First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Grade:

The remainder of this page has been left blank for your workings.

Indefinite Integrals

1. $\boxed{\text{12 marks}}$ Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate the indefinite integral $\int \frac{\ln x}{\sqrt{x}}\, dx$ for $x > 0$.

Answer: $2\sqrt{x}(\ln x - 2) + C$

Solution: We do integration by parts with:

$$u(x) = \ln x \Rightarrow u'(x) = \frac{1}{x},$$
$$v'(x) = x^{-1/2} \Rightarrow v(x) = 2x^{1/2}.$$

$$\int \frac{\ln x}{\sqrt{x}}\, dx = 2x^{1/2} \ln x - \int \frac{1}{x} 2x^{1/2}\, dx$$

The second term on the rhs is simply $2\int x^{-1/2}\, dx$, and we finally get:

$$\int \frac{\ln x}{\sqrt{x}}\, dx = 2\sqrt{x}(\ln x - 2) + C$$

(b) Calculate the indefinite integral $\int -2x\sqrt{3+2x}\, dx$ for $x > -3/2$.

Answer: $\frac{2}{5}(3+2x)^{3/2}(1-x) + C$

Solution: We take $u(x) = 3 + 2x$, then we have $u'(x) = 2$ and we replace $-2x$ by $3 - u(x)$, such that we write

$$\int -2x\sqrt{3+2x}\, dx = \frac{1}{2}\int 2(-2x)\sqrt{3+2x}\, dx = \frac{1}{2}\int (3-u)u^{1/2}u'\, dx$$

and apply substitution rule as:

$$\int (3-u)u^{1/2}u'\, dx = \left(\int 3u^{1/2} - u^{3/2}\, du\right)_{u=3+2x}$$

Anti-differentiating the simple polynomial function $3u^{1/2} - u^{3/2}$ and eventually substituting $u(x) = 3 + 2x$, we finally get:

$$\int -2x\sqrt{3+2x}\, dx = \left((3+2x)^{3/2} - \frac{1}{5}(3+2x)^{5/2}\right) + C = \frac{2}{5}(3+2x)^{3/2}(1-x) + C$$

Note that this problem can also be solved by IBP (but more challenging) with:

$$u(x) = -2x \Rightarrow u'(x) = -2,$$
$$v'(x) = (3+2x)^{1/2} \Rightarrow v(x) = \frac{2}{3}\left(\frac{1}{2}\right)(3+2x)^{3/2} = \frac{1}{3}(3+2x)^{3/2}.$$

such that

$$
\begin{aligned}
I &= -2x\frac{1}{3}(3+2x)^{3/2} - \int (-2)\frac{1}{3}(3+2x)^{3/2}\,dx \\
&= \left(-\frac{2}{3}x\right)(3+2x)^{3/2} + \frac{2}{3}\int (3+2x)^{3/2}\,dx
\end{aligned}
$$

Given that the anti-derivative of $\int (3+2x)^{3/2}\,dx$ is $\frac{2}{5}\left(\frac{1}{2}\right)(3+2x)^{5/2} + C = \frac{1}{5}(3+2x)^{5/2} + C$, we get:

$$
\begin{aligned}
I = (3+2x)^{3/2}\left(-\frac{2}{3}x + \frac{2}{15}(3+2x)\right) + C &= (3+2x)^{3/2}\left(\frac{6}{15} - \frac{2}{5}x\right) + C \\
&= \frac{2}{5}(3+2x)^{3/2}(1-x) + C
\end{aligned}
$$

(c) (A Little Harder): Calculate the indefinite integral $\int \frac{x^2+x+3}{x^3+4x-x^2-4}\, dx$.

Answer: $\ln|x-1| + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$

Solution: This is a partial fraction indefinite integral. We first recognize that $x = 1$ is a root of the denominator, such that we can write

$$x^3 + 4x - x^2 - 4 = (x-1)(ax^2 + bx + c) = ax^3 + (b-a)x^2 + (c-b)x - c$$

which gives $a = 1$, $b = 0$ and $c = 4$, such that we can now write:

$$\frac{x^2 + x + 3}{x^3 + 3x + x^2 + 3} = \frac{A}{x-1} + \frac{Bx + C}{x^2 + 4} = \frac{A(x^2+4) + (Bx + C)(x-1)}{x^3 + 4x - x^2 - 4}$$

which gives $A = 1$, $B = 0$ and $C = 1$. We can now calculate the indefinite integral as:

$$\int \frac{x^2 + x + 3}{x^3 + 4x - x^2 - 4}\, dx = \int \frac{1}{x-1} + \frac{1}{x^2+4}\, dx = \ln|x-1| + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$

Definite Integrals

2. $\boxed{\text{8 marks}}$ Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate $\int_{-\pi/2}^{\pi/2} 3\cos^3 x\,dx$.

Answer: 4

Solution: This is a trigonometric integral that is calculated as:

$$I = \int_{-\pi/2}^{\pi/2} 3\cos^3 x\,dx = 3\int_{-\pi/2}^{\pi/2} \cos x\ \cos^2 x\,dx = 3\int_{-\pi/2}^{\pi/2} \cos x(1-\sin^2 x)\,dx$$

which gives:

$$I = 3\left[\sin x - \frac{\sin^3 x}{3}\right]_{-\pi/2}^{\pi/2} = 4\left(\frac{2}{3}+\frac{2}{3}\right) = 4$$

(b) Calculate $\int_{-2}^{-1} \frac{x+2}{\sqrt{-4x-2-x^2}}\,dx$.

Answer: $\sqrt{2}-1$

Solution: We can rewrite $-4x-2-x^2$ as $2-(x+2)^2$ and use a trigonometric substitution as

$$x+2 = \sqrt{2}\sin\theta \quad , \quad x'(\theta) = \frac{dx}{d\theta} = \sqrt{2}\cos\theta,$$
$$x = -2 \Rightarrow \theta = 0 \quad , \quad x = -1 \Rightarrow \theta = \pi/4$$

to get:

$$I = \int_{-2}^{-1} \frac{x+2}{\sqrt{-4x-2-x^2}}\,dx = \int_0^{\pi/4} \frac{\sqrt{2}\sin\theta}{\sqrt{2-2\sin^2\theta}}\sqrt{2}\cos\theta\,d\theta$$

Now we replace $\sqrt{1-\sin^2\theta}$ by $\sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$ as $\cos\theta$ is positive on $[0,\pi/4]$ and finally calculate:

$$I = \sqrt{2}\int_0^{\pi/4} \sin\theta\,d\theta = \sqrt{2}[-\cos\theta]_0^{\pi/4} = \sqrt{2}-1$$

Note that this problem can also be solved by standard substitution: $u(x) = -4x-2-x^2$, $u'(x) = -4-2x = -2(x+2)$, $u(-2) = 2$, $u(-1) = 1$ as

$$\int_{-2}^{-1} \frac{x+2}{\sqrt{-4x-2-x^2}}\,dx = -\frac{1}{2}\int_{-2}^{-1} \frac{-2(x+2)}{\sqrt{-4x-2-x^2}}\,dx = -\frac{1}{2}\int_{-2}^{-1} u'u^{-1/2}\,dx$$

and then

$$-\frac{1}{2}\int_{-2}^{-1} u'u^{-1/2}\,dx = -\frac{1}{2}\int_{2}^{1} u^{-1/2}\,du = [-u^{1/2}]_2^1 = -1 + 2^{1/2}$$

Riemann Sum and FTC

3. $\boxed{\text{12 marks}}$ Each part is worth 4 marks. Please write your answers in the boxes.

(a) Which definite integral corresponds to $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\sqrt{i^2+9n^2}}{i^2}$?

(A) $\int_0^3 \frac{\sqrt{x^2+1}}{x^2} dx$

(B) $3\int_0^1 \frac{\sqrt{x^2+1}}{x^2} dx$

(C) $\frac{1}{3}\int_0^1 \frac{\sqrt{x^2+1}}{x^2} dx$

(D) $\int_0^1 \frac{\sqrt{x^2+9}}{x^2} dx$

(E) $\int_0^3 \frac{\sqrt{x^2+9}}{x^2} dx$

Answer: D

Solution: Pick $x_i = \frac{i}{n}$, so $x_0 = 0$, $x_n = 1$ and $\Delta x = \frac{1}{n}$. Then we can rewrite the summation as:

$$\sum_{i=1}^{n} \frac{\sqrt{n^2x_i^2 + 9n^2}}{n^2x_i^2} = \sum_{i=1}^{n} \frac{\sqrt{x_i^2+9}}{nx_i^2} = \sum_{i=1}^{n} \frac{\sqrt{x_i^2+9}}{x_i^2}\Delta x$$

which corresponds to the Right Riemann Sum for option (D).

(b) Define $F(x)$ and $g(x)$ by $F(x) = \int_x^{2x} \cos^2 t\, dt$ and $g(x) = x\,F(x)$. Calculate $g'(\pi)$.

Answer: $3\frac{\pi}{2}$

Solution: We use the product rule to get: $g'(x) = F(x) + x\,F'(x)$, and FTC I and the chain rule to calculate $F'(x) = 2\cos^2(2x) - \cos^2 x$. So we have:

$$g'(x) = F(x) + x(2\cos^2(2x) - \cos^2 x)$$

and $g'(\pi) = F(\pi) + \pi(2\cos^2(2\pi) - \cos^2\pi) = F(\pi) + \pi$. Now we calculate $F(\pi)$ as

$$F(\pi) = \int_\pi^{2\pi} \cos^2 t\, dt = \int_\pi^{2\pi} \frac{1}{2}\, dt + \int_\pi^{2\pi} \frac{cos(2t)}{2}\, dt = \frac{\pi}{2} + \left[\frac{sin(2t)}{4}\right]_0^\pi = \frac{\pi}{2}$$

and get $g'(\pi) = \frac{\pi}{2} + \pi = \frac{3\pi}{2}$.

(c) Let $F(x) = \int_{x^2}^{x^3} 7e^{t^2}\,dt$. Find the equation of the tangent line to the graph of $y = F(x)$ at $x = 1$. Tip: recall that the tangent line to the graph of $y = F(x)$ at $x = x_0$ is given by the equation $y = F(x_0) + F'(x_0)(x - x_0)$.

Answer: $y = 7e(x - 1)$

Solution: We first write $F(x)$ for any real number c as:

$$F(x) = -\int_c^{x^2} 7e^{t^2}\,dt + \int_c^{x^3} 7e^{t^2}\,dt$$

Then use FTC I and the chain rule to get:

$$F'(x) = -7e^{x^4}2x + 7e^{x^6}3x^2$$

Then we calculate $F(1)$ and $F'(1)$, we get $F(1) = \int_1^1 7e^{t^2}\,dt = 0$ and $F'(1) = 7e$, and finally the equation of the tangent $y - F(1) = F'(1)(x - 1)$ becomes

$$y = 7e(x - 1)$$

Areas and volumes

Please write your answers in the boxes. **Do not use absolute values in your expressions, always work out: (i) the outer function and the inner function for volumes or (ii) which function lies above the other function for areas.**

4. $\boxed{\text{4 marks}}$ Write a definite integral, with specified limits of integration, for the volume obtained by revolving the bounded region between $x = -(y-4)^2$ and $x = -2 - y$ about the vertical line $x = 1$. **Do not evaluate the integral.**

Answer: $\pi \int_2^7 (-3-y)^2 - (-1-(y-4)^2)^2 \, dy$

Solution: Intersection points are given by $-2 - y = -(y-4)^2$.

Solving for y, we determine the 2 intersection points

$$I_1 = (-4, 2) \quad , \quad I_2 = (-9, 7).$$

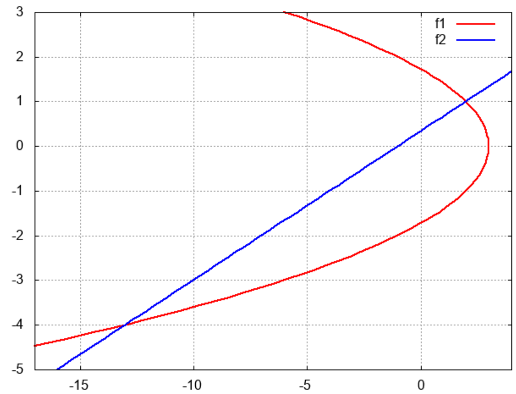
We integrate in y, hence we write x as a function of y for the 2 curves and apply a shift of -1, we finally establish:

$$\pi \int_2^7 (-3-y)^2 - (-1-(y-4)^2)^2 \, dy.$$

5. (a) $\boxed{\text{2 marks}}$ Sketch by hand the finite area enclosed by $y^2 = 3 - x$ and $3y = x + 1$

Answer:

Solution: The area is the region enclosed between the red and blue curves:



(b) $\boxed{\text{4 marks}}$ Write a definite integral with specific limits of integration that determines this finite area.

Answer: $\int_{-4}^{1} (-y^2 - 3y + 4)\ dy$

Solution: We first find the intersection between the two curves, given by the solution of:

$$3 - y^2 = 3y - 1 \Leftrightarrow (y + 4)(y - 1) = 0.$$

We then label the curve $x_R = 3 - y^2$ and $x_B = 3y - 1$ and notice that $x_B \leq x_R$ for $-4 \leq y \leq 1$. The area is therefore given by the following definite integral:

$$A = \int_{-4}^{1} \left(3 - y^2 - 3y + 1\right)\ dy = \int_{-4}^{1} \left(-y^2 - 3y + 4\right)\ dy$$

(c) $\boxed{\text{2 marks}}$ Evaluate the integral to compute the area enclosed.

Answer: $\frac{125}{6}$

Solution:

$$A = \left[-\frac{y^3}{3} - \frac{3y^2}{2} + 4y\right]_{-4}^{1} = \frac{125}{6}$$