First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Grade:

The remainder of this page has been left blank for your workings.

Indefinite Integrals

1. 12 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate the indefinite integral $\int x^2 \sin x \, dx$ for $x > 0$.

Answer: $-x^2 \cos x + 2(x \sin x + \cos x) + C$

Solution: We do integration by parts with:

$$u(x) = x^2 \Rightarrow u'(x) = 2x,$$
$$v'(x) = \sin x \Rightarrow v(x) = -\cos x.$$

$$\int x^2 \sin x \, dx = -x^2 \cos x - \int 2x(-\cos x) \, dx$$

and then again integration by parts on $\int x \cos x \, dx$ with $u(x) = x$ and $v'(x) = \cos x$, and finally get:

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2(x \sin x + \cos x) + C$$

(b) Calculate the indefinite integral $\int 4x\sqrt{3 - 4x} \, dx$ for $x < 3/4$.

Answer: $-\frac{2}{5}(3-4x)^{3/2}(\frac{1}{2}+x)+C$

Solution: We take $u(x) = 3 - 4x$, then we have $u'(x) = -4$ and we replace $4x$ by $3 - u(x)$, such that we write

$$\int 4x\sqrt{3 - 4x} \, dx = -\frac{1}{4} \int (-4)4x\sqrt{3 - 4x} \, dx = -\frac{1}{4} \int (3 - u)u^{1/2}u' \, dx$$

and apply substitution rule as:

$$-\int (3 - u)u^{1/2}u' \, dx = -\left(\int 3u^{1/2} - u^{3/2} \, du \right)_{u=3-4x}$$

Anti-differentiating the simple polynomial function $3u^{1/2} - u^{3/2}$ and eventually substituting $u(x) = 3 - 4x$, we finally get:

$$\int 4x\sqrt{3 - 4x} \, dx = -\frac{1}{2} \left((3 - 4x)^{3/2} - \frac{1}{5}(3 - 4x)^{5/2} \right) + C = -\frac{2}{5}(3-4x)^{3/2} \left(\frac{1}{2} + x \right) + C$$

Note that this problem can also be solved by IBP (but more challenging) with:

$$u(x) = 4x \Rightarrow u'(x) = 4,$$
$$v'(x) = (3 - 4x)^{1/2} \Rightarrow v(x) = \frac{2}{3} \left(\frac{-1}{4} \right) (3 - 4x)^{3/2} = -\frac{1}{6}(3 - 4x)^{3/2}.$$

such that

$$I = 4x\left(-\frac{1}{6}\right)(3-4x)^{3/2} - \int 4\left(\frac{-1}{6}\right)(3-4x)^{3/2}\,dx$$
$$= \left(-\frac{2}{3}x\right)(3-4x)^{3/2} + \frac{2}{3}\int (3-4x)^{3/2}\,dx$$

Given that the anti-derivative of $\int (3-4x)^{3/2}\,dx$ is $\frac{2}{5}\left(\frac{-1}{4}\right)(3-4x)^{5/2} + C = -\frac{1}{10}(3-4x)^{5/2} + C$, we get:

$$I = (3-4x)^{3/2}\left(-\frac{2}{3}x - \frac{2}{30}(3-4x)\right) + C = (3-4x)^{3/2}\left(-\frac{1}{5} - \frac{6}{15}x\right) + C$$
$$= -\frac{2}{5}(3-4x)^{3/2}\left(\frac{1}{2} + x\right) + C$$

(c) (A Little Harder): Calculate the indefinite integral $\int \frac{\sqrt{x^2-9}}{x^2}\,dx, x > 3$. Use the following known result: $\int \sec x\,dx = \ln|\sec x + \tan x| + C$. **Write your final answer without any trigonometric function**.

Answer: $\ln|x + \sqrt{x^2-9}| - \frac{\sqrt{x^2-9}}{x} + C$

Solution: We use the trigonometric substitution $x(\theta) = 3\sec\theta$, then $x'(\theta) = 3\sec\theta\tan\theta$, and we get

$$I = \int \frac{\sqrt{x^2-9}}{x^2}\,dx = \int \frac{\sqrt{9\sec^2\theta - 9}}{9\sec^2\theta} 3\sec\theta\tan\theta\,d\theta = \int \frac{\tan^2\theta}{\sec\theta}\,d\theta$$

$\frac{\tan^2\theta}{\sec\theta}$ can be written as $\frac{\sec^2\theta - 1}{\sec\theta} = \sec\theta - \cos\theta$. Using the known result, we get:

$$I = \ln|\sec\theta + \tan\theta| - \sin\theta + C$$

Drawing a right triangle with $\cos\theta = 3/x$, we can write $\sin\theta = \frac{\sqrt{x^2-9}}{x}$, give the expressions of $\sec\theta$ and $\tan\theta$ as function of x and eventually establish that:

$$I = \ln\left|\frac{1}{3}(x + \sqrt{x^2-9})\right| - \frac{\sqrt{x^2-9}}{x} + C$$

Since $x + \sqrt{x^2-9} > 0$ for $x > 3$ and $\ln(ab) = \ln a + \ln b$, we can simplify by absorbing $\ln\frac{1}{3}$ into the constant C as:

$$I = \ln|x + \sqrt{x^2-9}| - \frac{\sqrt{x^2-9}}{x} + C$$

Definite Integrals

2. 8 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate $\int_0^{\pi/3} \sec^{3/2} x \, \tan x \, dx$.

Answer: $\frac{2}{3}[2\sqrt{2} - 1]$

Solution: This is a trigonometric integral that is calculated as:

$$I = \int_0^{\pi/3} \sec^{3/2} x \, \tan x \, dx = \int_0^{\pi/3} \sec^{1/2} x \, \sec x \, \tan x \, dx = \frac{2}{3}[\sec^{3/2} x]_0^{\pi/3}$$

which gives:

$$I = \frac{2}{3}[2^{3/2} - 1] = \frac{2}{3}[2\sqrt{2} - 1]$$

(b) Calculate $\int_{-1}^{0} \frac{x+1}{\sqrt{-2x+1-x^2}} \, dx$.

Answer: $\sqrt{2} - 1$

Solution: We can rewrite $-2x + 1 - x^2$ as $2 - (x+1)^2$ and use a trigonometric substitution as

$$x + 1 = \sqrt{2}\sin\theta \quad , \quad x'(\theta) = \frac{dx}{d\theta} = \sqrt{2}\cos\theta,$$
$$x = -1 \Rightarrow \theta = 0 \quad , \quad x = 0 \Rightarrow \theta = \pi/4$$

to get:

$$I = \int_{-1}^{0} \frac{x+1}{\sqrt{-2x+1-x^2}} \, dx = \int_0^{\pi/4} \frac{\sqrt{2}\sin\theta}{\sqrt{2 - 2\sin^2\theta}} \sqrt{2}\cos\theta \, d\theta$$

Now we replace $\sqrt{1 - \sin^2\theta}$ by $\sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$ as $\cos\theta$ is positive on $[0, \pi/4]$ and finally calculate:

$$I = \sqrt{2}\int_0^{\pi/4} \sin\theta \, d\theta = \sqrt{2}[-\cos\theta]_0^{\pi/4} = \sqrt{2} - 1$$

Note that this problem can also be solved by standard substitution: $u(x) = -2x + 1 - x^2$, $u'(x) = -2 - 2x = -2(x+1)$, $u(-1) = 2$, $u(0) = 1$ as

$$\int_{-1}^{0} \frac{x+1}{\sqrt{-2x+1-x^2}} \, dx = -\frac{1}{2}\int_{-1}^{0} \frac{-2(x+1)}{\sqrt{-2x+1-x^2}} \, dx = -\frac{1}{2}\int_{-1}^{0} u' u^{-1/2} \, dx$$

and then

$$-\frac{1}{2}\int_{-1}^{0} u'u^{-1/2}\,dx = -\frac{1}{2}\int_{2}^{1} u^{-1/2}\,du = [-u^{1/2}]_2^1 = -1 + 2^{1/2}$$

Riemann Sum and FTC

3. [12 marks] Each part is worth 4 marks. Please write your answers in the boxes.

(a) Which definite integral corresponds to $\lim_{n\to\infty}\sum_{i=1}^{n}(\frac{i}{n}+1)e^{-2\frac{i^2}{n^2}}\frac{2}{n}$?

(A) $2\int_0^1 xe^{-2(x-1)^2}dx$

(B) $2\int_0^1 (x+1)e^{-2x^2}dx$

(C) $\int_1^2 xe^{-2(x-1)^2}dx$

(D) $\int_1^2 (x+1)e^{-2x^2}dx$

(E) $\int_0^1 (x+1)e^{-2x^2}dx$

Answer: B

Solution: Pick $x_i = \frac{i}{n}$, so $x_0 = 0$, $x_n = 1$ and $\Delta x = \frac{1}{n}$. Then we can rewrite the summation as:

$$\sum_{i=1}^{n}(x_i+1)e^{-2x_i^2}2\Delta x$$

which corresponds to the Right Riemann Sum for option (B).

(b) Define $F(x)$ and $g(x)$ by $F(x) = \int_0^x \sin^2 t\,dt$ and $g(x) = x\,F(x^3)$. Calculate $g'(\pi^{1/3})$.

Answer: $\frac{\pi}{2}$

Solution: We use the product rule to get: $g'(x) = F(x^3) + x\,F'(x^3)$, and the chain rule and FTC I to calculate $F'(x^3) = F'(y)y'(x) = 3x^2\sin^2(x^3)$ with $y(x) = x^3$. So we have:

$$g'(x) = F(x^3) + 3x^3\sin(x^3)$$

and $g'(\pi^{1/3}) = F((\pi^{1/3})^3) + 3(\pi^{1/3})^3\sin^2((\pi^{1/3})^3) = F(\pi) + 3\pi\sin^2(\pi) = F(\pi)$, as $sin\pi = 0$. Now we calculate $F(\pi)$ as

$$F(\pi) = \int_0^\pi \sin^2 t\,dt = \int_0^\pi \frac{1}{2}\,dt - \int_0^\pi \frac{cos(2t)}{2}\,dt = \frac{\pi}{2} - \left[\frac{sin(2t)}{4}\right]_0^\pi = \frac{\pi}{2}$$

and get $g'(\pi^{1/3}) = \frac{\pi}{2}$.

(c) Let $F(x) = \int_{x^2}^{x^3} 3e^{t^2}\,dt$. Find the equation of the tangent line to the graph of $y = F(x)$ at $x = 1$. Tip: recall that the tangent line to the graph of $y = F(x)$ at $x = x_0$ is given by the equation $y = F(x_0) + F'(x_0)(x - x_0)$.

Answer: $y = 3e(x - 1)$

Solution: We first write $F(x)$ for any real number c as:

$$F(x) = -\int_c^{x^2} 3e^{t^2}\,dt + \int_c^{x^3} 3e^{t^2}\,dt$$

Then use FTC I and the chain rule to get:

$$F'(x) = -3e^{x^4}2x + 3e^{x^6}3x^2$$

Then we calculate $F(1)$ and $F'(1)$, we get $F(1) = \int_1^1 3e^{t^2}\,dt = 0$ and $F'(1) = 3e$, and finally the equation of the tangent $y - F(1) = F'(1)(x - 1)$ becomes

$$y = 3e(x - 1)$$

Areas and volumes

Please write your answers in the boxes. **Do not use absolute values in your expressions, always work out: (i) the outer function and the inner function for volumes or (ii) which function lies above the other function for areas.**

4. 4 marks Write a definite integral, with specified limits of integration, for the volume obtained by revolving the bounded region between $y = (x-2)^2$ and $y = x + 4$ about the horizontal line $y = 10$. **Do not evaluate the integral.**

Answer: $\pi \int_0^5 ((x-2)^2 - 10)^2 - (x-6)^2 \, dx$

Solution: Intersection points are given by $(x-2)^2 = x + 4$.

Solving for x, we determine the 2 intersection points

$$I_1 = (0,4) \quad , \quad I_2 = (5,9).$$

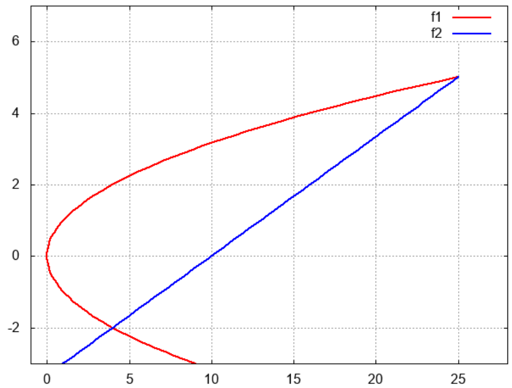
We integrate in x, hence we write y as a function of x for the 2 curves and apply a shift of -10, we finally establish:

$$\pi \int_0^5 ((x-2)^2 - 10)^2 - (x-6)^2 \, dx.$$

5. (a) 2 marks Sketch by hand the finite area enclosed by $y^2 - x = 0$ and $x - 3y = 10$

Answer:

Solution: The area is the region enclosed between the red and blue curves:



(b) 4 marks Write a definite integral with specific limits of integration that determines this finite area.

Answer: $\int_{-2}^{5} \left(-y^2 + 3y + 10\right)\, dy$

Solution: We first find the intersection between the two curves, given by the solution of:

$$y^2 = 3y + 10 \Leftrightarrow (y + 2)(y - 5) = 0.$$

We then label the curve $x_R = y^2$ and $x_B = 3y + 10$ and notice that $x_B \geq x_R$ for $-2 \leq y \leq 5$. The area is therefore given by the following definite integral:

$$A = \int_{-2}^{5} \left(3y + 10 - y^2\right)\, dy = \int_{-2}^{5} \left(-y^2 + 3y + 10\right)\, dy$$

(c) 2 marks Evaluate the integral to compute the area enclosed.

Answer: $\frac{343}{6}$

Solution:

$$A = \left[-\frac{y^3}{3} + \frac{3y^2}{2} + 10y\right]_{-2}^{5} = \frac{343}{6}$$