First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Grade:

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Indefinite Integrals

1. $\boxed{\text{12 marks}}$ Each part is worth 4 marks. Please write your answers in the boxes.

   (a) Calculate the indefinite integral $\int \cos x \ln(\sin x)\, dx$ for $\sin x > 0$.

   Answer: $\sin x(\ln(\sin x) - 1) + C$

   Solution: We do integration by parts with:

   $$u(x) = \ln(\sin x) \Rightarrow u'(x) = \frac{\cos x}{\sin x},$$
   $$v'(x) = \cos x \Rightarrow v(x) = \sin x.$$

   $$\int \cos x \ln(\sin x)\, dx = \sin x \ln(\sin x) - \int \frac{\cos x}{\sin x} \sin x\, dx$$

   The second term on the rhs is simply $-\int \cos x\, dx$, and we finally get:

   $$\int \cos x \ln(\sin x)\, dx = \sin x(\ln(\sin x) - 1) + C$$

   (b) Calculate the indefinite integral $\int 2x\sqrt{3-2x}\, dx$ for $x < 3/2$.

   Answer: $-\frac{2}{5}(3-2x)^{3/2}(1+x)+C$

   Solution: We take $u(x) = 3 - 2x$, then we have $u'(x) = -2$ and we replace $2x$ by $3 - u(x)$, such that we write

   $$\int 2x\sqrt{3-2x}\, dx = -\frac{1}{2}\int (-2)2x\sqrt{3-2x}\, dx = -\frac{1}{2}\int (3-u)u^{1/2}u'\, dx$$

   and apply substitution rule as:

   $$\int (3-u)u^{1/2}u'\, dx = \left(\int 3u^{1/2} - u^{3/2}\, du\right)_{u=3-2x}$$

   Anti-differentiating the simple polynomial function $3u^{1/2} - u^{3/2}$ and eventually substituting $u(x) = 3 - 2x$, we finally get:

   $$\int 2x\sqrt{3-2x}\, dx = -\left((3-2x)^{3/2} - \frac{1}{5}(3-2x)^{5/2}\right) + C = -\frac{2}{5}(3-2x)^{3/2}(1+x) + C$$

   Note that this problem can also be solved by IBP (but more challenging) with:

   $$u(x) = 2x \Rightarrow u'(x) = 2,$$
   $$v'(x) = (3-2x)^{1/2} \Rightarrow v(x) = \frac{2}{3}\left(\frac{-1}{2}\right)(3-2x)^{3/2} = -\frac{1}{3}(3-2x)^{3/2}.$$

such that

$$I = 2x\left(-\frac{1}{3}\right)(3-2x)^{3/2} - \int 2\left(-\frac{1}{3}\right)(3-2x)^{3/2}\,dx$$
$$= \left(-\frac{2}{3}x\right)(3-2x)^{3/2} - \frac{2}{3}\int (3-2x)^{3/2}\,dx$$

Given that the anti-derivative of $\int (3-2x)^{3/2}\,dx$ is $\frac{2}{5}\left(-\frac{1}{2}\right)(3-2x)^{5/2} + C = -\frac{1}{5}(3-2x)^{5/2} + C$, we get:

$$I = (3-2x)^{3/2}\left(-\frac{2}{3}x - \frac{2}{15}(3-2x)\right) + C = (3-2x)^{3/2}\left(-\frac{6}{15} - \frac{2}{5}x\right) + C$$
$$= -\frac{2}{5}(3-2x)^{3/2}(1+x) + C$$

(c) (A Little Harder): Calculate the indefinite integral $\int \frac{\sqrt{x^2-16}}{x^2}\,dx, x > 4$. Use the following known result: $\int \sec x\,dx = \ln|\sec x + \tan x| + C$. **Write your final answer without any trigonometric function**.

Answer: $\ln(x + \sqrt{x^2 - 16}) - \frac{\sqrt{x^2-16}}{x} + C$

Solution: We use the trigonometric substitution $x(\theta) = 4\sec\theta$, then $x'(\theta) = 4\sec\theta\tan\theta$, and we get

$$I = \int \frac{\sqrt{x^2 - 16}}{x^2}\,dx = \int \frac{\sqrt{16\sec^2\theta - 16}}{16\sec^2\theta} 4\sec\theta\tan\theta\,d\theta = \int \frac{\tan^2\theta}{\sec\theta}\,d\theta$$

$\frac{\tan^2\theta}{\sec\theta}$ can be written as $\frac{\sec^2\theta - 1}{\sec\theta} = \sec\theta - \cos\theta$. Using the known result, we get:

$$I = \ln|\sec\theta + \tan\theta| - \sin\theta + C$$

Drawing a right triangle with $\cos\theta = 4/x$, we can write $\sin\theta = \frac{\sqrt{x^2-16}}{x}$, give the expressions of $\sec\theta$ and $\tan\theta$ as function of x and eventually establish that:

$$I = \ln\left|\frac{1}{4}(x + \sqrt{x^2 - 16})\right| - \frac{\sqrt{x^2 - 16}}{x} + C$$

Since $x + \sqrt{x^2 - 16} > 0$ for $x > 4$ and $\ln(ab) = \ln a + \ln b$, we can simplify by absorbing $\ln\frac{1}{4}$ into the constant C as:

$$I = \ln(x + \sqrt{x^2 - 16}) - \frac{\sqrt{x^2 - 16}}{x} + C$$

Definite Integrals

2. $\boxed{\text{8 marks}}$ Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate $\int_1^5 \frac{x-1}{x^2(x+1)}\,dx$.

Answer: $-\frac{4}{5} + ln\left(\frac{25}{9}\right)$

Solution: This is a partial fraction integral. First we recognize that:

$$\frac{x-1}{x^2(x+1)} = -\frac{1}{x^2} - \frac{2}{x+1} + \frac{2}{x}$$

Then we calculate the integral as:

$$I = \int_1^5 \frac{x-1}{x^2(x+1)}dx = \int_1^5 \left(-\frac{1}{x^2} - \frac{2}{x+1} + \frac{2}{x}\right)dx = \left[\frac{1}{x} - 2ln(x+1) + 2ln(x)\right]_1^5$$

and eventually

$$I = \left[\frac{1}{x} + 2ln\left(\frac{x}{x+1}\right)\right]_1^5 = -\frac{4}{5} + 2\left[ln\left(\frac{5}{6}\right) - ln\left(\frac{1}{2}\right)\right] = -\frac{4}{5} + ln\left(\frac{25}{9}\right)$$

(b) Calculate $\int_2^3 \frac{x-2}{\sqrt{4x-2-x^2}}\,dx$.

Answer: $\sqrt{2}-1$

Solution: We can rewrite $4x-2-x^2$ as $2-(x-2)^2$ and use a trigonometric substitution as

$$x-2 = \sqrt{2}\sin\theta \quad , \quad x'(\theta) = \frac{dx}{d\theta} = \sqrt{2}\cos\theta,$$
$$x = 2 \Rightarrow \theta = 0 \quad , \quad x = 3 \Rightarrow \theta = \pi/4$$

to get:

$$I = \int_2^3 \frac{x-2}{\sqrt{4x-2-x^2}}\,dx = \int_0^{\pi/4} \frac{\sqrt{2}\sin\theta}{\sqrt{2-2\sin^2\theta}}\sqrt{2}\cos\theta\,d\theta$$

Now we replace $\sqrt{1-\sin^2\theta}$ by $\sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$ as $\cos\theta$ is positive on $[0,\pi/4]$ and finally calculate:

$$I = \sqrt{2}\int_0^{\pi/4} \sin\theta\,d\theta = \sqrt{2}[-\cos\theta]_0^{\pi/4} = \sqrt{2}-1$$

Note that this problem can also be solved by standard substitution: $u(x) = 4x - 2 - x^2$, $u'(x) = 4 - 2x = -2(x-2)$, $u(2) = 2$, $u(3) = 1$ as

$$\int_2^3 \frac{x-2}{\sqrt{4x-2-x^2}}\,dx = -\frac{1}{2}\int_2^3 \frac{-2(x-2)}{\sqrt{4x-2-x^2}}\,dx = -\frac{1}{2}\int_2^3 u'u^{-1/2}\,dx$$

and then

$$-\frac{1}{2}\int_2^3 u'u^{-1/2}\,dx = -\frac{1}{2}\int_2^1 u^{-1/2}\,du = [-u^{1/2}]_2^1 = -1 + 2^{1/2}$$

Riemann Sum and FTC

3. $\boxed{\text{12 marks}}$ Each part is worth 4 marks. Please write your answers in the boxes.

(a) Which definite integral corresponds to $\lim_{n\to\infty}\sum_{i=1}^{n}(\frac{6i}{n}+e^{9\frac{i^2}{n^2}})\sin(\frac{2i}{n}+1)\frac{1}{n}$?

(A) $\int_0^9 (x+e^{x^2})\sin(x+1)dx$
(B) $\int_0^6 (x+e^{\frac{1}{4}x^2})\sin(\frac{1}{3}x+1)dx$
(C) $\int_0^3 (2x+e^{x^2})\sin(\frac{2}{3}x+1)dx$
(D) $\int_0^2 (3x+e^{\frac{9}{4}x^2})\sin(x+1)dx$
(E) $\int_0^1 (6x+e^{9x^2})\sin(2x+1)dx$

Answer: E

Solution: Pick $x_i = \frac{i}{n}$, so $x_0 = 0$, $x_n = 1$ and $\Delta x = \frac{1}{n}$. Then we can rewrite the summation as:

$$\sum_{i=1}^{n}(6x_i + e^{9x_i^2})\sin(2x_i+1)\Delta x$$

which corresponds to the Right Riemann Sum for option (E).

(b) Define $F(x)$ and $g(x)$ by $F(x) = \int_1^x \ln t\,dt$ and $g(x) = (F(x^2))^2$ for $x > 1$. Calculate $g'(2)$. Give the answer as a function of $\ln 2$.

Answer: $16\ln\,2(8\ln 2 - 3)$

Solution: We use the derivative of the power of a function rule to get $g'(x) = 2F'(x^2)F(x^2)$, and the chain rule and FTC I to calculate $F'(x^2) = F'(y)y'(x) = 2x\ln x^2$ with $y(x) = x^2$. Hence we have:

$$g'(x) = 4x\ln x^2\,F(x^2)$$

and

$$g'(2) = 8\ln 2^2\,F(4) = 16\ln\,2\,F(4)$$

By IBP, we calculate:

$$F(4) = \int_1^4 \ln t\,dt = [t\ln t]_1^4 - \int_1^4 1\,dt = 4\ln 4 - 3 = 8\ln 2 - 3$$

and get $g'(2) = 16\ln\,2(8\ln 2 - 3)$.

(c) Let $F(x) = \int_{x^2}^{x^3} 2e^{t^2}\,dt$. Find the equation of the tangent line to the graph of $y = F(x)$ at $x = 1$. Tip: recall that the tangent line to the graph of $y = F(x)$ at $x = x_0$ is given by the equation $y = F(x_0) + F'(x_0)(x - x_0)$.

Answer: $y = 2e(x - 1)$

Solution: We first write $F(x)$ for any real number c as:

$$F(x) = -\int_c^{x^2} 2e^{t^2}\,dt + \int_c^{x^3} 2e^{t^2}\,dt$$

Then use FTC I and the chain rule to get:

$$F'(x) = -2e^{x^4}2x + 2e^{x^6}3x^2$$

Then we calculate $F(1)$ and $F'(1)$, we get $F(1) = \int_1^1 2e^{t^2}\,dt = 0$ and $F'(1) = 2e$, and finally the equation of the tangent $y - F(1) = F'(1)(x - 1)$ becomes

$$y = 2e(x - 1)$$

Areas and volumes

Please write your answers in the boxes. **Do not use absolute values in your expressions, always work out: (i) the outer function and the inner function for volumes or (ii) which function lies above the other function for areas.**

4. $\boxed{\text{4 marks}}$ Write a definite integral, with specified limits of integration, for the volume obtained by revolving the bounded region between $x = 10-(y-1)^2$ and $x = 2+(y-1)^2$ about the vertical line $x = 1$. **Do not evaluate the integral.**

Answer: $\pi \int_{-1}^{3} (9-(y-1)^2)^2 - (1+(y-1)^2)^2 \, dy$

Solution: Intersection points are given by $10-(y-1)^2 = 2+(y-1)^2$.

Solving for y, we determine the 2 intersection points

$$I_1 = (6, -1) \quad , \quad I_2 = (6, 3).$$

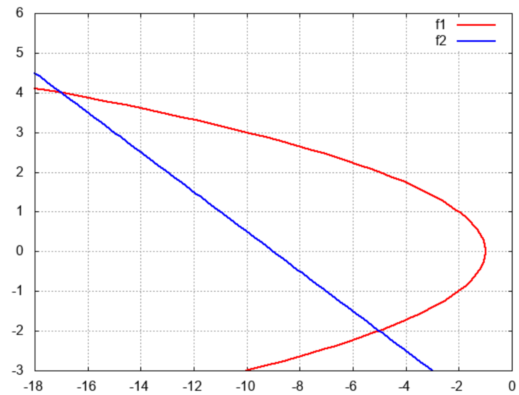
We integrate in y, hence we write x as a function of y for the 2 curves and apply a shift of -1, we finally establish:

$$\pi \int_{-1}^{3} (9-(y-1)^2)^2 - (1+(y-1)^2)^2 \, dy.$$

5. (a) $\boxed{\text{2 marks}}$ Sketch by hand the finite area enclosed by $y^2+1=-x$ and $2y=-9-x$

Answer:

Solution: The area is the region enclosed between the red and blue curves:



(b) $\boxed{\text{4 marks}}$ Write a definite integral with specific limits of integration that determines this finite area.

Answer: $\int_{-2}^{4}\left(-y^2+2y+8\right)\,dy$

Solution: We first find the intersection between the two curves, given by the solution of:

$$-1-y^2=-2y-9 \Leftrightarrow (y+2)(y-4)=0.$$

We then label the curve $x_R=-1-y^2$ and $x_B=-2y-9$ and notice that $x_B \leq x_R$ for $-2 \leq y \leq 4$. The area is therefore given by the following definite integral:

$$A=\int_{-2}^{4}\left(-1-y^2+2y+9\right)\,dy=\int_{-2}^{4}\left(-y^2+2y+8\right)\,dy$$

(c) $\boxed{\text{2 marks}}$ Evaluate the integral to compute the area enclosed.

Answer: 36

Solution:

$$A = \left[-\frac{y^3}{3} + \frac{2y^2}{2} + 8y \right]_{-2}^{4} = 36$$