we hold $\sec^2 u$, replace $\sec^2 u$ by $\tan^2 u + 1$, and do another substitution $v(u) = \tan u$, $v'(u) = \sec^2 u$ to get:

$$I = \frac{1}{8}\int (v^2+1)v^2v'\,du = \frac{1}{8}\left(\int (v^2+1)v^2\,dv\right)_{v=\tan u} = \frac{1}{8}\left[\frac{1}{5}v^5 + \frac{1}{3}v^3\right]_{v=\tan u} + C$$

Finally we substitute $v = \tan u$ and $u = 8x$, which boils down to substituting $v = \tan(8x)$ to establish that:

$$I = \frac{1}{24}\tan^3(8x) + \frac{1}{40}\tan^5(8x) + C$$

Definite Integrals

2. [8 marks] Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate $\int_0^{2\pi} (xe^{-x^2} + x\cos x)\,dx$.

Answer: $\frac{1-e^{-4\pi^2}}{2}$

Solution: The first part gives:

$$\int_0^{2\pi} xe^{-x^2}\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_0^{2\pi} = \frac{1-e^{-4\pi^2}}{2}$$

and the second part is calculated by IBP as:

$$\int_0^{2\pi} x\cos x\,dx = [x\sin x]_0^{2\pi} - \int_0^{2\pi} \sin x\,dx = [x\sin x + \cos x]_0^{2\pi} = 0$$

so $\int_0^{2\pi} (xe^{-x^2} + x\cos x)\,dx = \frac{1-e^{-4\pi^2}}{2}$.

(b) Calculate $\int_3^4 \frac{x-3}{\sqrt{6x-7-x^2}}\,dx$.

Answer: $\sqrt{2}-1$

Solution: We can rewrite $6x-7-x^2$ as $2-(x-3)^2$ and use a trigonometric substitution as

$$x-3 = \sqrt{2}\sin\theta \quad , \quad x'(\theta) = \frac{dx}{d\theta} = \sqrt{2}\cos\theta,$$
$$x = 3 \Rightarrow \theta = 0 \quad , \quad x = 4 \Rightarrow \theta = \pi/4$$

to get:

$$I = \int_3^4 \frac{x-3}{\sqrt{6x-7-x^2}}\,dx = \int_0^{\pi/4} \frac{\sqrt{2}\sin\theta}{\sqrt{2-2\sin^2\theta}}\sqrt{2}\cos\theta\,d\theta$$

Now we replace $\sqrt{1-\sin^2\theta}$ by $\sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$ as $\cos\theta$ is positive on $[0,\pi/4]$ and finally calculate:

$$I = \sqrt{2}\int_0^{\pi/4} \sin\theta\,d\theta = \sqrt{2}[-\cos\theta]_0^{\pi/4} = \sqrt{2}-1$$

Note that this problem can also be solved by standard substitution: $u(x) = 6x-7-x^2$, $u'(x) = 6-2x = -2(x-3)$, $u(3) = 2$, $u(4) = 1$ as

$$\int_3^4 \frac{x-3}{\sqrt{6x-7-x^2}}\,dx = -\frac{1}{2}\int_3^4 \frac{-2(x-3)}{\sqrt{6x-7-x^2}}\,dx = -\frac{1}{2}\int_3^4 u'u^{-1/2}\,dx$$

and then

$$-\frac{1}{2}\int_3^4 u'u^{-1/2}\,dx = -\frac{1}{2}\int_2^1 u^{-1/2}\,du = [-u^{1/2}]_2^1 = -1 + 2^{1/2}$$

Riemann Sum and FTC

3. 12 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Which definite integral corresponds to $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4i\ln(\frac{2i}{n}+3)}{n^2}$?

(A) $\int_0^4 x\ln(\frac{x}{2}+3)dx$

(B) $\int_0^2 (x-3)\ln(x)dx$

(C) $\int_3^5 (x-3)\ln(x)dx$

(D) $2\int_0^2 x\ln(x+3)dx$

(E) $\int_3^5 x\ln(x+3)dx$

Answer: C

Solution: Pick $x_i = \frac{2i}{n}+3$, so $x_0 = 3$, $x_n = 5$ and $\Delta x = \frac{2}{n}$. Then we can rewrite the summation as:

$$\sum_{i=1}^{n}(x_i-3)\ln(x_i)\Delta x$$

which corresponds to the Right Riemann Sum for option (C).

(b) Define $F(x)$ and $g(x)$ by $F(x) = \int_0^x te^t\,dt$ and $g(x) = 2x\,F(2x+1)$. Calculate $g'(0)$.

Answer: 2

Solution: We use the product rule to get: $g'(x) = 2F(2x+1)+2x\,F'(2x+1)$, and the chain rule and FTC I to calculate $F'(2x+1) = F'(y)y'(x) = 2(2x+1)e^{2x+1}$ with $y(x) = 2x+1$. So we have:

$$g'(x) = 2F(2x+1) + 4x(2x+1)e^{2x+1}$$

Note that you do not even need to calculate $F'(2x+1)$ as when you set $x = 0$ in the term $2x\,F'(2x+1)$, you already get 0 due to the factor $2x$.
Taking $x = 0$ we get $g'(0) = 2F(1) + 0\cdot 1\cdot e^1 = 2F(1)$. Now we calculate $F(1)$ by IBP as

$$F(1) = \int_0^1 te^t\,dt = [te^t]_0^1 - \int_0^1 1\cdot e^t\,dt = [(t-1)e^t]_0^1 = (1-1)e^1 - (0-1)e^0 = 1$$

and get $g'(0) = 2$.

(c) Let $F(x) = \int_{x^2}^{x^3} 4e^{t^2}\,dt$. Find the equation of the tangent line to the graph of $y = F(x)$ at $x = 1$. Tip: recall that the tangent line to the graph of $y = F(x)$ at $x = x_0$ is given by the equation $y = F(x_0) + F'(x_0)(x - x_0)$.

Answer: $y = 4e(x-1)$

Solution: We first write $F(x)$ for any real number c as:

$$F(x) = -\int_c^{x^2} 4e^{t^2}\,dt + \int_c^{x^3} 4e^{t^2}\,dt$$

Then use FTC I and the chain rule to get:

$$F'(x) = -4e^{x^4}2x + 4e^{x^6}3x^2$$

Then we calculate $F(1)$ and $F'(1)$, we get $F(1) = \int_1^1 4e^{t^2}\,dt = 0$ and $F'(1) = 4e$, and finally the equation of the tangent $y - F(1) = F'(1)(x-1)$ becomes

$$y = 4e(x-1)$$

Areas and volumes

Please write your answers in the boxes. **Do not use absolute values in your expressions, always work out: (i) the outer function and the inner function for volumes or (ii) which function lies above the other function for areas.**

4. 4 marks Write a definite integral, with specified limits of integration, for the volume obtained by revolving the bounded region between $x = y - 5$ and $x = y^2/36$ about the horizontal line $y = -2$. **Do not evaluate the integral.**

Answer: $\pi \int_1^{25} (6\sqrt{x} + 2)^2 - (x + 7)^2 \, dx$

Solution: Intersection points are given by $36x = (x + 5)^2$.

Solving for x, we determine the 2 intersection points

$$I_1 = (1, 6) \quad , \quad I_2 = (25, 30).$$

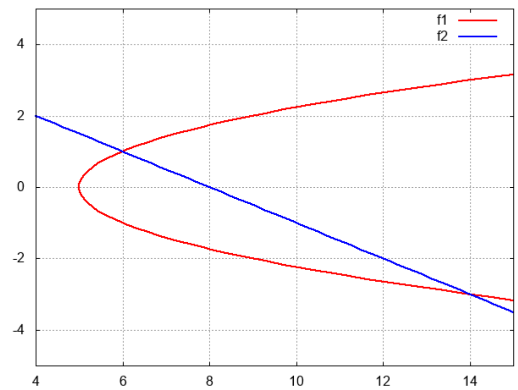
We integrate in x, hence we write y as a function of x for the 2 curves and apply a shift of $+2$, we finally establish:

$$\pi \int_1^{25} (6\sqrt{x} + 2)^2 - (x + 7)^2 \, dx.$$

5. (a) 2 marks Sketch by hand the finite area enclosed by $y^2 + 5 = x$ and $2y = 8 - x$

Answer:

Solution: The area is the region enclosed between the red and blue curves:



(b) 4 marks Write a definite integral with specific limits of integration that determines this finite area.

Answer: $\int_{-3}^{1} (-y^2 - 2y + 3)\ dy$

Solution: We first find the intersection between the two curves, given by the solution of:

$$y^2 + 5 = 8 - 2y \Leftrightarrow (y + 3)(y - 1) = 0.$$

We then label the curve $x_R = y^2 + 5$ and $x_B = 8 - 2y$ and notice that $x_B \geq x_R$ for $-3 \leq y \leq 1$. The area is therefore given by the following definite integral:

$$A = \int_{-3}^{1} \left(8 - 2y - y^2 - 5\right)\ dy = \int_{-3}^{1} \left(-y^2 - 2y + 3\right)\ dy$$

(c) 2 marks Evaluate the integral to compute the area enclosed.

Answer: $\frac{32}{3}$

Solution:

$$A = \left[-\frac{y^3}{3} - \frac{2y^2}{2} + 3y \right]_{-3}^{1} = \frac{32}{3}$$